29. *On Vector Lattice with a Unit.*

By Kôsaku YOSIDA.
Mathematical Institute, Osaka Imperial University.
(Comm. by T. TAKAGI, M.I.A., May 12, 1941.)

§ 1. *Introduction and the theorems.* The purpose of the present note is to give a new proof to Kakutani-Krein's lattice-theoretic characterisation of the space of continuous functions on a bicompact space[1]. We first represent algebraically the vector lattice as point functions and then introduce the topology to the represented function lattice, while Kakutani-Krein's (independent and different) proofs both make use of the assumed norm and hence the conjugate space of the vector lattice. Our treatment may be compared with Birkhoff-Stone-Wallman's representation of Boolean algebra as field of sets or with Gelfand's[2] representation of normed ring as function ring[3].

A *vector lattice* E is a partially ordered real linear space, some of whose elements f are "non-negative" (written $f \geqq 0$) and in which

(V 1): If $f \geqq 0$ and $\alpha \geqq 0$, then $\alpha f \geqq 0$.
(V 2): If $f \geqq 0$ and $-f \geqq 0$, then $f=0$.
(V 3): If $f \geqq 0$ and $g \geqq 0$, then $f+g \geqq 0$.
(V 4): E is a lattice by the semi-order relation $f \geqq g$.

In this note we further assume the *Archimedean axiom*

(V 5): If $f > 0$ and $\alpha_n \downarrow 0$, then $\alpha_n f \downarrow 0$ (in order-topology),

and the existence of a *unit* $I > 0$ satisfying

(V 6): For any $f \in E$ there exists $\alpha > 0$ such that $-\alpha I \leqq f \leqq \alpha I$.

A linear subspace N of E is called an *ideal*[4] if N contains with f all x satisfying $|x| \leqq |f|$. Here we put, as usual, $|f| = f^+ - f^-$, $f^+ = f \vee 0 = \sup(f, 0)$, $f^- = f \wedge 0 = \inf(f, 0)$. An ideal $N \neq E$ is called *maximal* if it is contained in no other ideal $\neq E$. Denote by $\mathfrak{N}$ the set of all the maximal ideals N of E. It is proved below that the residual class E/N of E mod any maximal ideal N is linear-lattice-isomorphic to the vector lattice of real numbers, the non-negative elements $\in E$ and I respectively being represented by non-negative numbers and the number 1. We denote by $f(N)$ the real number which

---

1) S. Kakutani: Proc. **16** (1940), 63-67. M. and S. Krein: C. R. URSS, **27** (1940), 427-430.

2) I. Gelfand: C. R. URSS, **23** (1939), 430-432.

3) After the present paper was completed, I knew, by Y. Kawada's remark, the paper of M. H. Stone: Proc. Nat. Acad. Sci. **27** (1941), 83-87, which arrived at our institute only recently. In this paper, Stone sketches a proof of Kakutani's theorem which is essentially the same as ours. (He seems to have not read Krein's paper.) Stone proves firstly the case of Banach vector lattice and then reduces the general case to it, while we first prove the algebraic case and then deduce from it the case of Banach vector lattice. It is to be noted that we do not, in the proof of the representation theorem, make use of the metrical nor order completeness other than the Archimedean axiom (V 5).

(4) Normal subspace in the terminology of Garrett Birkhoff: Lattice Theory, New York (1940).

corresponds to $f \in E$ by the homomorphism $E \to E/N$, $N \in \mathfrak{N}$. Then we have the

Theorem 1. By the correspondence $f \to f(N)$, E is linear-lattice-isomorphically mapped on the vector lattice $F(\mathfrak{N})$ of real-valued bounded functions on $\mathfrak{N}$ such that i) $|f| \to |f(N)|$, ii) $I(N) \equiv 1$ on $\mathfrak{N}$ and iii) $F(\mathfrak{N})$ is *separable* on $\mathfrak{N}$, viz.

(V 7): For any two different points N_1, $N_2 \in \mathfrak{N}$ and for any two real numbers α_1, α_2 there exists $f \in E$ satisfying $f(N_1) = \alpha_1$, $f(N_2) = \alpha_2$.

Next introduce a topology in $\mathfrak{N}$ by calling open the set of all points $N \in \mathfrak{N}$ which satisfy $|f_i(N) - f_i(N_0)| < \varepsilon$, $i = 1, 2, \dots, n$, where $N_0 \in \mathfrak{N}$, $\varepsilon > 0$, n and $f_i(-I \leqq f_i \leqq I)$ are arbitrary. Then $\mathfrak{N}$ is bicompact since it may be identified with a closed subset of a topological product (of the same potency as the cardinal number of elements $\in E$ between $-I$ and I) of the real intervals $(-1, +1)$. Each function $f(N) \in F(\mathfrak{N})$ is continuous on the bicompact space $\mathfrak{N}$ by the above topology. The set $C(\mathfrak{N})$ of all the real-valued continuous functions $C(N)$ on $\mathfrak{N}$ is a Banach vector lattice with the norm $\|c\| = \sup |c(N)|$, by calling "non-negative" the non-negative functions. As a precision to theorem 1 we have

Theorem 2. $F(\mathfrak{N})$ is dense in $C(\mathfrak{N})$ by the norm $\|c\|$.

Theorem 3. If we assume that E is a Banach space by the norm $\|f\| = \inf \alpha$, $-\alpha I \leqq f \leqq \alpha I$, then the representation $f \to f(N)$ is not only linear-lattice-isomorphism but also an isometry and $F(\mathfrak{N})$ coincides with the whole $C(\mathfrak{N})$.

The theorem 3 is the Kakutani-Krein's characterisation mentioned above.

§ 2. *The proofs.* The theorem 1 may be proved by the following series of lemmas.

Lemma 1. A linear subspace N of E determines a lattice homomorphism $E \to E/N$ if and only if N is an ideal.

Proof. See G. Birkhoff: loc. cit., p. 109.

Lemma 2. Unless $E \neq \{\alpha I\}$, $-\infty < \alpha < \infty$, E contains a non-trivial $(\neq 0, E)$ ideal.

Proof. Let $f \neq \gamma I$ for any γ. Let $\alpha = \inf \alpha'$, $\alpha' I \geqq f$, $\beta = \sup \beta'$, $\beta' I \leqq f$, then $\alpha I \geqq f \geqq \beta I$ and $\alpha > \beta$. Hence $(f - \delta I)^+ \neq 0$, $(f - \delta I)^- \neq 0$ for $\alpha > \delta > \beta$. The set N of all the elements $g \in E$ satisfying $|g| \leqq \eta (f - \delta I)^+$ is a non-trival ideal since $N \not\ni (f - \delta I)^-$.

Lemma 3. For any non-trivial ideal N_0, there exists a maximal ideal N containing N_0.

Proof. Construct a sequence of ideals

$$N_0, N_1, \dots, N_\eta, \dots \quad (N_\eta \subset N_{\eta+1} \neq E), \quad \eta < \omega .$$

If ω is a limit ordinal, define $f \equiv g \bmod N_\omega$ to mean $f \equiv g \bmod N_\eta$ for some $\eta < \omega$. That N_ω is an ideal follows from lemma 1, moreover $N_\omega \neq E$ since $I \not\equiv 0 \bmod N_\eta$, $\eta < \omega$. This process defines a transfinite sequence of linear-lattice-congruence relations on E, each more inclusive than the last. Hence it cannot continue indefinitely, for the number of different congruence relations are bounded. Thus, by the lemma 2, we would obtain the demanded maximal ideal $N \supset N_0$.

Lemma 4. If $f \neq 0$, then there exists a maximal ideal N not containing f.

Proof. If $|f| \geqq \alpha I$ for a certain $\alpha > 0$, then the lemma is evident from lemma 2 and 3. Let such α do not exist, and assume $f^+ > 0$ and $f^+ \geqq \alpha I$ for no $\alpha > 0$, without losing the generality. Let $\beta = \inf \gamma$, $\gamma I \geqq f^+$, then $\beta > 0$ by $f^+ > 0$. We have $\beta I - f^+ > 0$ and $\beta I - f^+ \geqq \delta f^+$ for no $\delta > 0$, by the definition of β. Thus the ideal N_0 of all the elements $g \in E$ satisfying $|g| \leqq \eta(\beta I - f^+)$ does not contain f^+. Hence the maximal ideal N containing N_0 does not contain f^+, since otherwise N would contain I and hence would coincide with E. A fortiori N does not contain f.

We have incidentally proved that for any two different points N_1. $N_2 \in \mathfrak{N}$, there exists $f \in E$ such that $f \equiv 0 \mod N_1$, $f \equiv 1 \mod N_2$. Hence the function lattice $F(\mathfrak{N})$ on the bicompact space $\mathfrak{N}$ satisfies the separability axiom (V 7). Thus, by M. and S. Krein's lemma in the cited paper[1], $F(\mathfrak{N})$ is dense in $C(\mathfrak{N})$ by the norm $\|c\|$. This proves theorem 2, and the proof of theorem 3 is now evident.

§ 3. *Determination of the maximal ideals.* Let T be a set of points t and let $E(T)$ be a real linear set of real-valued bounded functions $f(t)$ on T such that: i) $E(T)$ contains the constant function 1, ii) $E(T)$ is a lattice such that we have $|f| = |f(t)|$, iii) for any two points $t_1, t_2 \in T (t_1 \neq t_2)$ there exists a function $f(t) \in E(T)$ satisfying $f(t_1) = 0$, $f(t_2) = 1$ and iv) $E(T)$ is a Banach space by the norm $\|f\| = \sup |f(t)|$. Then any point $t \in T$ induces a linear-lattice-homomorphism $f \to f(t)$ of $E(T)$ on the vector lattice of real numbers. Hence each $t \in T$ defines a maximal ideal $N_t (f \to f(t)$ by $E(T) \to E(T)/N_t)$ of $E(T)$. The set $\{N_t\}$ is dense in the bicompact space $\mathfrak{N}$ of all the maximal ideals N of $E(T)$. For if an element $f \in E(T)$ satisfies $f(t) \equiv 0$ on T, then f is the zero-element of $E(T)$ and hence the continuous function on $\mathfrak{N}$, which represent f by the theorem 3, must be $\equiv 0$ on $\mathfrak{N}$. Thus we obtain, by making use of iii)

Theorem 4. T is mapped by $t \to N_t$ one-to-one on a dense subset of $\mathfrak{N}$.

If T is a *completely regular* topological space and if $E(T)$ is the set of all the real-valued bounded *continuous* functions on T, then $E(T)$ surely satisfies i)–iv). In this case we easily verify, by the definition of the topology of $\mathfrak{N}$ (§ 2), that the correspondence $t \leftrightarrow N_t$ is a homeomorphism. Thus, by theorem 4, we obtain Tychonoff's theorem[2] concerning the imbedding of the completely regular space in bicompact space[3]. In particular, when T is bicompact, T may be identified with $\mathfrak{N}$ by the homeomorphism $t \leftrightarrow N_t$. Hence two bicompact spaces T_1 and

---

1) They stated this lemma without proof. I express my thanks to T. Nakayama and M. Fukamiya for discussing the proof of this lemma.

2) A. Tychonoff: Math. Annalen, **102** (1930), 544–561.

3) Similar treatment, which makes use of the conjugate space of $E(T)$ (without regarding $E(T)$ as a lattice), was given by S. Kakutani: Isôsûgaku (Japanese), **2** (1940), 14–21.

T_2 are homeomorphic if and only if the vector lattices $E(T_1)$ and $E(T_2)$ are linear-lattice-isomorphic in such a way that the constant function 1 corresponds to the constant function 1. This may be considered as lattice-theoretic counterpart of Gelfand-Kolmogoroff's theorem[1].

---

1) I. Gelfand and A. Kolmogoroff: C. R. URSS, **22** (1939), 11-15.