PAPERS COMMUNICATED

28. *The Exceptional Values of Functions with the Set of Linear Measure Zero, of Essential Singularities.*

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1. Let $w=f(z)$ be a function which is meromorphic in a domain D except for a set E of essential singularities.

The object of this paper is to obtain some metrical property of the set of the values omitted by $f(z)$ near each point of E, provided that E is of Carathéodory's linear measure zero, the result of which is more precise than that given recently by M. L. Cartwright[1].

2. Let us begin by giving some preliminary definitions and notations. Given an increasing continuous function $h(t)$ defined for $t \geqq 0$ with $h(0)=0$, we shall denote, for each $\varepsilon>0$, by $L_\varepsilon^h(A)$ the lower bound of all the sums $\sum_i h[\delta(A_i)]$ where $\{A_i\}_{i=1,2,\dots}$ is an arbitrary partition of a point-set A into a sequence of sets whose diameters $\delta(A_i)$ are all less than ε and no two of which have common points.

Making ε tend to zero, $L_\varepsilon^h(A)$ tends monotonically to a limit (finite or infinite) which is called the h-measure of A and denoted by $L^h(A)$. In particular, in the case that $h(t)=t^p$ $(p>0)$, we call this h-measure p-dimensional measure and write $L^{(p)}$ instead of L^{t^p}. One dimensional measure is also called (Carathéothory's) linear measure. Evidently, any set of Lebesgue's plane measure zero is equivalent to that of 2-dimentional measure zero.

As is well known, all these h-measures have the property of Carathéodory's outer measure[2].

3. In his very important and interesting paper entitled "On sufficient conditions for a function to be analytic," Besicovitch has shown among others that, *if E is of linear measure zero, then the function $f(z)$ is unbounded near each point of E.* He also proved from this that *the set of values taken by $f(z)$ near each point of E is everywhere dense*[3]. This is evidently a generalization of Weierstrass' classical theorem.

One might expect that this could be extended so far as to the theorem of Picard's type. Seidel has shown by an example that this is

1) M. L. Cartwright. The exceptional values of functions with a non-enumerable set of essential singularities, Quart. J. Math., Vol. **8** (1937).

2) F. Hausdorff. Dimension und äusseres Mass, Math. Ann., Vol. **79** (1919).

3) A. S. Besicovitch. On sufficient conditions for a function to be analytic, and on behaviour of analytic functions in the neighbourhood of non-isolated singular points, Proc. London Math. Soc. (2), Vol. **32** (1931).

Though his result was first proved for regular functions, it may be needless to say that the same holds even for meromorphic functions.

not the case[1], while M. L. Cartwright has obtained the following result[2]:

If the set E is of linear measure zero, then $f(z)$ takes all finite values except perhaps a set of plane measure zero near each point of E.

Now the theorem that we are to prove is:

Theorem. *If the set E is of linear measure zero, then, near each point of E, $f(z)$ takes all finite values except perhaps those belonging to a set of h-measure zero, if only*

$$\lim_{\sigma \to +0} \int_\sigma^\tau h(t)/t^2 dt < \infty \quad \textit{for some} \quad \tau > 0 . \tag{1}$$

In particular, given any positive ε and η however small, there exists a sequence of circles $\{C_i\}$ such that the interiors of all the C_i together cover the set of the exceptional values and that

$$\sum_i [\delta(C_i)]^{1+\eta} < \varepsilon .$$

To prove our theorem, we require the following lemma which is based on some knowledge of potential theory.

Lemma. *If a bounded and closed set F lying in Euclidean plane ω is of Newtonian capacity positive, then there exists a one-valued analytic function bounded and non-constant outside F.*

Proof of Lemma. From the definition of the Newtonian capacity[3], there exists a non-negative mass distribution μ on F with $\mu(F)=1$ such that the Newtonian potential

$$H(P) = \int_F \frac{d\mu(Q)}{r_{PQ}}$$

is bounded in $\omega - F$, where r_{PQ} is the Euclidean distance between the points P and Q, the integral being taken in the sense of Lebesgue-Stieltjes-Radon[4].

Let us consider now the following one-valued analytic function of $w=u+iv$ in $\omega - F$,

$$\varphi(w) = \int_F \frac{d\mu(\zeta)}{\zeta - w} = \int_F \frac{d\mu(Q)}{r_{PQ} e^{i\theta_{PQ}}}$$
$$= \int_F (\cos\theta_{PQ}/r_{PQ})\, d\mu(Q) - i\int_F (\sin\theta_{PQ}/r_{PQ})\, d\mu(Q) ,$$

where we put $\zeta - w = r_{PQ} \cdot e^{i\theta_{PQ}}$, P and Q being points represented by the complex number $w \in \omega - F$ and $\zeta \in F$ respectively, and so θ_{PQ} is the angle made by the vector $\overrightarrow{PQ}$ with the positive semi-axis of u.

1) W. Seidel. On the distribution of values of bounded analytic functions, Trans. of Amer. Math. Soc., Vol. **34**. esp. Sec. 16.

2) Cartwright. Loc. cit. Theorem 1.

3) As regards the definition of capacity with respect to the function $1/r^\alpha$ ($\alpha > 0$), namely, of order α, see, for example, O. Frostman, Potentiel d'équilibre et capacité des ensembles. Lund. (1935) Thése, pp. 42–52. The case where $\alpha=1$ is the Newtonian.

4) Frostman. Loc. cit. Chap. I. S. Saks. Theory of the Integral (1937), Chap. I.

Hence

$$|\varphi(w)| \leqq \int_F r_{PQ}^{-1} d\mu(Q) = H(P),$$

from which follows the boundedness of $\varphi(w)$ in $\omega - F$.

It remains only to prove that $\varphi(w)$ is non-constant. For this purpose, it is sufficient to show the real part of $\varphi(w)$, namely, $\varphi_r(P) = \int_F \cos\theta_{PQ} \cdot r_{PQ}^{-1} d\mu(Q)$ is not a constant.

Since the set F is bounded, we can enclose it with a circle C whose centre and radius will be denoted by S and ρ respectively. The straight line s passing through S and parallel to the axis of u is divided by S into two half-lines s_1 and s_2 issuing from S and making angles 0 and π respectively with the positive semi-axis of u.

Denoting by P_1 and P_2 the points on s_1 and s_2 with distance 2ρ from S respectively, we find easily, for all $Q \in F$

$$r_{P_iQ} \leqq 3\rho \qquad (i=1,2) \tag{2}$$

and

$$|\theta_{P_1Q}| \geqq \frac{5\pi}{6}\left(>\frac{\pi}{2}\right) \quad \text{and} \quad |\theta_{P_2Q}| \leqq \frac{\pi}{6}\left(<\frac{\pi}{2}\right),$$

so that we have for all $Q \in F$

$$\cos\theta_{P_1Q} \leqq -\frac{\sqrt{3}}{2} = -k \quad \text{and} \quad \cos\theta_{P_2Q} \geqq \frac{\sqrt{3}}{2} = k$$

which shows

$$\begin{aligned} \varphi_r(P_1) &= \int_F \cos\theta_{P_1Q} \cdot r_{P_1Q}^{-1} d\mu(Q) \leqq -k\int_F r_{P_1Q}^{-1} d\mu(Q) \\ &\leqq -k\int_F (3\rho)^{-1} d\mu(Q) = -k \cdot (3\rho)^{-1} < 0 \end{aligned} \tag{3}$$

and

$$\varphi_r(P_2) = \int_F \cos\theta_{P_2Q} \cdot r_{P_2Q}^{-1} d\mu(Q) \geqq k\int_F r_{P_2Q}^{-1} d\mu(Q) \geqq k \cdot (3\rho)^{-1} > 0. \tag{3'}$$

From (3) and (3′) it follows that $\varphi_r(P)$ is not a constant, which completes the proof of our lemma.

Proof of Theorem. Suppose that our Theorem were false. Then there would exist a point z_0 of E such that the set A of all the values omitted by $f(z)$ near z_0 were of positive h-measure for some $h(t)$ satisfying the condition (1).

Since we can represent the plane ω as the sum of an enumerable number of closed squares K_i $(i=1, 2, \ldots)$, we have $A = A\sum K_i = \sum AK_i$, so that

$$0 < L^h(A) \leqq \sum L^h(AK_i)$$

from which follows that there exists an integer i_0 such that

$$0 < L^h(AK_{i_0}).$$

Writing $F=AK_{i_0}$, we find this bounded and also closed since the set A is a complementary set of an open set. Then, by the theorem of Myrberg-Frostman, it follows that the set F is of Newtonian capacity positive[1].

Constructing the function $\varphi(w)$ stated in our Lemma, let us now consider the following function:

$$\Phi(z)=\varphi[f(z)].$$

Since $\varphi(w)$ is bounded and analytic outside F whose values are omitted by $f(z)$ near z_0, $\Phi(z)$ is surely bounded, and also regular in a certain neighbourhood of z_0 except perhaps for a set of linear measure zero.

Besicovitch's theorem already cited shows that the function $\Phi(z)$ becomes regular analytic throughout the neighbourhood just considered, if defined properly at the points where it has not been defined.

As $\varphi(w)$ is not a constant, we can find two values w' and w'' of $\omega-F$ such that

$$\varphi(w') \neq \varphi(w''). \tag{4}$$

But since the values taken by $f(z)$ near the point z_0 is known to be everywhere dense in the plane ω by Besicovitch's theorem, there exist two sequences $\{z_i'\}$ and $\{z_i''\}$ both tending to the point z_0 and satisfying

$$f(z_i')\to w' \quad \text{and} \quad f(z_i'')\to w'' \quad \text{as} \quad i\to\infty.$$

Then it would follow

$$\Phi(z_0)=\lim_{i\to\infty}\Phi(z_i')=\lim_{i\to\infty}\varphi[f(z_i')]=\varphi(w')$$

and also

$$\Phi(z_0)=\lim_{i\to\infty}\Phi(z_i'')=\varphi(w''),$$

which is impossible on account of (4), and this completes the proof of the first half of our Theorem.

The second half is an immediate corollary of the result obtained above, since putting $h(t)=t^{1+\eta}$ $(\eta>0)$ and $\tau=1$, the integral in (1) satisfies

$$\lim_{\sigma\to 0}\int_\sigma^1 \frac{t^{1+\eta}}{t^2}dt=\lim_{\sigma\to 0}\int_\sigma^1 t^{\eta-1}dt=\lim_{\sigma\to 0}\frac{1}{\eta}[t^\eta]_\sigma^1=\frac{1}{\eta}<\infty,$$

from which follows $L^{(1+\eta)}(A)=0$.

4. In case that the set E is sufficiently small, then the set of values omitted is known to be much smaller[2].

But we do not know yet whether the set of exceptional values in our case is of even linear measure zero or not.

1) Frostman. Loc. cit. p. 86, Theorem 1. His definition of h-measure is a little different from ours, which needs no essential change in our argument. Cf. also, G. Bouligand. Ensembles impropres et nombre dimensionnel, Bull. des Sc. math., Vol. **52** (1928), p. 364.

2) For example, see R. Nevanlinna, Eindeutige Funktionen, p. 135, Theorem 3.