93. On Analytic Functions in Abstract Spaces.

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§ 1. The purpose of the present paper is to extend some of the theorems of H. Cartan[1] on functions of several complex variables to the case of functions whose domain and range both lie in complex Banach spaces.[*]

Let E and E' be two complex Banach spaces, and let $x'=f(x)$ be an E'-valued function defined on a certain neighborhood $V(x_0)$ of a point $x_0 \in E$. $x'=f(x)$ is said to admit a *variation* or a *Gateaux differential* at $x=x_0$ if

$$(1) \qquad \lim_{\alpha\to 0} \frac{f(x_0+\alpha y)-f(x_0)}{\alpha}$$

exists strongly for any $y \in E$ (α is a complex number).

An E'-valued function $x'=f(x)$ defined on a domain D of E is *analytic* in D if it is strongly continuous on D and if it admits a Gateaux differential at every point of D. It is clear that, in case both E and E' are the field of complex numbers, this definition coincides with the usual definition of a complex-valued analytic function of a single complex variable. Further, if E is the field of complex numbers while E' is an arbitrary complex Banach space, then our definition coincides with that of a Banach-space-valued analytic function of a single complex variable given by E. Hille and N. Dunford.[2]

An E'-valued function $x'=p(x)$ defined on E is a *polynomial of degree n* if the following conditions are satisfied: 1) $p(x)$ is strongly continuous at each point of E, 2) for each x and y in E, and for any complex number α, $p(x+\alpha y)$ can be expressed as

$$(2) \qquad p(x+\alpha y)=\sum_{k=1}^{n}\alpha^k p_k(x, y),$$

where $p_k(x, y)$ are arbitrary E'-valued functions of two variables x and y, 3) $p_n(x, y)\neq 0$ for some x and y. If, in addition to these, $p(\alpha x)=\alpha^n p(x)$, then the function $p(x)$ is called a *homogeneous polynomial of degree n*. It is clear that an E'-valued polynomial defined on E is analytic on E.

We shall state a theorem of A. E. Taylor[3] which we shall need in the following discussions:

Let E and E' be two complex Banach spaces. If an E'-valued

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1) H. Cartan, Sur les groupes des transformations analytiques, Actualités, Paris, 1938.

2) Cf. E. Hille, Semi-group of linear transformations, Annals of Math., **40** (1939).

3) A. E. Taylor, On the properties of analytic functions in abstract spaces, Math. Annalen, **115** (1938).

function $x'=f(x)$ *is defined and is analytic in the sphere* $S_\rho=\{x \mid \|x\|<\rho\}$ *of* E, *then it may be expanded into the series*

(3) $$f(x)=f(0)+\sum_{n=1}^{\infty}f_n(x),$$

where $f_n(x)$ *is an* E'*-valued homogeneous polynomial of degree* n *given by*

(4) $$f_n(x)=\frac{1}{2\pi i}\int\frac{f(\alpha x)}{\alpha^{n+1}}d\alpha.$$

the integral being taken in the positive sense on the circle $|\alpha|=\rho'<1$. *The series on the right hand side of* (3) *converges absolutely and uniformly in the sphere* $S_{\rho'}=\{x \mid \|x\|\leqq\rho'\}$, where ρ' is a sufficiently small positive number.

§2. *Theorem 1. Let* E, E' *and* E'' *be three complex Banach spaces and let* D *and* D' *be two domains in* E *and* E' *respectively. If* $x'=f(x)$ *is an* E'*-valued analytic function defined on* D *whose value lies in* D', *and if* $x''=g(x')$ *is an* E''*-valued analytic function defined on* D', *then* $x''=g(f(x))$ *is an* E''*-valued analytic function defined on* D.

Proof. It is clear that $x''=g(f(x))$ is strongly continuous on D. So it suffices to show that

(5) $$\lim_{\alpha\to 0}\frac{g(f(x_0+\alpha y))-g(f(x_0))}{\alpha}$$

exists for any $x_0\in D$ and for any $y\in E$. Without loss of generality we may assume that $x_0=0$, $f(0)=0'$ and $g(0')=0''$, where 0, $0'$ and $0''$ denote the origin of E, E' and E'' respectively. Thus we have only to show that

(6) $$\lim_{\alpha\to 0}\frac{g(f(\alpha y))}{\alpha}$$

exists for any $y\in E$, which we shall assume given and fixed.

Since $x'=f(x)$ is analytic at $x=0$, so there exist two positive constants δ and M such that

(7) $$f(\alpha y)=\alpha f_1(y)+\alpha^2R(y,\alpha)$$

with $\|R(y,\alpha)\|\leqq M$ for any α with $|\alpha|\leqq\delta$. Further, since $x''=g(x')$ is analytic at $x'=0'$, so there exist two positive constants $\delta'(\leqq\delta)$ and M' such that

(8) $$g(\alpha z)=\alpha g_1(z)+\alpha^2S(z,\alpha)$$

with $\|S(z,\alpha)\|\leqq M'$ for any z and α with $\|z\|\leqq\|f_1(y)\|+\delta M$ and $|\alpha|\leqq\delta'$. Consequently, $|\alpha|\leqq\delta'$ implies

(9) $$\begin{aligned}g(f(\alpha y))&=g(\alpha f_1(y)+\alpha^2R(y,\alpha))\\&=\alpha g_1(f_1(y)+\alpha R(y,\alpha))+\alpha^2S(f_1(y)+\alpha R(y,\alpha),\alpha)\end{aligned}$$

Since $g_1(z)$ is strongly continuous, it follows from (9) that the limi (6) exists and is equal to $g_1(f_1(y))$ for any $y\in E$.

Exactly in the same way, we may prove the following

Theorem 2. In addition to the assumptions in Theorem 1, let us assume that $0 \in D$, $0' \in D'$, $f(0)=0'$ *and* $g(0')=0''$, *where* 0, $0'$ *and* $0''$ *denote the origin of* E, E' *and* E'' *respectively. Let further*

$$f(x)=\sum_{n=m}^{\infty} f_n(x), \tag{10}$$

$$g(x')=\sum_{n=p}^{\infty} g_n(x') \tag{11}$$

be the Taylor expansions of $x'=f(x)$ *and* $x''=g(x')$ *at* $x=0$ *and* $x'=0'$ *respectively which begin with the m-th term and the p-th term respectively. Then* $x''=h(x)=g\big(f(x)\big)$ *is an analytic function defined on D, and the Taylor expansion*

$$h(x)=\sum_{n=mp}^{\infty} h_n(x) \tag{12}$$

of $x''=h(x)$ *begins with the mp-th term* $h_{mp}(x)=g_p\big(f_m(x)\big)$.

§3. *Theorem 3. Let E be a complex Banach space, and let* $x'=f(x)$ *be an E-valued analytic function defined on the unit sphere* $S=\{x \mid \|x\|<1\}$ *of E which maps S into itself. If the Taylor expansion of* $f(x)$ *at* $x=0$ *is of the form:*

$$f(x)=x+\sum_{n=2}^{\infty} f_n(x), \tag{13}$$

then $x'=f(x)$ *must be the identity mapping:* $f(x)\equiv x$.

Proof. It suffices to show that $f_n(x)\equiv 0$ for $n=2, 3, \dots$. Assume the contrary, and let $f_m(x)$ $(m \geqq 2)$ be the first term which does not vanish identically. i.e. $f_n(x)\equiv 0$ for $n=2, \dots, m\text{-}1$, and $f_m(x_0) \neq 0$ for some $x_0 \in S$.

Let us define a sequence $\{f^{(k)}(x) \mid k=1, 2, \dots\}$ of E-valued functions $f^{(k)}(x)$ recurrently by

$$f^{(k)}(x)=f\big(f^{(k-1)}(x)\big), \qquad k=2, 3, \dots; \qquad f^{(1)}(x)=f(x). \tag{14}$$

Then, from Theorem 1 follows that each $f^{(k)}(x)$ gives an analytic mapping of S into itself. Further, it is not difficult to see, by appealing to Theorem 2, that the Taylor expansion of $x'=f^{(k)}(x)$ at $x=0$ is of the form:

$$f^{(k)}(x)=x+kf_m(x)+\sum_{n=m+1}^{\infty} f_n^{(k)}(x). \tag{15}$$

In fact, (15) is clear for $k=1$, and the case for general k may be proved by mathematical induction.

The integration formula (4) then gives

$$kf_m(x)=\frac{1}{2\pi i}\int \frac{f^{(k)}(\alpha x)}{\alpha^{m+1}}\,d\alpha, \tag{16}$$

the integral being taken in the positive sense on the circle $|\alpha|=\rho<1$. From (16) follows immediately

$$k\|f_m(x_0)\| \leqq \frac{1}{2\pi}\int_0^{2\pi} \frac{\|f^{(k)}(\alpha x_0)\|}{\rho^{m+1}}\rho\, d\theta \leqq \frac{1}{\rho^m} \tag{17}$$

for $k=1, 2, \dots$, in contradiction to our assumption that $f_m(x_0) \neq 0$. This completes the proof of Theorem 3.

§ 4. Let D and D' be two domains in two complex Banach spaces E and E' respectively. If $x'=f(x)$ is a one-to-one mapping of D onto D' such that both $x'=f(x)$ and its inverse $x=f^{-1}(x')$ are analytic functions in D and D' respectively, then $x'=f(x)$ is called an *analytical mapping* of D onto D'.

Theorem 4. Let E and E' be two complex Banach spaces, and let $x'=f(x)$ be an analytic mapping of the unit sphere $S=\{x \mid \|x\|<1\}$ of E onto the unit sphere $S'=\{x' \mid \|x'\|<1\}$ of E'. If the origin 0 of E is mapped to the origin $0'$ of E' by $x'=f(x)$, then $x'=f(x)$ is a linear and isometric mapping.

Proof. For any $\theta (0 \leqq \theta < 2\pi)$, let us consider an analytic mapping $x'=h_\theta(x)$ of S onto itself given by

$$h_\theta(x)=e^{-i\theta}f^{-1}\big(e^{i\theta}f(x)\big). \tag{18}$$

It is clear that

$$h_\theta(0)=0, \qquad h_0(x)\equiv x. \tag{19}$$

Further, let us consider the Taylor expansions of $f(x)$, $g(x)=f^{-1}(x)$ and $h_\theta(x)$ at $x=0$:

$$f(x)=\textstyle\sum_{n=1}^{\infty} f_n(x),. \tag{20}$$

$$g(x)=\textstyle\sum_{n=1}^{\infty} g_n(x), \tag{21}$$

$$h_\theta(x)=\textstyle\sum_{n=1}^{\infty} h_{\theta,n}(x). \tag{22}$$

Then Theorem 2 implies

$$h_{\theta,1}(x)=e^{-i\theta}g_1\big(e^{i\theta}f_1(x)\big)=g_1\big(f_1(x)\big). \tag{23}$$

Hence $h_{\theta,1}(x)$ is independent of θ, and so by (19),

$$h_{\theta,1}(x)\equiv x. \tag{24}$$

Thus Theorem 3 is applicable, and we see that $h_\theta(x)\equiv x$, or equivalently that $f(e^{i\theta}x)\equiv e^{i\theta}f(x)$ for any $\theta(0\leqq 0<2\pi)$ and for any $x\in S$. From this follows immediately by (4) that $f_n(x)\equiv 0$ for $n\geqq 2$. Thus we see $f(x)=f_1(x)$, and this shows that $f(x)$ is linear.[1] Further, since every $y\in S$ is mapped by $x'=f(x)$ to an element $f(y)\in S'$, so we see that $\|f(x)\|=\big\|(\|x\|+\varepsilon)\; f\big(x/(\|x\|+\varepsilon)\big)\big\|\leqq \|x\|+\varepsilon$ for any $\varepsilon>0$, from which follows that $\|f(x)\|\leqq\|x\|$. Since the inverse inequality $\|x\|=\big\|f^{-1}\big(f(x)\big)\big\| \leqq \|f(x)\|$ may be obtained in a similar way, so we finally see that $\|f(x)\|=\|x\|$. This completes the proof of Theorem 4.

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1) It is easy to see that a homogeneous polynomial of degree 1 is linear. It only suffices to show that a homogeneous polynomial $p(x)$ of degree 1 satisfies $p(x+y)=p(x)+p(y)$. In fact, by definition, $p(x)$ satisfies a relation $p(x+\alpha y)=p_0(x,y)+\alpha p_1(x,y)$, for any x, y and α. It is easy to see that $p_0(x,y)=p(x)$, and so $p_1(x,y)=\frac{1}{\alpha}p(x+\alpha y)-\frac{1}{\alpha}p(x)=p\left(\frac{1}{\alpha}x+y\right)-p\left(\frac{1}{\alpha}x\right)$. If we now let $\alpha\to\infty$, then the continuity of $p(x)$ implies that $p_1(x,y)=p(y)$.