47. Notes on Fourier Series (XII). On Fourier Constants.

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1. G. H. Hardy proved[1] that, if a_n are the Fourier sine or cosine coefficients of a function of $L_p(0, 2\pi)$, $p > 1$, then their arithmetic means $\frac{1}{n}\sum_1^n a_k$ are also Fourier coefficients of some function of L_p.

In this section, the author considers the another combination of Fourier coefficients instead of arithmetic means. From the well known K. Knopp's inequality

$$\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\frac{a_k}{k}\right)^2 \leqq 4\sum_{n=1}^{\infty}a_n^2, \tag{1}$$

it readily results that if a_n are Fourier coefficients of a function of L_2, then

$$\sum_{k=n}^{\infty}\frac{a_k}{k} \tag{2}$$

are convergent[2] and are Fourier coefficients of a function of L_2. We ask here whether the similar results will hold for a function of L_p, $p \geqq 1$. With regard to this we obtain the following theorem.

Theorem 1. Let $p > 1$ and a_n be the Fourier sine coefficients of a function of L_p. Then (2) *are the Fourier sine coefficients of some function of L_p.*

We have

$$a_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin kx dx, \qquad f(x) \in L_p(0, \pi), \qquad p > 1.$$

Thus

$$\sum_{k=n}^{\infty}\frac{a_k}{k} = \sum_{k=n}^{\infty}\frac{1}{k}\cdot\frac{2}{\pi}\cdot\int_0^{\pi} f(x)\sin kx dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sum_{k=n}^{\infty}\frac{\sin kx}{k}dx,$$

where the change of order of integration and summation is legitimate since the series $\sum(\sin kx)/k$ is boundedly convergent.

Now since

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n} = \frac{\pi - x}{2} \quad (0 < x < \pi)$$

we get

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1) G. H. Hardy, On some points in the integral calculus 66. The arithmetic mean of Fourier constants, Messenger of Math. 58 (1928-29).

2) It is well known that if a_n are Fourier sine coefficients, then $\sum a_n/n$ is convergent.

$$\sum_{k=n}^{\infty}\frac{a_k}{k}=\frac{2}{\pi}\int_0^{\pi}f(x)\,\frac{\pi-x}{2}\,dx-\frac{2}{\pi}\int_0^{\pi}f(x)\sum_{k=1}^{n}\frac{\sin kx}{k}\,dx$$

which is, by the integration by parts on the last integral, equal to

$$\frac{2}{\pi}\int_0^{\pi}f(x)\,\frac{\pi-x}{2}\,dx+\frac{2}{\pi}\int_0^{\pi}\sum_{k=1}^{n-1}\cos kx\cdot\int_0^{x}f(t)dtdx$$

$$=\frac{2}{\pi}\int_0^{\pi}\frac{\sin\left(n-\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\,dx\int_0^{x}f(t)dt$$

$$=\frac{2}{\pi}\int_0^{\pi}\sin nxdx\cdot\frac{1}{2\tan(x/2)}\int_0^{x}f(t)dt-\frac{1}{\pi}\int_0^{\pi}\cos nxdx\int_0^{x}f(t)dt$$

$$=I_n+J_n\,,$$

say. Since J_n are the Fourier coefficients of an integral, we get $J_n=o(n^{-1})$. Therefore they are Fourier sine ceofficients of a function of L_p for any $p>1$.

I_n are clearly the Fourier sine coefficients of $\dfrac{1}{2\tan(x/2)}\displaystyle\int_0^{x}f(t)dt$ which is a function of L_p, $p>1$[1].

By the similar arguments we have also the theorem.

Theorem 2. If a_n are the Fourier sine coefficients of a function of Zygmund class, then (2) *are the Fourier sine coefficients of an integrable function.*

2. We shall add a following theorem.

Theorem 3. Let $a_n>0$. Then the series

$$\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\frac{a_k}{k}\right)\sin nx \tag{3}$$

is a Fourier sine series, if and only if

$$\sum_{n=2}^{\infty}\frac{a_n}{n}\log n<\infty\,. \tag{4}$$

This readily follows from the following fact[2]: If $a_n\downarrow 0$, the series $\sum a_n\sin nx$ is a Fourier sine series of an integrable function if and only if $\sum a_n\log n<\infty$.

We take $a_n=\sum_{k=n}^{\infty}\frac{a_k}{k}$ whose convergence is a consequence of (4). Thus the sufficiency is immediate Next if (3) is a sine series then $\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=n}^{\infty}\frac{a_k}{k}$ converges and from this $\sum_{n=1}^{\infty}\Delta\left(\sum_{k=n}^{\infty}\frac{a_k}{k}\right)\log n<\infty$ follows, which is (4).

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1) A. Zygmund, Trigonometrical series (1935), Warsaw p. 244. This is a special case of Hardy-Littlewood's maximal theorem.

2) A. Zygmund, ibid., p. 112.

We note that if $a_n > 0$ and (3) is Fourier sine series then it is also cosine series. This follows by the monotoneness of the sequence $\{\sum_{k=n}^{\infty} a_k/k\}$ and the fact that monotone sine coefficients are also cosine coefficients[1].

3. If $\Delta^2 a_n > 0$, $a_n \downarrow 0$, then the series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx \tag{5}$$

is a Fourier series of a non-negative integrable function. This is well known. We discuss here the additional conditions in order that the series (5) is a Fourier series of a function of L_p, $p>1$.

Let $\sigma_n(x)$ be the (C, 1) means of (5) and $\Delta^2 a_n > 0$, $a_n \downarrow 0$. Since

$$\begin{aligned} S_n(x) &= \frac{1}{2}a_0+\sum_{k=1}^{n} a_k \cos kx \\ &= \sum_{n=0}^{n} \Delta^2 a_k \cdot (k+1)K_k(x) + K_n(x)\,(n+1)\Delta a_{n+1} + D_n(x)\cdot a_{n+1}, \end{aligned}$$

where K_n and D_n denotes Fejèr's and Dirichlet's kernels, we have, by partial summation,

$$\begin{aligned} (N+1)\sigma_N(x) &= \sum_{n=0}^{N} S_n(x) = \sum_{n=0}^{N}\sum_{k=0}^{n} \Delta^2 a_k (k+1)K_k(x) \\ &\quad + \sum_{n=0}^{N} K_n(x)\cdot(n+1)\Delta a_{n+1} + \sum_{n=0}^{N} D_n(x)\cdot a_{n+1} \\ &= \sum_{n=0}^{N}\sum_{k=0}^{n} \Delta^2 a_k (k+1)K_k(x) + 2\sum_{n=0}^{N-1}(n+1)K_n(x)\Delta a_{n+1} \\ &\quad + 2(N+1)K_N(x)\cdot a_{N+1}. \end{aligned} \tag{6}$$

If (5) is a Fourier series of a function of L_p, $p>1$, then $\int_0^{2\pi}|\sigma_N(x)|^p dx < C$, C beidg a constant inependent of N. Hereafter C denotes a constant which may differ on can occurrence. Since every term of the right hand side of (6) is non-negtive, we have

$$\begin{aligned} C &> \int_0^{2\pi} |\sigma_n(x)|^p dx \geqq \frac{1}{N^p}\int_0^{2\pi}\{\sum_{n=0}^{N-1}\Delta a_{n+1}\cdot(n+1)K_n(x)\}^p dx \\ &\geqq \frac{1}{N^p}\sum_{k=0}^{N-1}\int_{(k\pi+\alpha)/N}^{((k+1)\pi-\alpha)/\pi}\{\sum_{n=0}^{N-1}\Delta a_{n+1}(n+1)K_n(x)\}^p dx \\ &\geqq \frac{1}{N^p}\sum_{k=0}^{N-1}\int_{(k\pi+\alpha)/N}^{((k+1)\pi-\alpha)/N}\{\sum_{n>N\cdot\beta_k}^{N-1}\Delta a_{n+1}\cdot(n+1)K_n(x)\}^p dx, \end{aligned}$$

where $0<\alpha<\frac{\pi}{2}$, $\beta_k = \dfrac{k\pi+\varepsilon}{k\pi+\alpha}$, $0>\varepsilon>\alpha$. But for every x such that $\{(k+1)\pi-\alpha\}/N > x > (k\pi+\alpha)/N$, we have

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1) A. Zygmund, ibid., p. 129, ex. 5.

$$(k+1)\pi-a>\frac{\big((k+1)\pi-a\big)n}{N}>nx>\frac{n(k\pi+a)}{N}>k\pi+\varepsilon,$$

$(n\geqq\beta_k\cdot N)$. Thus for such x there exists a constant C such that $\sin^2 nx>C$. Therefore we have

$$\begin{aligned}C&\geqq C\frac{1}{N^p}\sum_{k=0}^{N-1}\int_{(k\pi+a)/N}^{((k+1)\pi-a)/N}\Big\{\sum_{n>N\beta_k}^{n-1}\Delta a_{n+1}\cdot\frac{1}{x^2}\Big\}^p dx\\&\geqq C\frac{1}{N^p}\sum_{k=0}^{N-1}(\Delta a_N)^p\cdot\{N(1-\beta_k)\}^p\int_{(k\pi+a)/N}^{((k+1)\pi-a)/N}\frac{dx}{x^{2p}}\\&\geqq C(\Delta a_N^p)\sum_{k=1}^{N-1}\frac{1}{k^p}\cdot\frac{1}{N}\Big(\frac{N}{k}\Big)^{2p}\\&\geqq CN^{2p-1}(\Delta a_N)^p.\end{aligned}$$

Thus we get the inequality

$$\Delta a_n=O(n^{-2+\frac{1}{p}}). \tag{7}$$

Similarly or more simply we get

$$a_n=O(n^{-1+\frac{1}{p}}),\qquad \Delta^2 a_n=O(n^{-3+\frac{1}{p}}) \tag{8}$$

which, however, does not need for our purposes

Next we assume that

$$\sum\Delta a_n\cdot n^{1-\frac{1}{p}}<\infty \tag{9}$$

and that (7) holds. Then

$$\sum\Delta^2 a_n n^{2-\frac{1}{p}}<\infty \tag{10}$$

which is evident by partial summation.

Now by (6) we have

$$\begin{aligned}\Big\{\int_0^{2\pi}|\sigma_N(x)|^p dx\Big\}^{\frac{1}{p}}&\leqq\frac{C}{N}\Big[\int_0^{2\pi}\{\sum_{n=0}^{N-1}\sum_{k=0}^{n}\Delta^2 a_k(k+1)K_k(x)\}^p dx\Big]^{\frac{1}{p}}\\&+\frac{C}{N}\Big[\int_0^{2\pi}\{\sum_{n=0}^{N-1}(n+1)K_n(x)\cdot\Delta a_{n+1}\}^p dx\Big]^{\frac{1}{p}}\\&+\frac{C}{N}\Big[\int_0^{2\pi}(N+1)^p K_N^p(x)a_{N+1}^p dx\Big]^{\frac{1}{p}},\end{aligned}$$

which does not exceed by Minkowski inequality

$$\begin{aligned}&\frac{C}{N}\Big[\sum_{n=0}^{N-1}\sum_{k=0}^{n}\Delta^2 a_k\Big\{\int_0^{2\pi}(k+1)^p K_k^p(x)dx\Big\}^{\frac{1}{p}}\\&+\sum_{n=0}^{N-1}\Delta a_{n+1}\Big\{\int_0^{2\pi}(n+1)^p K_n^p(x)dx\Big\}^{\frac{1}{p}}\\&+a_{N+1}\Big\{\int_0^{2\pi}(N+1)^p K_N^p(x)dx\Big\}^{\frac{1}{p}}\end{aligned}$$

Sinc

$$\int_0^{2n}(n+1)^p K_n^p(x)dx = O\left(\int_0^{2\pi}\left(\frac{\sin^2 nx}{x^2}\right)^p dx\right)$$

$$O=\left(n^{2p-1}\int_0^{\infty}\left(\frac{\sin^2 u}{u^2}\right)^p du\right)=(n^{2p-1}),$$

we have

$$\left(\int_0^{2\pi}|\sigma_N(x)|^p dx\right)^{\frac{1}{p}} = O\left(\frac{1}{N}\sum_{n=0}^{N}\sum_{k=0}^{n}\Delta^2 a_k\cdot k^{2-\frac{1}{p}}\right). \tag{11}$$

$$+O\left(\frac{1}{N}\sum_{n=0}^{N-1}\Delta a_{n+1}\cdot n^{2-\frac{1}{p}}\right)+O(a_{N+1}\cdot N^{1-\frac{1}{p}}).$$

The first term of the right is $O(1)$ by (10), the second term is also $O(1)$ by (9), Since

$$a_N=\sum_{n=N}^{\infty}\Delta a_n = O\left(\sum_{n=N}^{\infty}\frac{1}{n^{2-\frac{1}{p}}}\right)$$

by (6) which is $O(N^{-1+\frac{1}{p}})$, the last term is also $O(1)$. Thus

$$\int_0^{2\pi}\{\sigma_N(x)\}^p dx = O(1)$$

from which, by the known result, it results that the series (5) is a Fourier series of a function of L_p. Thus we get the theorem.

Theorem 4. Let $\Delta^2 a_n\cdot a_n \downarrow 0$. *If the series* (5) *is a Fourier series of a function of* L_p, $p>1$, *then* (7) *holds. Conversely if* (7) *and* (9) *hold, then* (5) *is a Fourier series of a function of* L_p, $p>1$.

As a corollary we obtain

Cor. Let $\Delta^2 a_n\cdot a_n\downarrow 0$. *If* (7) *holds then* (5) *is a Fourier series of a function of* $L_{p'}$ *where* $1<p'<p$.

This is trivial, since (6) implies the convergence of $\sum \Delta a_n\cdot n^{1-\frac{1}{p}}$.