87. On the Representation of Boolean Algebra.

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1. Representation theory of Boolean algebra was developed by Stone, Wallman and many writers. Wallman's[1] method is simpler than that of Stone[2] in the point that the notion of ideal is not used. The method of Livenson[3] is complicated than that of Wallman. But if we replace the regular table of Livenson by the set satisfying conditions (1°), (2°) and (4°) in § 2, then the maximal regular table becomes a ideal basis. Further we can prove that the representation space of Livenson becomes a T_2-space satisfying the first countability axiom.

2. Let L be a distributive lattice including 0 and 1. That is, L is a lattice having zero element 0 and unit element 1 and for any three elements a, b and c

$$a(b \vee c)=ab \vee ac \quad \text{and} \quad a \vee bc=(a \vee b)(a \vee c)\,.$$

Now we consider a subset $\{g\}$ of L satisfying the following conditions:

(1°) $0 \bar{\in} \{g\}$.

(2°) If $g_1, g_2 \in \{g\}$ then there exists g_3 such that $g_3 < g_1 g_2$.

In such two sets $\{g\}$ and $\{g'\}$, if for any $g \in \{g\}$ there exists $g' \in \{g'\}$ such that $g' < g$ then we write

$$\{g\} \prec \{g'\}\,.$$

Further we will introduce two conditions concerning $\{g\}$ in L:

(3°) For $\{g\}$ and any two elements a and b such as $g(a \vee b)=g$ there exists $g_1 \in \{g\}$ such that $g_1 a=g_1$ or $g_1 b=g_1$.

(4°) For $\{g\}$ and any $a \in L$ there exists $g \in \{g\}$ satisfying $ag=g$ or $ag=0$.

Lemma 1. Under (1°) and (2°), (4°) implies (3°).

Suppose that $\{g\}$ satisfies (1°), (2°) and (4°) and a and b are any elements satisfying $(a \vee b)g=g$ for some $g \in \{g\}$. Then there exist g_1 and g_2 such that $ag_1=g_1$ or $ag_1=0$ and $bg_2=g_2$ or $bg_2=0$. If $ag_1=bg_2=0$, then $g_3 < g < a \vee b$ for $g_3 < gg_1g_2$. Hence $0=ag_3 \vee bg_3=(a \vee b)g_3=g_3$. This is a contradiction.

Lemma 2. Suppose that $\{g\}$ satisfies (3°) (or (4°)) and $\{g\} \prec \{g'\} \prec \{g\}$. Then $\{g'\}$ satisfies (3°) (or (4°)).

Suppose that $\{g\}$ satisfies (3°) and that a and b are any two elements satisfying $(a \vee b)g'=g'$ for some $g' \in \{g'\}$. If $gg'=g$ then $g(a \vee b)=g$. Consequently there exists $g_1 \in \{g\}$ such that $ag_1=g_1$ or

1) H. Wallman, Lattice and topological Spaces (Ann. Math., Vol. 39 (1938)).

2) H. Stone, Topological Representations of Distributive Lattice and Browerian Logics. (Casopic pro pestovani matematiky a fysiky 1939).

3) E. Livenson, On the realization of Boolean algebras by algebras of sets (Rec. Math. de la Soc. Math. de Moscou (1940)).

$bg_1=g_1$. If $g'_1g_1=g'_1\in\{g'\}$, then $ag'_1=g'_1$ or $bg'_1=g'_1$. The proof of remaining part is easy.

Lemma 3. For any $\{g\}$ satisfying (1°) and (2°), there exists $\{g'\}\supset\{g\}$ satisfying the condition (4°).

Let $ag\neq g$ and $ag\neq 0$ for some a and for every $g\in\{g\}$. $\{g\}^1\equiv(g, ag; g\in\{g\})$ satisfies (1°) and (2°). If $\{g\}^1$ satisfies (4°) then it is a desired one. Otherwise we next construct $\{g\}^2$ from $\{g\}^1$ similarly as $\{g\}^1$ obtained from $\{g\}$. We suppose that $\{g\}^\alpha$ is defined for all $\alpha<\beta$, where β is an ordinal, such that $\{g\}^\alpha\subset\{g\}^{\alpha'}$ $(\alpha<\alpha'<\beta)$ and each $\{g\}^\alpha$ satisfies (1°) and (2°). When β is an isolated number, we can construct $\{g\}^\beta$ from $\{g\}^{\beta-1}$ as above. When β is a limiting number, we define $\{g\}^\beta$ as the set of all elements in $\{g\}^\alpha$ $(\alpha<\beta)$. Evidently $\{g\}^\beta$ satisfies (1°) and (2°). Thus we get a transfinite sequence $\{g\}^\gamma$ containing $\{g\}$. On the other hand, since L has a fixed cardinal number and $L-\{0\}$ satisfies (1°), (2°) and (4°), this process stops at some γ. Then $\{g\}^\gamma$ is the desired one.

Lemma 4. In the above lemma we can replace (4°) by (3°).

Proof is easy.

Let $\mathfrak{G}$ be a set of all $\{g\}$ satisfying (1°) and (2°). If $\{g\}<\{g'\}<\{g\}$ in $\mathfrak{G}$ we write by $\{g\}\equiv\{g'\}$. For $a\in L$ the set of all $\{g\}\in\mathfrak{G}$ such as $g<a$ for some $g\in\{g\}$, is denote by $\mathfrak{E}(a)$. $\mathfrak{E}$ is a transformation from L onto subset of $\mathfrak{G}$. We have

Lemma 5. $\mathfrak{E}$ is a lattice-homomorphism, and $\mathfrak{E}(0)=0$ and $\mathfrak{E}(1)=\mathfrak{G}$.

Proof is easy.

If we define the closed set in $\mathfrak{G}$ by the product of finite or infinite $\mathfrak{E}(a)$, then we have.

Theorem 1. If every element $\{g\}$ of $\mathfrak{G}$ satisfies (3°), then $\mathfrak{G}$ is a T_0-space.

Proof. Evidently $\mathfrak{G}$ is a T-space. In order to prove that $\mathfrak{G}$ is T_1-space, it is sufficient to prove that $\{g\}\neq\{g'\}$ implies $\overline{\{g\}}\neq\overline{\{g'\}}$. Let $\overline{\{g\}}=\overline{\{g\}}$, then $\{g\}$, $\{g'\}\in\overline{\{g\}}=\overline{\{g'\}}$. If $g\in\{g\}$ then $\{g\}\in\mathfrak{E}(g)$ and $\mathfrak{E}(g)$ is a closed set. Hence $\{g'\}\in\mathfrak{E}(g)$ and $g'<g$ for some $g'\in\{g'\}$. By the same way for any $g'\in\{g'\}$ there exists $g\in\{g\}$ satisfying $g<g'$. Thus we have $\{g\}\equiv\{g'\}$. This is impossible.

Theorem 2. If every $\{g\}\in\mathfrak{G}$ satisfies (3°), then $\mathfrak{G}$ is a T_1-space when and only when every $\{g\}$ satisfies (4°).

Proof. Since $\mathfrak{G}$ is a T_0-space, it is sufficient to prove that $\overline{\{g\}}=\{g\}$ for every $\{g\}\in\mathfrak{G}$. Now let $\{g\}\neq\{g'\}$ and $\{g\}$, $\{g'\}\in\overline{\{g\}}$. Then $\{g'\}\in\mathfrak{E}(\bar{g})$ for any $g\in\{g\}$, or there exists $g'\in\{g'\}$ such as $g'<g$ i.e. $\{g\}<\{g'\}$. On the other hand for any $g'\in\{g'\}$ there exist $g_1\in\{g\}$ such that $g_1g'=g_1$ or $g_1g'=0$. If $g_1g'=0$ then we have g'_1 and g'_2 such that $g'_1g_1=g'_1$ and $g'_2<g'_1g'$. Since $g'_2=g'_2$ $g_1<g'_1$ $g'g_1=g'_1g'=0$. This is impossible. Hence $g_1g'=g_1$, or $\{g\}>\{g'\}$. i.e. $\{g\}\equiv\{g'\}$, which is a contradiction. Conversely let $\mathfrak{G}$ be a T_1-space, and let $ag\neq g$ and $ag=0$ for $\{g\}\in\mathfrak{G}$ and some a. $(ag, g; g\in\{g\})$ satisfies (1°) and (2°). Hence there exists $\{g'\}\supset\{g\}$ satisfying (3°). Evidently $\{g\}\leqq\{g'\}$. Since $\{g\}\bar{\in}\mathfrak{E}(a)$ and $\{g'\}\in\mathfrak{E}(a)$, $\overline{\{g'\}}=\overline{\{g\}}$ or $\{g\}=\{g'\}$. This is a contradiction.

Theorem 3. If every $\{g\} \in \mathfrak{G}$ satisfies (3°), then $\mathfrak{G}$ is the bicompact space.

Proof is analogous to the Wallman's corresponding theorem.

Theorem 4. If L is a Boolean algebra and every $\{g\}$ of $\mathfrak{G}$ satisfies (3°), then $\mathfrak{G}$ is a T_2-space.

Proof. If L is a Boolean algebra, $\mathfrak{E}'(a)=\mathfrak{E}(a')$ is evident, where $\mathfrak{E}'(a)$ is a complement of $\mathfrak{E}(a)$. Hence each $\mathfrak{E}(a)$ is an open and closed set simultaneously. Let $\{g\} \neq \{h\}$. Since $\mathfrak{G}$ is a T_0-space there exists a neighbourhood of $\{g\}$ (or $\{h\}$) which does not contain $\{h\}$ (or $\{g\}$). For instance let a neighbourhood $\left(\Pi\, \mathfrak{E}(a)\right)'$ of $\{h\}$ does not contain $\{g\}$. Then

$$\{h\} \in \left(\Pi\, \mathfrak{E}(a)\right)' = \sum \mathfrak{E}'(a) = \sum \mathfrak{E}(a')$$

or $\{h\} \in \mathfrak{E}(a')$ for some a'. On the other hand $\{g\} \bar{\in} \left(\Pi\, \mathfrak{E}(a)\right)' = \sum \mathfrak{E}(a')$, or $\{g\} \bar{\in} \mathfrak{E}(a')$, or $\{g\} \in \mathfrak{E}(a)$. Consequently $\mathfrak{G}$ is a T_2-space.

Theorem 5. Let L be a Boolean algebra, and $\mathfrak{G}$ be the set of $\{g\}$ satisfying (1°) and (2°). Then $\mathfrak{G}$ satisfies the first countability axiom if and only if each $\{g\}$ of $\mathfrak{G}$ contains at least countable elements.

Proof. By theorem 4 $\{\mathfrak{E}(a)\,;\, a \in L\}$ is an open basis of $\mathfrak{G}$. Let $\{g\} \equiv \{g_n\}$ and $\{g\} \in \mathfrak{E}(a)$, then $\{g_n\} \in \mathfrak{E}(a)$. Hence $\{g_n\} \in \mathfrak{E}(g_n) \subset \mathfrak{E}(a)$. That is, $\{g_n\}$ has a complete system of countable neighbourhood. Conversely if for each $\{g\} \in \mathfrak{G}$ there corresponds a complete system of countable neighbourhoods $\{\mathfrak{E}(g_n)\}$, then $\{g\} \in \mathfrak{E}(g_n)$ $(n=1, 2, \ldots)$. Since $\mathfrak{E}(g_n) \neq 0$ for $n=1, 2, \ldots$, $g_n \neq 0$, for any $\mathfrak{E}(g_{n_1})$ and any $\mathfrak{E}(g_{n_2})$ there exists $\mathfrak{E}(g_{n_3})$ such that $\mathfrak{E}(g_{n_1})$. $\mathfrak{E}(g_{n_2}) \supset \mathfrak{E}(g_{n_3})$ or equivalently $g_{n_1} g_{n_2} > g_{n_3}$.

If $g_n(a \vee b) = g_n$ for some a and b, then $\{g\} \in \mathfrak{E}(g_n) \subset \mathfrak{E}(a) \dotplus \mathfrak{E}(b)$. Hence $\{g\} \in \mathfrak{E}(a)$ or $\{g\} \in \mathfrak{E}(b)$. Equivalently $g < a$ or $g < b$. By the above consideration $\{g_n\} \in \mathfrak{G}$ and then we can easily prove that $\{g_m\} \equiv \{g\}$. For, since $\{g\} \in \mathfrak{E}(g_n)$ there exists $\{g\} \in \{g\}$ such as $g < g_n$. Conversely, since $\{\mathfrak{E}(g_n)\}$ is a complete system of neighbourhoods, for any open set $\mathfrak{E}(g)$ there exists $\mathfrak{E}(g_n)$ such that $\mathfrak{E}(g_n) \subset \mathfrak{E}(g)$. Or equivalently $g_n < g$. That is, $\{g\} \equiv \{g_n\}$.