PAPERS COMMUNICATED

36. *Notes on Banach Space (VI): Abstract Integrals and Linear Operations.*

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The object of this paper is to give general representation theorems of linear operations from a Banach space to another where one is a concrete Banach space. In this direction there are many results due to Gelfand[1], Kantorovitch-Vulich[2], Kantorovitch[3] and Phillips[4], etc. In §§ 3–4 their results are all generalized and simplified. Our problem is closely connected with the integration theory. In § 2 we define abstract integrals using idea of von Neumann and Dunford[5]. These integrals are used in the representation theorems. In § 1 we state notations and theorems which are used throughout the paper.

1. *Notations and theorems due to Dunford.*

Let X be a Banach space of numerical functions $\phi(t)$, where t ranges over an abstract set T such that

1°. if $\phi_1(t)+\phi_2(t)=\phi(t)$ for all t in T, then $\phi_1+\phi_2=\phi$,

2°. if $c\phi_1(t)=\phi(t)$ for a constant c and for all t in T, then $c\phi_1=\phi$,

3°. if $\phi_n\rightarrow\phi$ and $\phi_n(t)\rightarrow\phi^*(t)$ for all t in T, then $\phi=\phi^*$,

4°. if $\phi_n\rightharpoonup\phi$, then $\phi_n(t)\rightarrow\phi(t)$ for all t in T.

Examples of such X are c, l^p $(1\leqq p\leqq\infty)$, C, B, AC and V^p $(1\leqq p\leqq\infty)$ where V^1 denotes the space of all completely additive set functions on an abstract set. In the following X denotes always such Banach space. But in §§ 1–2 we need not the condition 4°. Since L^p $(1\leqq p\leqq\infty)$ satisfies conditions 1°–3°, the results in §§ 1–2 are applicable to such spaces.

Let Y be an arbitrary Banach space and Γ a closed linear manifold in $\bar{Y}$. The linear space $\mathfrak{X}\equiv X(Y,\Gamma)$ is, by definition, the space of all abstract functions $y(.)=y(t)$ on T to Y such that $\gamma f(.)$ lies in X for every γ in Γ. $y(.)$ and $\gamma y(.)$ represent points in the function space from T to Y and X, respectively.

The following theorems are due to Dunford. We prove them for the sake of completeness.

(1.1) *If $y(.)\in X(Y,\Gamma)$, then $\gamma y(.)$ is a linear operation on Γ to X. In other words there exists a smallest non-negative number $\|y(.)\|$ such that*

$$\|\gamma y(.)\|_X\leqq\|y(.)\|\cdot\|\gamma\| \qquad (\gamma\in\Gamma).$$

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1) Gelfand, Recueil Math., Moscou, **4** (1938), pp. 235–284.

2) Kantorovitch-Vulich, Comp. Math., **5** (1937), pp. 119–165.

3) Kantorovitch, Recueil Math., Moscou, **7** (1940), pp. 207–279.

4) Phillips, Trans. Amer. Math., Soc., **44** (1940), pp. 516–541.

5) Dunford, Ibidem, **44** (1936), pp. 305–356.

Proof. If we put $U(\gamma)\equiv\gamma y(.)$, then by 1° and 2°, U is an additive operation from Γ to X. 3° shows that if $\gamma_n\to\gamma^*$ and $U(\gamma_n)\to\phi$ then $U(\gamma^*)=\phi$. Thus U is closed and then is continuous by a well known theorem.

(1.2) *If $y(.)\in X(Y,\Gamma)$, $\gamma\in\Gamma$ and $\bar{x}\in\bar{X}$, then $\bar{x}\gamma y(.)$ is linear in γ and $|\bar{x}\gamma y(.)|\leqq\|\bar{x}\|\cdot\|\gamma\|\cdot\|y(.)\|$.*

Proof. By (1.1)

$$|\bar{x}\gamma y(.)|\leqq\|\bar{x}\|\cdot\|\gamma y(.)\|_X\leqq\|\bar{x}\|\cdot\|\gamma\|\cdot\|y(.)\|.$$

Let $\bar{x}\gamma y(.)\equiv\bar{\gamma}(\gamma)$. then $\bar{\gamma}$ is in $\bar{\Gamma}$ by (1.2). Let $\mathfrak{X}_0\equiv X_0(Y,\Gamma)$ be the space of $y(.)\in\mathfrak{X}$ such that for every $\bar{x}\in\bar{X}$ there is a y such that

$$\bar{x}\gamma y(.)=\gamma(y)\qquad(\gamma\in\Gamma).$$

Then we have

(1.3) *A necessary and sufficient condition that $y(.)\in X(Y,\bar{Y})$ lies in $X_0(Y,\bar{Y})$ is that for every $\bar{x}_0$ in a fundamental set $F\subset\bar{X}$ there exists a y_0 such that $\bar{x}_0\gamma y(.)=\gamma(y_0)$ $(\gamma\in\bar{Y})$.*

Proof. The necessity is ovious. To prove the sufficiency we suppose that the condition of the theorem is satisfied. Then $\bar{x}\gamma y(.)=\gamma(y_0)$ $(\gamma\in\bar{Y})$ for every $\bar{x}$ in a dense set $D\subset\bar{X}$. But for every $\bar{x}\in\bar{X}$ there are $\bar{x}_n\in D$ $(n=1,2,3,\ldots)$ such that $\|\bar{x}_n-\bar{x}\|\to0$. Since $\bar{x}_n\gamma y(.)=\gamma y_n$ $(\gamma\in\Gamma)$, we have by (1.2)

$$\|y_m-y_n\|=\sup_{|\gamma|=1}|(\bar{x}_m-\bar{x}_n)\gamma y(.)|\leqq\|\bar{x}_m-\bar{x}_n\|\cdot\|y(.)\|\to0.$$

Therefore $y_n\to y$ implies $\bar{x}\gamma y(.)=\gamma(y)$.

2. *Abstract integrals.*

Let Δ be a set in $\bar{X}$ such that for every $\delta\in\Delta$

$$\delta\phi=\int_T\psi(t)d\phi(t),\qquad(\phi\in X),\tag{1}$$

the integral being taken in the Lebesgue-Stieltjes or Riemann-Stieltjes senses. If we put $\delta\gamma y(.)\equiv\big(\delta y(.)\big)(\gamma)$, then $\delta y(.)$ is linear on Γ by (1.2). Since δ is given by (1), we define the abstract Stieltjes integral by

$$\delta y(.)\equiv\int_T\psi(t)dy(t).\tag{2}$$

If instead of (1) $\delta\phi$ is given by the Radon intergral

$$\delta\phi=\int_T\psi(t)d\phi(\tau)\qquad(\phi\in X),$$

then we define

$$\delta y(.)\equiv\int_T\psi(t)dy(\tau)\tag{3}$$

where $y(\tau)$ is an abstract set-function. This is called abstract Radon integral. In the same way, if we take the Hildebrandt integral in the place of (1), we can define the abstract Hildebrandt integral. It is obvious that the values of these integrals are contained in Y by (1.3).

Abstract Riemann-Stieltjes integral is given by Gowurin[1) and Dunford. The abstract Radon integral in the case $X\equiv V^q$, $\varDelta\equiv L^p$ $(1/p+1/q=1)$, is given by Phillips. The abstract Hildebrandt integral in the case $\varDelta\equiv L^\infty$, $X\equiv\bar{L}^\infty$ is given by Gowurin. Their definitions are all constractive and then is not so simple as ours.

Our principal idea lies in defining the integral of Banach space-valued functions by means of the integral of numerical functions in the same domain. This permits us to derive the representations of linear operations on a given Banach space by means of the representation of linear functionals on it.

We will state without proof some elementary properties of the integral of abstract Lebesgue-Stieltjes for example.

(2.1) *The integral* $\int_T \psi(t)dy(t)$ *is a linear operation on* $\mathfrak{X}$ *to* $\bar{\Gamma}$.

In the following let $\varDelta$ be a closed linear manifold in $\bar{X}$. Then $\varDelta$ is a Banach space by the norm in $\bar{X}$.

(2.2) *If* $\gamma\in\Gamma$, $\delta\equiv\psi(t)\in\varDelta$ *and* $y(.)\in\mathfrak{X}$, *then*

$$\left|\int_T \psi(t)d\gamma y(.)\right| \leqq \|\delta\|\cdot\|\gamma\|\cdot\|y(.)\|.$$

(2.3) *If* $\|\psi_e\|\to 0$ *as* $|e|\to 0$, *where*

$$\psi_e(t)\equiv\psi(t)\quad(t\in e),\quad \equiv 0\quad(t\bar{\in} e),$$

then $\int_e \psi(t)dy(t)$ *is completely additive and absolutely continuous set-function of e.*

3. *Representation of linear operations.*

(3.1) *If Y is an arbitrary Banach space, then the general form of the linear operations* $U(y)$ *from Y to X is given by*

$$U(y)=\bar{y}(.)(y),$$

where $\bar{y}(.)\in X(\bar{Y}, Y)$ and $\|U\|=\|\bar{y}(.)\|$.

Proof. Let $\bar{x}_t$ be a functional on X such that $\bar{x}_t\phi(.)=\phi(t)$. By 4°, $\bar{x}_t\in\bar{X}$. Since $\bar{x}_tU(y)=\big(\bar{x}_t(U)\big)(y)$, we have $\bar{x}_t(U)\in\bar{Y}$. So that there exists $\bar{y}(.)\in X(\bar{Y}, Y)$ such that

$$\bar{x}_tU(y)=\bar{y}(t)y.$$

Thus we have $U(y)=\bar{y}(.)(y)$. It is obvious by (1.1) that $\bar{y}(.)(y)$ is linear and $\|U\|=\|\bar{y}(.)\|$. Thus the theorem is proved.

If we take $X\equiv V^p$, we have the general form of the linear operation from Y to V^p. This is Phillips result. Especially taking $T=(0, 1)$, the linear operation from Y to L^p is given by

$$U(y)=\frac{d}{dt}\bar{y}(t)(y)$$

where $\bar{y}(.)\in V^p(\bar{Y}, Y)$ and $\|U\|=\|\bar{y}(.)\|$. This is the result of Kantorovitch and Vulich. Thus we get almost all known representation

1) Gowurin, Fund. Math., **27** (1936), pp. 254–268.

theorem of linear operations from arbitrary Banach space to concrete space.

(3.2) *Suppose that $\bar{\Delta}$ is a closed linear manifold in a Banach space and $\bar{\Delta}$ may be embeded into X. Then the general form of the linear operation $U(\delta)$ from Δ to an arbitrary Banach space Y is given by*

$$U(\delta)=\int_T \psi dy\,,\qquad \delta=\psi(.)\,,$$

where $y(.)\in\bar{X}(Y,\bar{Y})$ and $\|U\|=\|y(.)\|$.

Proof. Put $\gamma\in\bar{Y}$, $\delta\equiv\psi(.)$, then $\gamma U(\delta)=\mu\psi(.)$, $\mu\in\bar{\Delta}$. Moreover we denote $\mu=\phi(.)$ and ν_t the operation which corresponds $\phi(t)$ to $\phi(.)$ for a fixed t, that is $\nu_t\phi(.)=\phi(t)$. Then by 4°, ν_t is a linear functional. But we have $\phi(.)=\gamma U$, that is,

$$\phi(t)=\nu_t(\gamma U)=(\nu_t\bar{U})(\gamma)\,.$$

So that $U=\nu_t\bar{U}$ is a mapping from T to $\bar{\bar{Y}}$. But since $U(\delta)\in Y$, the range of U is Y. Thus we have

$$U\equiv y(.)\in V(Y,\bar{Y})\,,$$

and then
$$\gamma U(\delta)=\gamma U\psi=\gamma\big(\delta y(.)\big)$$

that is
$$U(\delta)=\int_T \psi dy\,.$$

The remaining part of the theorem is immediate.

In the case $X=V^q$, $\Delta=L^p$, the theorem (3.2) is proved by Phillips. By some change of conditions we get representation theorems of linear operations by abstract Hildebrandt integrals. From these we can derive almost all known representation theorems of linear operations from concrete Banach space to arbitrary Banach space. Especially taking Δ as a space BV^*[1] equivalent to BV, we get the representation of operation from BV^* to an arbitrary Banach space by the abstract Hildebrandt integral.

4. *Representation of H_t^0-operations in a semi-ordered Banach space.*

In this section we suppose that X is a K_5^+-space[2] such that

5°. if $\phi\geqq 0$, then $\phi(t)\geqq 0$ $(t\in T)$.

then we have

(4.1) *The general form of the H_t^0-operation from an arbitrary Banach space to X is given by*

$$U(y)=\bar{y}(.)(y)\,,\quad \|\bar{y}(t)\|\in X\,,\quad |U|=\|\bar{y}(t)\|\,,$$

where $|U|$ denotes the abstract norm.

Proof. By $H_t^0\subset H_t^t$ and (3.1) we get $U(y)=\bar{y}(.)(y)$ and $\bar{y}(.)\in X(\bar{Y},Y)$. Now the necessary and sufficient condition that $U\in H_t^0$ is that there exists x_0 such that

1) Hildebrandt, Bull. Amer. Math. Soc., **44** (1938), p. 75.

2) Kantorovitch, loc. cit.

$$\sup_{\|y\|\leq 1}(|U(y)|/\|y\|)=x_0.$$

By
$$\sup_{\|y\|\leq 1}|U(y)|/\|y\|=\sup_{\|y\|\leq 1}|\bar{y}(t)(y)|/\|y\|=\|\bar{y}(t)\|,$$

we get $\|\bar{y}(t)\|\in X$. The remaining part is obvious.

If $X=V^q$ $(q>1)$ or AC, then

$$U(y)=\bar{y}(.)(y), \quad \|\bar{y}(\tau)\|\in V^q \ \text{ or } \ AC, \quad |U|=\|\bar{y}(\tau)\|.$$

When $\bar{Y}$ is locally weakly compact, then we have

$$\bar{y}(\tau)=\int_\tau \bar{y}(t)dt$$

where the integral is in the Bochner sense[1]. Thus we have

(4.2) *When $\bar{Y}$ is locally weakly compact, the general form of the H_t^0-operation U from Y to L^p $(p\geqq 1)$ is given by*

$$U(y)=\bar{y}(t)(y), \qquad \|\bar{y}(t)\|\in L^p \ (p\geqq 1), \quad |U|=\|\bar{y}(t)\|.$$

Especially if $Y\equiv L^\rho$ $(1<\rho<\infty)$, then the general form of the H_t^0-operation U from L^ρ to L^p is given by

$$U(y)=\int K(t,s)y(s)ds$$

where
$$\left[\int\left[\int|K(t,s)|^\sigma ds\right]^{\frac{p}{\sigma}}dt\right]^{\frac{1}{p}}=\||U|\|<\infty \qquad \left(\frac{1}{\rho}+\frac{1}{\sigma}=1\right).$$

The special case of $T=(0,1)$ was proved by Kantorovitch and Vulich.

By the same idea we can prove a theorem corresponding to (3.2). In this case we can also derive (4.2) as a special case.

1) Phillips, loc. cit. 小笠原, 全國紙上談話會, **235** (昭和 17 年), pp. 1014-1018.