17. *Normed Rings and Spectral Theorems, III.*

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§1. *A Spectral theorem.* Let $\boldsymbol{R}$ be a function ring of real-valued continuous functions $S(M)$ on a bicompact Hausdorff space $\mathfrak{M}$. We assume that $\boldsymbol{R}$ satisfies

(1) for any pair $M_1, M_2 \in \mathfrak{M}$ there exists $S(M) \in \boldsymbol{R}$ such that $S(M_1) \neq S(M_2)$.

Then, by Gelfand-Silov's abstraction of Weierstrass' polynomial approximation theorem[1],

(2) every continuous function on $\mathfrak{M}$ may be uniformly approximated by functions $\in \boldsymbol{R}$.

Next let $F(S)$ be a linear functional on $\boldsymbol{R}$:

$$(3) \quad \begin{cases} F(\alpha S+\beta U)=\alpha F(S)+\beta F(U) \quad (\alpha, \beta = \text{scalars}), \\ F(S_n) \to F(S) \ (n\to\infty) \ \text{ if } \ \sup\limits_M |S_n(M)-S(M)| \to 0 \ (n\to\infty). \end{cases}$$

We further assume that

$$(4) \qquad F(S) \geqq 0 \quad \text{if} \quad S(M) \geqq 0 \quad \text{on } \mathfrak{M},$$

$$(5) \qquad F(I)=1, \quad \text{where} \quad I(M) \equiv 1 \quad \text{on } \mathfrak{M}.$$

Then, by (2)–(5) and Riesz-Markoff-Kakutani's theorem[2], we have the representation:

$$(6) \qquad F(S)=\int_{\mathfrak{M}} S(M)\varphi(dM) \qquad S\in \boldsymbol{R},$$

where φ is a non-negative, continuous from above set function countably additive on Borel sets $\leqq \mathfrak{M}$ and $\varphi(\mathfrak{M})=1$. Here the continuity from above means that the value of the function on a set is equal to the infimum of its values on open sets covering this set.

We have, from (6),

$$(7) \quad \begin{cases} F(T)=\int_{\mathfrak{M}} T(M)\varphi(dM)=\int_{\lambda_0}^{\lambda_1}\lambda d\tau(\lambda), \ \ \lambda_0=\inf\limits_M T(M), \ \ \lambda_1=\sup\limits_M T(M), \\ \tau(\lambda)=\varphi(M;\, T(M)<\lambda). \end{cases}$$

Put

$$(8) \qquad \mu=\sup_\lambda \big(\lambda\,;\ \tau(\lambda+\varepsilon)-\tau(\lambda-\varepsilon)>0 \quad \text{for all} \quad \varepsilon>0\big).$$

μ may be called as *the maximal spectrum of F referring to T.*

We will prove the

1) Rec. Math., **9** (1941), Cf. also H. Nakano: 全國紙上數學談話會, **218** (1941).

2) A. Markoff: Rec. Math., **4** (1938). S. Kakutani: Proc. **19** (1943).

Theorem. Let λ_0 be >0, then we have

(i) $$\sqrt{\frac{F(T^{2n})}{F(T^{2(n+1)})}} \geqq \sqrt{\frac{F(T^{2(n+1)})}{F(T^{2(n+2)})}} \geqq \frac{1}{\mu} \qquad (n \geqq 0)$$

(ii) $$\sqrt{\frac{F(T^{2n})}{F(T^{2(n+1)})}} \geqq \frac{F(T^{2n+1})}{F(T^{2(n+1)})} \geqq \frac{1}{\mu} \qquad (n \geqq 0)$$

(iii) $$\lim_{n\to\infty} \sqrt{\frac{F(T^{2n})}{F(T^{2(n+1)})}} = \lim_{n\to\infty} \frac{F(T^{2n+1})}{F(T^{2(n+1)})} = \frac{1}{\mu}$$

(iv) $$0 \leqq \frac{F(T^{2n+1})}{F(T^{2(n+1)})} - \frac{1}{\mu} \leqq \sqrt{\frac{F(T^{2n})}{F(T^{2(n+1)})} - \frac{F(T^{2n+1})^2}{F(T^{2(n+1)})^2}} \ (n \to \infty).$$

Proof. By Schwarz's inequality, we have

$$F(T^m) = \int \lambda^m d\tau(\lambda) \leqq \sqrt{\int \lambda^{2k} d\tau(\lambda)} \sqrt{\int \lambda^{2(m-k)} d\tau(\lambda)} = \sqrt{F(T^{2k})}\sqrt{F(T^{2(m-k)})}.$$

Thus, by taking $m=2k+2$ and $m=2k+1$, we obtain the first parts of (i) and of (ii). We have, by (8),

$$\sqrt{\frac{F(T^{2n})}{F(T^{2(n+1)})}} \geqq \frac{F(T^{2n+1})}{F(T^{2(n+1)})} = \frac{\int \lambda^{2n+1} d\tau(\lambda)}{\int \lambda^{2(n+1)} d\tau(\lambda)} \geqq \frac{\int \lambda^{2n+1} d\tau(\lambda)}{\mu \int \lambda^{2n+1} d\tau(\lambda)} = \frac{1}{\mu},$$

and, by (i) and

$$\lim_{n\to\infty} \sqrt[2n]{F(T^{2n})} = \lim_{n\to\infty} \sqrt[2n]{\int \lambda^{2n} d\tau(\lambda)} = \mu,$$

$$\sqrt{\frac{F(T^{2n})}{F(T^{2(n+1)})}} = \sqrt{\frac{\int \lambda^{2n} d\tau(\lambda)}{\int \lambda^{2(n+1)} d\tau(\lambda)}} \to \frac{1}{\mu} \qquad (n \to \infty).$$

Next put

$$\frac{F(T^{2n+1})}{F(T^{2(n+1)})} = \alpha_{n+1}, \quad \sqrt{\frac{F(T^{2n})}{F(T^{2(n+1)})}} = \beta_{n+1}, \quad F\left((T^{-1} - \alpha_{n+1} I)^2 \cdot \frac{T^{2(n+1)}}{F(T^{2(n+1)})}\right) = \gamma_{n+1}.$$

We have

(9) $$\gamma_{n+1} = \frac{F(T^{2n} - 2\alpha_{n+1} T^{2n+1} + \alpha_{n+1}^2 T^{2n+2})}{F(T^{2(n+1)})} = \beta_{n+1}^2 - 2\alpha_{n+1} \cdot \alpha_{n+1} + \alpha_{n+1}^2 \cdot 1 = \beta_{n+1}^2 - \alpha_{n+1}^2.$$

Since

(10) $$F_{n+1}(S) = F\left(S \cdot \frac{T^{2(n+1)}}{F(T^{2(n+1)})}\right), \qquad S \in R,$$

satisfies (3), (4) and (5), we have

(11) $$F_{n+1}(S) = \int_{\mathfrak{M}} S(M) \varphi_{n+1}(dM), \qquad F_{n+1}(T) = \int \lambda d\tau_{n+1}(\lambda)$$

as in (6) and (7). We have

(12) $\mu_{n+1}=\sup\Big(\lambda\,;\,\tau_{n+1}(\lambda+\varepsilon)-\tau_{n+1}(\lambda-\varepsilon)>0\quad\text{for all}\quad\varepsilon>0\Big)=\mu\,,$

for, by applying (iii) to $F_{n+1}(S)$,

$$\frac{1}{\mu_{n+1}}=\lim_{k\to\infty}\sqrt{\frac{F_{n+1}(T^{2k})}{F_{n+1}(T^{2(k+1)})}}=\lim_{k\to\infty}\sqrt{\frac{F(T^{2k+2(n+1)})}{F(T^{2(k+1)+2(n+1)})}}=\frac{1}{\mu}\,.$$

Therefore, by (iii), (9)-(12), we obtain, for any $\varepsilon>0$ ($\mu>\varepsilon$),

$$\beta_{n+1}^2-\alpha_{n+1}^2=F_{n+1}\big((T^{-1}-\alpha_{n+1}I)^2\big)=\int\Big(\frac{1}{\lambda}-\alpha_{n+1}\Big)^2 d\tau_{n+1}(\lambda)$$
$$\geqq\Big(\frac{1}{\mu-\varepsilon}-\alpha_{n+1}\Big)^2\tau_{n+1}(\mu-\varepsilon)\,,$$

as $n\to\infty$. Since $\tau_{n+1}(\nu)=\tau\Big(\sqrt[2n+2]{\nu\int\lambda^{2(n+1)}d\tau(\lambda)}\Big)$ we have $\lim\limits_{n\to\infty}\tau_{n+1}(\mu-\varepsilon)=1$.

§ 2. *Application to Hilbert space.* Let $\boldsymbol{B}$ denote the totality of bounded, self-adjoint operators in Hilbert space $\mathfrak{H}$ and $T\in\boldsymbol{B}$ be positive-definite:

$$\lambda_1\|f\|^2\geqq(Tf,f)\geqq\lambda_0\|f\|^2\,,\quad f\in H\,,\quad\lambda_0>0\,.$$

Let $(T)'$ denote the totality of operators $\in\boldsymbol{B}$ that commute with T, and $(T)''$ be the totality of operators $\in\boldsymbol{B}$ that commute with every operators $\in(T)'$. $(T)''$ is a (real) normed ring by the norm $\|S\|=\sup\limits_{\|f\|=1}\|Sf\|$ and the unit I (=the identity operator). In the first note[1], it is proved that $(T)''$ is ring-isomorphic (with real multipliers) and linear-isometric with the function ring $\boldsymbol{R}$ of all the real-valued continuous functions $S(M)$ on the bicompact Hausdorff space $\mathfrak{M}$ of all the maximal ideals M of $(T)''$:

$$(T)''\ni S\leftrightarrow S(M)\in R\,,\quad I\leftrightarrow I(M)\equiv1\,,\quad\|S\|=\sup_M|S(M)|\,,$$

and moreover

$S(M)$ is non-negative if and only if $(Sf,f)\geqq0$, $f\in\mathfrak{H}$.

Thus, by putting

$F(S)=(Sf_0,f_0)$, (f_0 is any element $\in\mathfrak{H}$ subject to the condition $\|f_0\|=1$),

$$\mu=\inf_\lambda(\lambda\,;\,E_\lambda\cdot f_0=f_0)\,,\qquad T=\int\lambda dE_\lambda\,,$$

we may apply the theorem to $(T)''$ (=$\boldsymbol{R}$). In this way the well-known procedure of *E.* Schmidt concerning completely continuous self-adjoint operators is extended in a more precise form[3].

1) Proc. **19** (1943).

3) Math. Ann. **63** (1907).

3) Concerning (iv), cf. N. Kryloff-N. Bogoliouboff: Bult. de l'acad. Sci. URSS. (1929). D. H. Weinstein: Proc. Nat. Acad. Sci., **20** (1934). S. Huruya: Mem. Fac. Sci. Kyûsyû Imp. Univ. **1** (1941).