13. *An Abstract Integral, VII.*

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Introduction. This is the preliminary report of analysis of functions with range in a complete vector lattice. This subject was firstly studied by S. Bochner[1]. § 1 contains the definitions and theorems of measurable functions. The definition of the measurability is that of S. Bochner. § 2 is the integration theory. Integral is defined by the idea of McNeille[2]. § 3 contains some remarks on integrals. Some related integrals are introduced and a modified integral is shown to coincide with the Bochner integral[3] when the range is the Banach lattice. § 4 is the Fourier series theory. Here the Bessel inequality is proved. This is not true for the Bochner integral[3] with range in the Banach space. This point is a reason why we develop the analysis of functions with range in a lattice in stead of a Banach space. § 5 is a generalization of § 1 and § 2. The content of § 5 shows that the theory of integral and that of measure can be placed under a general theory. In the ordinary theory one of those theories is derived from the other[4].

§ **1.** *Measurable functions*[5].

[1.1] I is a fixed finite interval in an Euclidean space.

[1.2] V is a fixed σ-complete vector lattice.

We will consider functions with domain I and with range in V and will denote them by $f(x)$ and $g(x)$, etc. Such functions are supposed to be defined uniquely in a full set of I and need not be defined in the complementary null set.

[1.3] $f(x)$ is called a simple function if there are an integer n, a set of real numbers $(a_1, a_2, \dots, a_n)$ and a set of disjoint measurable sets $(E_1, E_2, \dots, E_n)$ such that

$$I=\sum_{k=1}^{n} E_k,\quad f(x)=a_k \text{ in } E_k \ (k=1, 2, \dots, n).$$

[1.4] $f(x)$ is called to be measurable if there is a sequence of simple functions $f_n(x) (n=1, 2, \dots)$ such that $f_n(x)$ tends to $f(x)$ relative uniformly almost everywhere, that is, there are sequences of functions $\lambda_n(x)$, $g_n(x) (n=1, 2, \dots)$ such that $\lambda_n(x)$ tends to zero monotonously (by the order topology) almost everywhere as $n \to \infty$ and $|f_n(x)-f(x)| \leqq \lambda_n(x) g(x)$ almost everywhere. We write $f_n(x) \to f(x)$ (r. u.) a. e. or $f(x)=$(r. u)-$\lim f_n(x)$, a. e.

If $f(x)$ is measurable, then we write $f(x) \in M$.

1) S. Bochner, Proc. Nat. Academy, (1939).
2) McNeille, ibidem (1941).
3) S. Bochner, Fund. Math., **20** (1930).
4) cf. S. Izumi, An Abstract integral IV, Proc. Imp. Acad. of Japan, (1941).
5) [ ], ( ) and { } denote definition, theorem and axiom respectively.

We can prove easily that

(1.5) $f(x) \in M$ implies $|f(x)| \in M$.

(1.6) Linear sum of measurable functions is also measurable.

Further we introduce the assumption

{1.7} For any sequence $(u_k;\ k=1, 2, \dots)$ in V, there is a sequence of real numbers $(\lambda_k;\ k=1, 2, \dots)$ such that $\sum_{k=1}^{\infty} \lambda_k u_k$ converges relative uniformly. We denote such sum by (r. u.)-$\sum_{k=1}^{\infty} \lambda_k u_k$.

Then we can prove that

(1.8) If $f_n(x) \in M (n=1, 2, \dots)$ and $f_n(x) \to f(x)$ (r. u.) a. e., then $f(x) \in M$.

§ **2.** *Lebesgue integral.*

[2.1] If $f(x)$ is a simple function in [1.3], then the integral of $f(x)$ is defined by $\sum_{k=1}^{n} |E_k| a_k$ and is denoted by $\int f(x)dx$.

[2.2] Let $f(x)$ be $\in M$ and non-negative. $f(x)$ is called to be integrable if there is a sequence of simple functions $(u_n(x);\ n=1, 2, \dots)$ such that (1) $u_n(x) \geqq 0 (n=1, 2, \dots)$, (2) $f(x) = (\text{r. u.}) \sum_{n=1}^{\infty} u_n(x)$, a. e. and (3) $(\text{r. u.}) \sum_{n=1}^{\infty} \int u_n(x)dx$ exists.

[2.3] If $f(x) \geqq 0$ and is integrable, then we define the integral of $f(x)$ by $(\text{r. u.}) \sum_{n=1}^{\infty} \int u_n(x)dx$ and denote it by $\int f(x)\ dx$.

Then we can prove that

(2.4) Let $f_n(x) (n=1, 2, \dots)$ be $\geqq 0$ and integrable. If $f(x) = (\text{r. u.}) \sum_{n=1}^{\infty} f_n(x)$, a. e. and $(\text{r. u.}) \sum_{n=1}^{\infty} \int f_n(x)dx$ exists, then $f(x)$ is integrable and $\int f(x)dx = (\text{r. u.}) \sum_{n=1}^{\infty} \int f_n(x)dx$.

[2.5] Let $f(x) \in M$, then $f(x)$ is called to be integrable if $f^+(x)$ and $f^-(x)$ are integrable. In this case we write $f(x) \in L$.

[2.6] If $f(x) \in L$, then the integral of $f(x)$ is defined by $\int f^+(x)dx - \int f^-(x)dx$ and is denoted by $\int f(x)dx$.

We can easily prove that

(2.7) L is a linear space.

(2.8) $f(x) \in L$ implies $|f(x)| \in L$.

Further we can prove the convergence theorems of Fatou and Lebesgue:

(2.9) If $f_n(x) (n=1, 2, \dots)$ are $\geqq 0$ and $\in L$, and $f_n(x) \leqq f_{n+1}(x)$, then $\int (\text{r. u.})\text{-}\lim f_n(x)dx = (\text{r. u.})\text{-}\lim \int f_n(x)dx$, provided that the latter limit exists.

(2.10) If $f_n(x) (n=1, 2, \dots)$ are $\in L$ and tend to $f(x)$ relative uniformly almost everywhere, further there is a function $g(x) \in L$ such as $|f_n(x)| \leqq g(x) (n=1, 2, \dots)$, then $f(x) \in L$ and $(\text{r. u.})\text{-}\lim \int f_n(x)dx = \int f(x)dx$.

§ **3.** *Remarks.*

1°. In the definition of the Lebesgue integral we can drop the notion of measurability as McNeille says. We can also define the integral directly for general functions.

2°. In the above definitions and theorems, we can replace relative uniform convergence by relative uniform star convergence. " $(u_n; n=1, 2, \dots)$ is relative uniformly star convergent " means that any subsequence (u_{n_k}) of (u_n) contains a relatively uniform convergent subsequence. Using this convergence, we can define another integral which we call (*)-integral. (*)-integral is also a Lebesgue integral. If V is the Banach lattice, then the relative uniform star convergence becomes the norm convergence. And then the (*)-integral becomes the Bochner integral. (*)-integral has the advantage that the case $V=(S)$, set of all measurable functions, is contained.

3°. In the definitions of integral the concept of " absolute convergence " is used. That is, $\sum_{n=1}^{\infty} u_n(x)$ is absolutely convergent when $\sum_{n=1}^{\infty} |u_n(x)|$ converges. If this notion is replaced by the different ones, then we get different integrals. For example by unconditional convergence and Moore-Smith convergence. " $\sum u_n$ converges unconditionally " means that for every rearrangement $(u_{n'})$ of (u_n) $\sum u_{n'}$ converges in the relative uniform sense or relative uniform star sense and " $\sum u_n$ converges in the Moore-Smith sense " means that if σ is a finite subset of integers then $\sum (u_n; n \in \sigma)$ converges in the Moore-Smith sense.

§ **4.** *Fourier series.*

[4.1] Let I be $(0, 2\pi)$.

(4.2) If $f(x) \in L$, then there exist $\frac{1}{\pi}\int f(x)dx$, $\frac{1}{\pi}\int f(x)\cos nx\, dx$ and $\frac{1}{\pi}\int f(x)\sin nx\, dx$ $(n=1, 2, \dots)$. We call them Fourier coefficients of $f(x)$ and denote them by $a_0, a_n, b_n (n=1, 2, \dots)$.

[4.3] We call the series $\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ as Fourier series of $f(x)$ and write $f(x) \sim \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ whose convergence is indeferent.

[4.4] We denote a fixed element >0 in V by 1 and $\lambda 1$ by λ.

[4.5][6] If $c \in V$, then we put $c^2 = \sup(2\lambda c - \lambda^2; -\infty < \lambda < \infty)$ provided that the right hand side term exists.

Then we can prove that

(4.6) If $f(x) \in L$ and $f^2(x)$ exists almost everywhere, then there exist $a_0^2, a_n^2, b_n^2 (n=1, 2, \dots)$.

Bessel's inequality holds, that is,

(4.7) If $f^2(x)$ exists almost everywhere and $f^2(x) \in L$, then we have

$$\frac{a_0^2}{4}+\sum_{n=1}^{\infty}(a_n^2+b_n^2) \leqq \frac{1}{\pi}\int f^2(x)dx .$$

6) cf. F. Riesz, Acta de Szeged, 1941.

[4.8] If $f^2(x)$ exists almost everywhere and $f^2(x) \in L$, we denote $f(x) \in L^2$.

[4.9] If $|f(x)-f(y)| \leqq A|x-y|^a$ for all x, y in I, then we denote $f(x) \in \mathrm{Lip}^a$. And if $\sum_{i=1}^{n-1} |f(x_i)-f(x_{i+1})|$ is bounded for any partition $(x_1, x_2, \dots, x_n)$ of I, then we write $f(x) \in BV$.

Then we can prove the following theorems.

(4.10) If $f(x) \in L^2$ and $\in \mathrm{Lip}^a$ for $a > \frac{1}{2}$, then $\sum_{n=1}^{\infty}(|a_n|+|b_n|)$ converges.

(4.11) If $f(x) \in L^2$, $f(x) \in BV$ and $f(x) \in \mathrm{Lip}^a$ for $a > 0$, then $\sum_{n=1}^{\infty}(|a_n|+|b_n|)$ converges.

(4.12) If $f(x)$ is continuous, $f(0)=f(2\pi)$ and Fourier coefficients of $f(x)$ are all non-negative, then the Fourier series of $f(x)$ converges relative uniformly.

§ **5.** *Generalization.*

In the above theory we can drop the function-concept.

[5.1] V_1 and V_2 are σ-complete vector lattices.

[5.2] F is a subspace of V_1 such that there is a function $s(u)$ with domain F and with range in V_2, satisfying the conditions of the Riemann type.

[5.3] If $f \in V_1$, $f \geqq 0$, then f is integrable provided that there is a sequence $(u_n;\ n=1, 2, \dots)$ in F such that (1) $u_n \geqq 0 (n=1, 2, \dots)$, (2) $f = (\text{r. u.}) \sum_{n=1}^{\infty} u_n$ (or $(0.) \sum_{n=1}^{\infty} u_n$) and (3) $\sum_{n=1}^{\infty} s(u_n)$ converges relative uniformly.

[5.4] The integral of such function $f(x)$ is defined by $(\text{r. u.}) \sum_{n=1}^{\infty} s(u_n)$ and is denoted by $\int f$.

Thus proceeding we can develop the integration theory independent of function-concept. This theory contains theory of measure as a particular case. For, if we suppose that the theory of null sets is established which is quitely elementary as F. Riesz emphasizes, and we take F as the class of closed sets, then we get the measure theory.

In the ordinary theory, integration theory is founded on the measure theory or this is derived from that. But the above theory contains both simultaneously.