119. *Normed Rings and Spectral Theorems, VI.*

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(Comm. by T. TAKAGI, M.I.A., Oct. 12, 1944.)

1. *Introduction.* The arguments in 3 of the fifth note[1] was insufficient since the lemma 2 is valid for $z \in L_1(G)$ only. The purpose of the present note is i) to give a complete proof of (19)—the Plancherel's theorem—in 3 of the fifth note and ii) to show that the Bochner-Raikov's representation theorem[2] may be obtained easily from the Plancherel's theorem. In this way, the Fourier analysis may be subsumed under the operator theory in Hilbert space formulated in terms of the normed ring.

We will make use of, in this note, the results and the notations in the fifth note.

2. *Proof of (19).* The set $\left\{\varphi_z(\chi)=\lambda+\int_G x(g)\chi(g)dg\,;\; z=\lambda e+x,\; x \in L_1(G)\right\}$ is dense in the space $C(X \cup \chi_\infty)$ of continuous functions $T(\chi)$ on the character group X of G compactified by adjoining the formal character χ_∞. This results from the Gelfand-Silov's abstraction[3] of Weierstrass' polynomial approximation theorem. We have

$$\| T_z \| = \lim_{n\to\infty} \sqrt[n]{\| T_z^n \|} \leqq \lim_{n\to\infty} \sqrt[n]{\| z^n \|_1} = \sup_x | \varphi_z(\chi) |$$

by $\| T_z \| \leqq \| z \|_1$. Hence,

$$(*)\quad \begin{cases} \text{for any continuous function } T(\chi) \text{ on } X \cup \chi_\infty, \text{ there exists one} \\ \text{and only one operator } T \text{ such that } \lim_{n\to\infty} \sup_\chi | \varphi_{z_n}(\chi) - T(\chi) | = 0 \\ \text{implies } \lim_{n\to\infty} \| T_{z_n} - T \| = 0. \end{cases}$$

Let $x \in L_1(G) \frown C(G)$ and put

$$J\big(\varphi_x(\chi)\big) = x(0).$$

J is additive, homogeneous and positive on $\{\varphi_x(\chi)\}$:

$$\varphi_x(\chi) \geqq 0 \qquad \text{implies} \qquad J\big(\varphi_x(\chi)\big) \geqq 0.$$

The proof was given by the lemma 3. The Plancherel's theorem may be proved if we show that

$$(**)\qquad J\big(\varphi_x(\chi)\big) = \int_X \varphi_x(\chi) d\chi, \quad d\chi = \text{the Haar's measure on } X.$$

This formula together with the definition

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1) Proc. **20** (1944), 269.
2) C. R. URSS: **28**, 4 (1940).
3) Rec. Math., **9** (51), (1941).

$$\varphi_x(\chi)=\int_G x(g)\chi(g)dg\,, \qquad dg=\text{the Haar's measure on G.}$$

constitutes (19).

Let $\mathfrak{R}=\{\varphi_x(\chi);\ x\in L_1(G)\frown C(G)\}$ and let $\Gamma=\{\varphi(\chi);\ \varphi(\chi)$ is continuous and $=0$ in a certain vicinity U_φ of $\chi_\infty\}$ and denote by $\mathfrak{R}_1$ the linear envelope of $\mathfrak{R}$ and Γ. We first show that J may be extended additive homogeneous and positive on $\mathfrak{R}_1$. This extension is surely possible if the following conditions are satisfied:

$$(***)\ \begin{cases}\text{Let } \tilde{\varphi}(\chi)\in\Gamma, \text{ then} \\ \inf\limits_{\substack{\varphi(\chi)\in\mathfrak{R}\\ \varphi(\chi)\geqq\tilde{\varphi}(\chi)}} J\big(\varphi(\chi)\big)=\sup\limits_{\substack{\varphi(\chi)\in\mathfrak{R}\\ \varphi(\chi)\leqq\tilde{\varphi}(\chi)}} J\big(\varphi(\chi)\big).\end{cases}$$

For the proof of (***) let $X_0=\{\chi;\tilde{\varphi}(\chi)\neq 0\}$ and take $y\in L_1(G)\cap C(G)$ such that the closure $\overline{X}_0\subseteqq\{\chi;\ \varphi_y(\chi)\neq 0\}$. Put $T(\chi)=\tilde{\varphi}(\chi)/|\varphi_y(\chi)|^2$ and let

$$\begin{cases}\lim\limits_{n\to\infty}\sup\limits_{x}|\varphi_{z_n}(\chi)-T(\chi)|=0, & \varphi_{z_n}(\chi)\geqq T(\chi), \\ \lim\limits_{n\to\infty}\sup\limits_{\chi}|\varphi_{w_n}(\chi)-T(\chi)|=0, & \varphi_{w_n}(\chi)\leqq T(\chi), \\ z_n=\lambda_n e+x_n,\ x_n\in L_1(G)\frown C(G), & w_n=\mu_n e+y_n,\ y_n\in L_1(G)\frown C(G).\end{cases}$$

Then, by (*) there exists one and only one operator T such that $\lim\limits_{n\to\infty}\|T_{z_n}-T\|=0$, $\lim\limits_{n\to\infty}\|T_{w_n}-T\|=0$.

We have thus

$$\begin{aligned}&\lim_{n\to\infty}J\big(\varphi_{z_n}(\chi)\,|\varphi_y(\chi)|^2\big)=\lim_{n\to\infty}z_n*y*y^*(0)\\ &=\lim_{n\to\infty}(T_{z_n}\cdot y,y)=\lim_{n\to\infty}(T_{w_n}\cdot y,y)=\lim_{n\to\infty}w_n*y*y^*(0)\\ &=\lim_{n\to\infty}J\big(\varphi_{w_n}(\chi)\,|\varphi_y(\chi)|^2\big).\end{aligned}$$

Since z_n*y*y^* and $w_n*y*y^*\in L_1(G)\frown C(G)$, (***) is completely proved.

Therefore[1], remembering $J\big(\varphi_x(\chi+\chi')\big)=\big(\chi'(g)x(g)\big)(0)=x(0)=J\big(\varphi_x(\chi)\big)$, we have for $\tilde{\varphi}(x)\in\Gamma$,

$$(**)'\qquad J\big(\tilde{\varphi}(\chi)\big)=\int_X\tilde{\varphi}(\chi)d\chi,\quad d\chi=\text{the Haar measure on } X.$$

Hence, by the proof of (***), we have

$$\begin{cases}J\big(T(\chi)\,|\varphi_y(\chi)|^2\big)=(\mathrm{T}\cdot y,y)=\int_X T(\chi)\,|\varphi_y(\chi)|^2d\chi, \\ T(\chi)\in\Gamma,\qquad y\in L_1(G)\frown C(G).\end{cases}$$

Thus, by letting $T(\chi)$ tend to 1, we see that $\varphi_y(\chi)\in L_2(X)$ and hence

$$(**)''\ \begin{cases}J\big(\varphi_x(\chi)\,|\varphi_y(\chi)|^2\big)=(T_x\cdot y,y)=\int_X\varphi_x(\chi)\,|\varphi_y(\chi)|^2d\chi, \\ x,\ y\in L_1(G)\frown C(G).\end{cases}$$

1) M. Krein: C. R. URSS, **30**, 6 (1941). The closure of the set $\{x;\tilde{\varphi}(x)\neq 0\}$ is compact in X.

Since $L_1(G)$ is dense in $L_1(G) \frown L_2(G)$, $(**)''$ also holds good for $y \in L_1(G) \frown L_2(G)$. Let V_n be a symmetric and compact neighbourhood in G of 0 such that $0 < \int_{V_n} dg < \infty$, $\lim_{n\to\infty} V_n = 0$ and put $y_n(g)=$ characteristic function of V_n divided by $\int_{V_n} dg$. Then, we have, from $(**)''$,

$$x(0) = \lim_{n\to\infty} (T_x y_n, y_n) = \lim_{n\to\infty} \int_X \varphi_x(\chi) \,|\, \varphi_{y_n}(\chi)\,|^2 d\chi = \int_X \varphi_x(\chi) d\chi,$$

since[1] $\chi(g)$ is uniformly continuous in the aggregate of variables χ, g such that $\chi(0)=1$. Q. E. D.

3. *Bochner-Raikov's theorem.* A measurable function $f(g)$ is called pesitive definite if $f(-g)=\overline{f(g)}$ and

$$\iint f(g-h)x(g)\overline{x(h)}dgdh \geqq 0$$

for every $x \in L_1(G)$. Such $f(g)$ is essentially bounded and we may assume

$$f(0) = \operatorname*{ess.\ sup}_{g} |f(g)|.$$

We will prove Bochner-Raikov's theorem to the effect that

$$f(g) = \int_X \chi(g) F(d\chi) \qquad \text{a.e. on } G$$

with a uniquely determined continuous from above measure countably additive on Borel sets $\subseteqq X$.

Proof. Let X_i = symmetric and compact neighbourhood in X of the zero 0 such that $\int_{X_i} d\chi \neq 0$, $\lim_{i\to\infty} X_i = 0$. Then, since $\chi(g)f(g)$ is positive definite with $f(g)$,

$$f_i(g) = f(g)\Big(\int_{X_i} \chi(g) d\chi\Big)^2 \Big/ \Big(\int_{X_i} d\chi\Big)^2$$

is also positive definite, viz.

$$(f_i * y, y) \geqq 0 \qquad \text{for every} \qquad y \in L_1(G).$$

By the Plancherel's theorem, the continuous function $\Big(\int_{X_i} \chi(g) d\chi\Big)^2 \in L_1(G)$ and hence $\in L_2(G)$. Thus $f_i(g) \in L_1(G) \frown L_2(G)$. Therefore, again by Plancherel's theorem,

$$(****) \qquad f_i(g) = \int_X \chi(g) \varphi_{f_i}(\chi) d\chi \qquad \text{a.e. on } G \text{ in the mean.}$$

Moreover we have

$$(f_i * y, y) = \int_X \varphi_{f_i}(\chi) \,|\, \varphi_y(\chi)\,|^2 d\chi \geqq 0$$

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1) See 2).

by the positive-definiteness of $f_i(g)$ and the Plancherel's theorem. Since $\{\varphi_y(\chi)\,;\, y\in L_1(G)\}$ is dense in the space of continuous functions on $X\cup\chi_\infty$, vanishing at χ_∞, we have $\varphi_{f_i}(\chi)\geqq 0$ on X. By taking average of (****) over a vicinity of 0 in G, we see that $\int_X \varphi_{f_i}(\chi)d\chi \leqq f_i(0)=f(0)$. Hence the set $\left\{\int\varphi_{f_i}(\chi)d\chi\right\}$ is compact as a set of measures. Therefore, there exists a subsequence $\{i'\}$ of $\{i\}$ such that

$$F(d\chi)=\lim_{i'\to\infty}\ \varphi_{f_{i'}}(\chi)d\chi \qquad \text{(limit as measures)},$$

$$\lim_{i'\to\infty} f_{i'}(g)=f(g)=\int_X \chi(g)F(d\chi) \qquad \text{a.e. on } G$$

Q. E. D.