Dissipative property for non local evolution equations

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Abstract

In this work we consider the non local evolution problem

$$\begin{cases} \partial_t u(x,t) = -u(x,t) + g(\beta K(f \circ u)(x,t) + \beta h), \ x \in \Omega, \ t \in [0,\infty[; \\ u(x,t) = 0, \ x \in \mathbb{R}^N \setminus \Omega, \ t \in [0,\infty[; \\ u(x,0) = u_0(x), \ x \in \mathbb{R}^N, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$; $g, f : \mathbb{R} \to \mathbb{R}$ satisfying certain growing condition and K is an integral operator with symmetric kernel, $Kv(x) = \int_{\mathbb{R}^N} J(x,y)v(y)dy$. We prove that Cauchy problem above is well posed, the solutions are smooth with respect to initial conditions, and we show the existence of a global attractor. Furthermore, we exhibit a Lyapunov's functional, concluding that the flow generated by this equation has the gradient property.

*Partially supported by CAPES-Brazil
†Partially supported by CAPES-Brazil
Received by the editors in May 2017 - In revised form in July 2018.
Communicated by D. Bonheure.
2010 *Mathematics Subject Classification* : 45J05, 45M05, 37B25.
Key words and phrases : Non local equation; Well-posedness; Smoothness orbit; Global Attractor; Lyapunov's functional.

1 Introduction

We consider the non local evolution problem

$$\begin{cases} \partial_t u(x,t) = -u(x,t) + g(\beta K(f \circ u)(x,t) + \beta h), \ x \in \Omega, \ t \in [0,\infty[, \\ u(x,t) = 0, \ x \in \mathbb{R}^N \setminus \Omega, \ t \in [0,\infty[, \\ u(x,0) = u_0(x), \ x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $u(x,t)$ is a real function on $\mathbb{R}^N \times [0,\infty[$, Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 1)$; h and β are non negative constants; $f, g : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous satisfying some growth conditions and K is an integral operator with symmetric nonnegative kernel, given by

$$Kv(x) := \int_{\mathbb{R}^N} J(x,y)v(y)dy, \tag{1.2}$$

where J is a symmetric non negative function of class $\mathscr{C}^1$, with

$$\int_{\mathbb{R}^N} J(x,y)dy = \int_{\mathbb{R}^N} J(x,y)dx = 1.$$

The dynamics of non local evolution Equations like in (1.1) has attracted the attention of many researchers in the last years; see for instance [1, 2, 3, 5, 6, 8, 9, 10, 14, 15, 16, 20, 21, 23, 25, 28, 30] and [31]. However, the model considered here presents innovation, because it includes the model considered in [3, 23, 24] and [25], which can be obtained as a particular case of (1.1) with f being the identity, as well as it includes the model considered in [8, 9, 10, 20, 23, 28, 30] and [31], which can be obtained as a particular case of (1.1) where g is the identity, $\beta = 1$ and the integral operator K is the convolution product. When g and f are identity, $\beta = 1$ and the integral operator K is the convolution product, we also obtain as particular case of (1.1) the model considered in [4].

The approach considered here was motivated by similar approaches in [3, 12] and [27], whose basic idea is to find an abstract way to impose Dirichlet boundary conditions in non local evolution equations.

The paper is organized as follows. In Section 2, assuming a growth condition on the functions g and f, we prove that (1.1) is well posed with globally defined solution, (see Proposition 2.2, Proposition 2.3 and Corollary 2.5) that generalize Proposition 2.4 and Corollary 2.6 in [13]. Furthermore, according to our assumptions, the results presented in this section are also extensions of Proposition 2.2 and Corollary 2.3 proved in [25]; Proposition 2.1 and Corollary 2.2 proved in [3]; and Proposition 2 and Corollary 3 obtained in [10]. In Section 3 we prove that (1.1) generates a $\mathscr{C}^1$ flow in a space X which is isometric to $L^p(\Omega)$ (see Proposition 3.2), which extends Proposition 2.4 in [3] and Proposition 3.1 in [11]. In Section 4, we prove existence of a global attractor, (see Theorem 4.3) that extends the following results: Theorem 3.3 in [3]; Theorem 8 in [10]; Theorem 3.3 in [25] and Theorem 3.2 in [13]. In Section 5, we prove comparison and boundedness results for the solutions of (1.1), (see Theorem 5.2), which extends Theorem 2.7 in

[24] and Theorem 4.2 in [25]. Finally, in Section 6, we exhibit a continuous Lyapunov's functional for the flow generated by (1.1), and we use it to prove that this flow has the gradient property in the sense of [18], extending Theorem 5.2 and Proposition 5.5 obtained in [25], as well as Theorem 4.4 and Proposition 4.6 in [11], and Theorem 4.3 and Proposition 4.5 obtained in [13].

2 Well posedness

In this section, we prove that the Cauchy problem (1.1) is well posed in the suitable phase space

$$X = \left\{ u \in L^p\left(\mathbb{R}^N\right) \;:\; u(x) = 0, \text{ if } x \in \mathbb{R}^N \setminus \Omega \right\}$$

with the induced norm of $L^p\left(\mathbb{R}^N\right)$. In order to this, in addition to the hypotheses from introduction, we assume that the functions g and f satisfy the "suitable" following growth conditions: *there exist non negative constants k_1, k_2, c_1 and c_2 such that*

$$|g(x)| \leq k_1|x| + k_2, \ \forall\, x \in \mathbb{R} \tag{2.3}$$

and

$$|f(x)| \leq c_1|x| + c_2, \ \forall\, x \in \mathbb{R}. \tag{2.4}$$

The space X is canonically isometric to $L^p(\Omega)$ and we usually identify the two spaces, without further comment. We also use the same notation for a function in $\mathbb{R}^N$ and its restriction to Ω for simplicity, wherever we believe the intention is clear from the context.

In order to obtain well posedness of (1.1), we consider the Cauchy problem

$$\begin{cases} \partial_t u = -u + F(u), \\ u(t_0) = u_0, \end{cases} \tag{2.5}$$

where the map $F : X \to X$ is defined by

$$F(u)(x) = \begin{cases} g(\beta K(f \circ u)(x) + \beta h), & x \in \Omega, \\ 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.6}$$

Depending on the properties assumed for J, the map given by (1.2) is well defined as a bounded linear operator in various functions spaces and, in particular, it will be well defined in X.

To prove that F given in (2.6) is well defined, under the conditions given in (2.3) and (2.4), we need the estimates below for the map K, which have been proven in [25].

Lemma 2.1. *Let K be the map defined by (1.2) and $\|J\|_r := \sup_{x\in\Omega} \|J(x,\cdot)\|_{L^r(\Omega)}$, $1 \leq r \leq \infty$. If $u \in L^p(\Omega)$, $1 \leq p \leq \infty$, then $Ku \in L^\infty(\Omega)$,*

$$|Ku(x)| \leq \|J\|_q \|u\|_{L^p(\Omega)}, \ \forall\, x \in \Omega, \tag{2.7}$$

where $1 \leq q \leq \infty$ is the conjugate exponent of p, and

$$\|Ku\|_{L^p(\Omega)} \leq \|J\|_1 \|u\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)}. \tag{2.8}$$

Moreover, if $u \in L^1(\Omega)$, then $Ku \in L^p(\Omega)$, $1 \leq p \leq \infty$, and

$$\|Ku\|_{L^p(\Omega)} \leq \|J\|_p \|u\|_{L^1(\Omega)}. \tag{2.9}$$

Proposition 2.2. *In addition to the hypotheses from Lemma 2.1, suppose that the functions g and f satisfy the two growth conditions (2.3) and (2.4). Then the function F given by* (2.6) *is well defined in $L^p(\Omega)$.*

Proof. Consider $1 \leq p < \infty$ and let $u \in L^p(\Omega)$. Then, using Hölder inequality (see [17]) and (2.4), we obtain

$$\|f(u)\|_{L^1(\Omega)} \leq \int_\Omega [c_1|u(x)| + c_2]dx \leq c_1|\Omega|^{\frac{1}{q}}\|u\|_{L^p(\Omega)} + c_2|\Omega|, \tag{2.10}$$

where q denotes the conjugate exponent of p.

From estimates (2.9) and (2.10), it follows that

$$\begin{aligned} \|Kf(u)\|_{L^p(\Omega)} &\leq \|J\|_p \|f(u)\|_{L^1(\Omega)} \\ &\leq \|J\|_p (c_1|\Omega|^{\frac{1}{q}}\|u\|_{L^p(\Omega)} + c_2|\Omega|) \\ &= c_1\|J\|_p|\Omega|^{\frac{1}{q}}\|u\|_{L^p(\Omega)} + \|J\|_p c_2|\Omega|. \end{aligned} \tag{2.11}$$

Thus, using (2.11), it follows that

$$\begin{aligned} \|F(u)\|_{L^p(\Omega)} &= \|g(\beta|Kf(u)| + \beta h)\|_{L^p(\Omega)} \\ &\leq \left(\int_\Omega [\beta k_1 |K((f(u))(x)| + k_1\beta h + k_2]^p dx \right)^{\frac{1}{p}} \\ &\leq \|\beta k_1 |Kf(u)| + (k_1\beta h + k_2)\|_{L^p(\Omega)} \\ &\leq \beta k_1 \|Kf(u)\|_{L^p(\Omega)} + \|k_1\beta h + k_2\|_{L^p(\Omega)} \\ &\leq \beta k_1 (c_1\|J\|_p|\Omega|^{\frac{1}{q}}\|u\|_{L^p(\Omega)} + \|J\|_p c_2|\Omega|) + (k_1\beta h + k_2)|\Omega|^{\frac{1}{p}} \\ &= \beta k_1 c_1 \|J\|_p |\Omega|^{\frac{1}{q}}\|u\|_{L^p(\Omega)} + \beta k_1\|J\|_p c_2|\Omega| + (k_1\beta h + k_2)|\Omega|^{\frac{1}{p}}, \end{aligned} \tag{2.12}$$

showing that, in this case, F is well defined.

The proof for $p = \infty$ is straightforward, because if $u \in L^\infty(\Omega)$, from (2.4) it follows that $f(u) \in L^\infty(\Omega)$ and, consequently

$$|K(f(u)(x))| \leq \|J\|_1 \|f(u)\|_\infty = \|f(u)\|_\infty.$$

Thus, using (2.4), we obtain

$$\|Kf(u)\|_{L^\infty(\Omega)} \leq c_1\|u\|_\infty + c_2.$$

Hence, from (2.3), we have

$$\begin{aligned} \|F(u)\|_{L^\infty(\Omega)} &\leq k_1\beta\|Kf(u)\|_{L^\infty(\Omega)} + k_1\beta h + k_2 \\ &\leq \beta k_1(c_1\|u\|_\infty + c_2) + k_1\beta h + k_2. \end{aligned}$$

Thus, we conclude the result. ∎

Proposition 2.3. *Suppose, in addition to the hypotheses from Proposition 2.2, that the function f satisfies*

$$|f(x) - f(y)| \leq c_0(1 + |x|^{p-1} + |y|^{p-1})|x - y|, \text{ for any } (x, y) \in \mathbb{R} \times \mathbb{R}. \quad (2.13)$$

Then the function F given by (2.6) *is Lipschitz continuous on bounded sets of $L^p(\Omega)$, $1 \leq p \leq \infty$.*

Proof. Initially, suppose $1 < p < \infty$. Then, for any $u \in L^p(\Omega)$, using (2.7) and (2.4), we have

$$\begin{aligned}
|Kf(u)(x)| &\leq \|J\|_q \|f(u)\|_{L^p(\Omega)} \\
&= \|J\|_q \left(\int_\Omega |f(u(x))|^p dx \right)^{\frac{1}{p}} \\
&\leq \|J\|_q \left(\int_\Omega [c_1|u(x)| + c_2]^p dx \right)^{\frac{1}{p}} \\
&\leq \|J\|_q (c_1 \|u\|_{L^p(\Omega)} + \|c_2\|_{L^p(\Omega)}) \\
&= c_1 \|J\|_q \|u\|_{L^p(\Omega)} + c_2 \|J\|_q |\Omega|^{\frac{1}{p}}.
\end{aligned}$$

In particular, if u is in a ball centered at origin of $L^p(\Omega)$ with radius r, it follows that

$$|Kf(u)(x)| \leq c_1 \|J\|_q r + c_2 \|J\|_q |\Omega|^{\frac{1}{p}}.$$

Then, if $l = \beta(c_1\|J\|_q r + c_2 \|J\|_q |\Omega|^{\frac{1}{p}} + h)$ and N denotes the Lipschitz constant of g in the interval $[-l, l] \subset \mathbb{R}$, for $u, v \in L^p(\Omega)$ with $\|u\|_{L^p(\Omega)} \leq r$ and $\|v\|_{L^p(\Omega)} \leq r$, we have

$$\begin{aligned}
\|F(u) - F(v)\|_{L^p(\Omega)} &= \|g(\beta K f(u) + \beta h) - g(\beta K f(v) + \beta h)\|_{L^p(\Omega)} \\
&= \left(\int_\Omega |g(\beta K f(u) + \beta h)(x) - g(\beta K f(v) + \beta h)(x)|^p dx \right)^{\frac{1}{p}} \\
&\leq \left(\int_\Omega |N\beta|^p |K f(u)(x) - K f(v)(x)|^p dx \right)^{\frac{1}{p}} \\
&= N\beta \|K(f(u) - f(v))\|_{L^p(\Omega)}. \qquad (2.14)
\end{aligned}$$

Now, using (2.13) and Hölder Inequality, it follows that

$$
\begin{aligned}
&\|f(u)-f(v)\|_{L^1(\Omega)}\\
&\quad\leq \int_{\Omega} c_0(1+|u(x)|^{p-1}+|v(x)|^{p-1})|u(x)-v(x)|dx\\
&\quad\leq c_0\left[\int_{\Omega}\left(1+|u(x)|^{p-1}+|v(x)|^{p-1}\right)^q dx\right]^{\frac{1}{q}}\left[\int_{\Omega}|u(x)-v(x)|^p dx\right]^{\frac{1}{p}}\\
&\quad\leq c_0\left[\|1\|_{L^q(\Omega)}+\|u^{p-1}\|_{L^q(\Omega)}+\|v^{p-1}\|_{L^q(\Omega)}\right]\|u-v\|_{L^p(\Omega)}\\
&\quad\leq c_0[|\Omega|^{\frac{1}{q}}+\|u\|^{\frac{p}{q}}_{L^p(\Omega)}+\|v\|^{\frac{p}{q}}_{L^p(\Omega)}]\|u-v\|_{L^p(\Omega)}, \qquad (2.15)
\end{aligned}
$$

where q is the conjugate exponent of p. Thus, using (2.9) and (2.15), it follows that

$$
\begin{aligned}
\|Kf(u)-Kf(v)\|_{L^p(\Omega)} &\leq \|J\|_p\|f(u)-f(v)\|_{L^1(\Omega)}\\
&\leq c_0\|J\|_p[|\Omega|^{\frac{1}{q}}+\|u\|^{\frac{p}{q}}_{L^p(\Omega)}+\|v\|^{\frac{p}{q}}_{L^p(\Omega)}]\|u-v\|_{L^p(\Omega)}. \qquad (2.16)
\end{aligned}
$$

From (2.14) and (2.16), it follows that, for $u,v\in L^p(\Omega)$ with $\|u\|_{L^p(\Omega)}<r$ and $\|v\|_{L^p(\Omega)}<r$, we have

$$
\begin{aligned}
\|F(u)-F(v)\|_{L^p(\Omega)} &\leq N\beta c_0[\|J\|_p[|\Omega|^{\frac{1}{q}}+\|u\|^{\frac{p}{q}}_{L^p(\Omega)}+\|v\|^{\frac{p}{q}}_{L^p(\Omega)}]\|u-v\|_{L^p(\Omega)}]\\
&\leq N\beta c_0\|J\|_p[|\Omega|^{\frac{1}{q}}+2\|r\|^{\frac{p}{q}}_{L^p(\Omega)}]\|u-v\|_{L^p(\Omega)},
\end{aligned}
$$

showing that F is Lipschitz on bounded sets of $L^p(\Omega)$.

If $p=1$ the proof is more simpler. In fact, for $u,v\in L^1(\Omega)$, with $\|u\|_{L^1(\Omega)}\leq r$ and $\|v\|_{L^1(\Omega)}\leq r$, from (2.4), it follows that

$$|Kf(u)(x)|\leq\|J\|_\infty\|f(u)\|_{L^1}\leq\|J\|_\infty(c_1\|u\|_{L^1}+c_2|\Omega|),$$

and from (2.13), it follows that

$$|f(x)-f(y)|\leq c_0|x-y|,\ \text{for any } (x,y)\in\mathbb{R}^N\times\mathbb{R}^N.$$

Thus

$$|K(f(u)-f(v))(x)|\leq c_0\|J\|_\infty\|u-v\|_{L^1}.$$

Hence, if N denotes the Lipschitz constant of g in the interval $[-l,l]\subset\mathbb{R}$, where now $l=\beta\|J\|_\infty(c_1r+c_2|\Omega|)+\beta h$, we have

$$|F(u)(x)-F(v)(x)| \leq N\beta c_0\|J\|_\infty\|u-v\|_{L^1(\Omega)}.$$

Then

$$\|F(u)-F(v)\|_{L^1(\Omega)} \leq N\beta c_0\|J\|_\infty|\Omega|\|u-v\|_{L^1(\Omega)}.$$

Suppose, finally, that $\|u\|_{L^\infty(\Omega)} \leq r, \ \|v\|_{L^\infty(\Omega)} \leq r$. Then

$$\begin{aligned} |Kf(u)(x)| &\leq \|J\|_1 \|f(u)\|_\infty \\ &\leq \|J\|_1 [c_1 \|u\|_\infty + c_2] \\ &\leq \|J\|_1 [c_1 r + c_2]. \end{aligned}$$

Now, if M denotes the Lipschitz constant of f in the interval $[-r, r] \subset \mathbb{R}$, we have

$$|Kf(u)(x) - Kf(v)(x)| \leq \|J\|_1 \|f(u) - f(v)\|_\infty \leq \|J\|_1 M \|u - v\|_\infty.$$

Thus, if N denotes the Lipschitz constant of g in the interval $[-l, l] \subset \mathbb{R}$, where now $l = \beta \|J\|_1 (c_1 r + c_2) + \beta h$, it follows that

$$\|F(u) - F(v)\|_{L^\infty(\Omega)} \leq \beta N M \|J\|_1 \|u - v\|_\infty. \qquad \blacksquare$$

From Proposition 2.3, it follows from well known results, on ordinary differential equation in Banach space, that the problem (2.5) has a local solution for arbitrary initial condition in X. For the global existence, we need the following result ([22] - Theorem 5.6.1).

Theorem 2.4. *Let X be a Banach space, and suppose that $g : [t_0, \infty[\times X \to X$ is continuous and $\|g(t, u)\| \leq h(t, \|u\|)$; $\forall \ (t, u) \in [t_0, \infty[\times X$, where $h : [t_0, \infty[\times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $h(t, r)$ is non decreasing in $r \geq 0$, for each $t \in [t_0, \infty[$. Then, if the maximal solution $r(t, t_0, r_0)$ of the scalar initial value problem*

$$r' = h(t, r), \ r(t_0) = r_0,$$

exists throughout $[t_0, \infty[$, the maximal interval of existence of any solution $u(t, t_0, u_0)$ of the initial value problem

$$\frac{du}{dt} = g(t, u), \ t \geq t_0, \ u(t_0) = u_0,$$

with $\|u_0\| \leq r_0$, also contains $[t_0, \infty[$.

Corollary 2.5. *Suppose the same hypotheses from Proposition 2.3. Then the problem (2.5) has a unique globally defined solution for arbitrary initial condition in X, which is given, for $t \geq t_0$, by the "variation of constants formula"*

$$u(t, x) = \begin{cases} e^{-(t - t_0)} u_0(x) + \displaystyle\int_{t_0}^{t} e^{-(t-s)} g(\beta K f(u(s, \cdot))(x) + \beta h) ds, & x \in \Omega, \\ 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.17}$$

Proof. From Proposition 2.3, it follows that the right-hand-side of (2.5) is Lipschitz continuous in bounded sets of X and, therefore, the Cauchy problem (2.5) is well posed in X, with a unique local solution $u(t, x)$, given by (2.17) (see [7]).

If $1 \leq p < \infty$, from (2.12), we obtain that the right-hand-side of (2.5) satisfies

$$\begin{aligned} &\| -u + F(u)\|_{L^p(\Omega)} \leq \\ &\qquad (1 + \beta k_1 c_1 \|J\|_p |\Omega|^{\frac{1}{q}}) \|u\|_{L^p(\Omega)} + \beta k_1 \|J\|_p c_2 |\Omega| + (k_1 \beta h + k_2) |\Omega|^{\frac{1}{p}}. \end{aligned}$$

If $p = \infty$, we have that the right-hand-side of (2.5) satisfies

$$\| -u + F(u)\|_\infty \leq (1 + k_1\beta c_1)\|u\|_\infty + k_1(\beta c_2 + \beta h) + k_2.$$

Hence, defining $h : [t_0, \infty[\times \mathbb{R}^+ \to \mathbb{R}^+$, by

$$h(t,r) = (1 + \beta k_1 c_1 \|J\|_p |\Omega|^{\frac{1}{q}})r + \beta k_1 \|J\|_p c_2 |\Omega| + (k_1 \beta h + k_2)|\Omega|^{\frac{1}{p}},$$

if $1 \leq p < \infty$ or by

$$h(t,r) = (1 + k_1\beta c_1)r + k_1(\beta c_2 + \beta h) + k_2,$$

in the case $p = \infty$, it follows that (2.5) satisfies the hypotheses from Theorem 2.4 and the global existence follows immediately. The variation of constants formula can be easily verified by direct derivation. ∎

3 Smoothness of the solutions

In this section, in addition the hypotheses from previous section, we assume that the functions $g, f \in \mathscr{C}^1(\mathbb{R})$, and g' and f' are locally Lipschitz and there exist non negative constants $k_3,\ k_4,\ c_3$ and c_4, such that

$$|g'(x)| \leq k_3|x| + k_4,\ \forall,\ x \in \mathbb{R}, \tag{3.18}$$

$$|f'(x)| \leq c_3|x| + c_4,\ \forall,\ x \in \mathbb{R}. \tag{3.19}$$

The following result has been proven in [26].

Proposition 3.1. *Let X and Y be normed linear spaces, $F : X \to Y$ a map and suppose that the Gateaux's derivative of F, $DF : X \to \mathcal{L}(X, Y)$ exists and is continuous at $x \in X$. Then the Fréchet's derivative F' of F exists and it is continuous at x.*

Using Proposition 3.1, we have the following result:

Proposition 3.2. *Suppose, in addition to the hypotheses of Corollary 2.5 that the functions g and f have derivatives satisfying (3.18) and (3.19), respectively. Then F is continuously Fréchet differentiable on X with derivative given by*

$$DF(u)v(x) := \begin{cases} -v(x) + g'(\beta K f(u)(x) + \beta h)\beta K f'(u(x))v(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Proof. From a simple computation, using the fact that f is continuously differentiable on $\mathbb{R}$, it follows that the Gateaux's derivative of F is given by

$$DF(u)v(x) := \begin{cases} -v(x) + g'(\beta K f(u)(x) + \beta h)\beta K f'(u(x))v(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

The operator $DF(u)$ is clearly a linear operator in X.

Suppose $1 \leq p < \infty$ and q the conjugate exponent of p. Then, if $u \in L^p(\Omega)$, using (3.18) and (2.7), it follows that

$$\begin{aligned}
&\|g'(\beta K f(u) + \beta h)\beta K f'(u)v\|_{L^p(\Omega)} \\
&\leq \left\{ \int_\Omega |g'(\beta K(f(u)(x)) + \beta h)\beta K(f'(u(x)))v(x)|^p dx \right\}^{\frac{1}{p}} \\
&\leq \left\{ \int_\Omega \left[k_3\beta |K(f(u)(x))| + k_3\beta h + k_4\right]^p \beta^p |K(f'(u(x)))v(x)|^p dx \right\}^{\frac{1}{p}} \\
&\leq \left\{ \int_\Omega [k_3\beta \|J\|_q \|f(u)\|_{L^p(\Omega)} + k_3\beta h + k_4]^p \beta^p [\|J\|_q \|f'(u)\|_{L^p(\Omega)} |v(x)|^p dx \right\}^{\frac{1}{p}}.
\end{aligned}$$

Thus, from (2.4) and (3.19), we have

$$\begin{aligned}
&\|g'(\beta K f(u) + \beta h)\beta K f'(u)v\|_{L^p(\Omega)} \leq \\
&\leq \left\{ \int_\Omega [k_3\beta \|J\|_q (c_1 \|u\|_{L^p(\Omega)} + c_2 |\Omega|^{\frac{1}{p}}) \right. \\
&\qquad \left. + k_3\beta h + k_4]^p \beta^p [\|J\|_q (c_3 \|u\|_{L^p(\Omega)} + c_4 |\Omega|^{\frac{1}{p}}) |v(x)|^p dx \right\}^{\frac{1}{p}} \\
&= (k_3\beta \|J\|_q (c_1 \|u\|_{L^p(\Omega)} + c_2 |\Omega|^{\frac{1}{p}}) \\
&\qquad + k_3\beta h + k_4)\beta \|J\|_q (c_3 \|u\|_{L^p(\Omega)} + c_4 |\Omega|^{\frac{1}{p}}) \|v\|_{L^p(\Omega)}. \qquad (3.20)
\end{aligned}$$

From (3.20), we have

$$\begin{aligned}
\|DF(u)v\|_{L^p(\Omega)} = &\left(k_3\beta \|J\|_q \left(c_1 \|u\|_{L^p(\Omega)} + c_2 |\Omega|^{\frac{1}{p}}\right)\right. \\
&\left. + k_3\beta h + k_4\right) \beta \|J\|_q \left(c_3 \|u\|_{L^p(\Omega)} + c_4 |\Omega|^{\frac{1}{p}}\right) \|v\|_{L^p(\Omega)},
\end{aligned}$$

showing that $DF(u)$ is a bounded operator. In the case $p = \infty$, we have that

$$\begin{aligned}
\|DF(u)v\|_{L^\infty(\Omega)} &= \|g'(\beta K f(u) + \beta h)\beta K f'(u)v\|_\infty \\
&\leq (k_3\beta \|K f(u)\|_\infty + k_3\beta h + k_4)\beta \|K \circ (f'(u))\|_\infty \|v\|_\infty \\
&\leq (k_3\beta \|J\|_1 (c_1 \|u\|_{L^\infty(\Omega)} + c_2) \\
&\qquad + k_3\beta h + k_4)\beta \|J\|_1 (c_3 \|u\|_{L^\infty(\Omega)} + c_4) \|v\|_\infty \\
&\leq (k_3\beta (c_1 \|u\|_{L^\infty(\Omega)} + c_2) + k_3\beta h + k_4)\beta (c_3 \|u\|_{L^\infty(\Omega)} + c_4) \|v\|_\infty
\end{aligned}$$

showing the boundedness of $DF(u)$ also in this case.

Suppose now that u_1, u_2 and v belong to $L^p(\Omega)$, $1 \leq p < \infty$. Then

$$\begin{aligned}
&\|(DF(u_1) - DF(u_2))v\|_{L^p(\Omega)} = \\
&= \|g'(\beta K f(u_1) + \beta h)\beta K f'(u_1)v - g'(\beta K f(u_2) + \beta h)\beta K f'(u_2)v\|_{L^p(\Omega)} \\
&\leq I + II,
\end{aligned}$$

where

$$I = \|[g'(\beta K f(u_1) + \beta h) - g'(\beta K f(u_2) + \beta h)]\beta K f'(u_1)v\|_{L^p(\Omega)}$$

and

$$II = \|g'(\beta K f(u_2) + \beta h)\beta K([f'(u_1) - f'(u_2)])v\|_{L^p(\Omega)}.$$

Fixed $u_1 \in L^p(\Omega)$ and letting $u_2 \to u_1$ in $L^p(\Omega)$ it follows that $\beta K f(u_2) + \beta h$ is in a ball of L^∞ centered at $\beta K f(u_1) + \beta h$. Then, since g' is locally Lipschitz, there exists $C > 0$, such that

$$\begin{aligned} |g'(\beta K f(u_1) + \beta h)(x) - g'(\beta K f(u_2) + \beta h)(x)| &\leq C\beta |K[f(u_1) - f(u_2)](x)| \\ &\leq C\beta \|J\|_q \|u_1 - u_2\|_{L^p(\Omega)}. \end{aligned}$$

Thus, using (2.7), we have that

$$\begin{aligned} I &\leq \left(\int_\Omega |(C\beta \|J\|_q \|u_1 - u_2\|_{L^p(\Omega)})^p \beta^p |K f'(u_1)(x)|^p |v(x)|^p \right)^{\frac{1}{p}} \\ &\leq C\beta \|J\|_q \|u_1 - u_2\|_{L^p(\Omega)} \beta \left(\int_\Omega |K f'(u_1)(x)|^p |v(x)|^p \right)^{\frac{1}{p}} \\ &\leq C\beta^2 \|J\|_q \|u_1 - u_2\|_{L^p(\Omega)} \left(\int_\Omega [\|J\|_q \|f'(u_1)\|_{L^p(\Omega)}]^p |v(x)|^p \right)^{\frac{1}{p}}. \end{aligned}$$

But, from (3.19) it follows that

$$\|f'(u_1)\|_{L^p(\Omega)} \leq c_3 \|u_1\|_{L^p(\Omega)} + c_4 |\Omega|^{\frac{1}{p}}.$$

Hence,

$$I \leq C\beta^2 \|J\|_q \|u_1 - u_2\|_{L^p(\Omega)} \|J\|_q (c_3 \|u_1\|_{L^p(\Omega)} + c_4 |\Omega|^{\frac{1}{p}}) \|v\|_{L^p(\Omega)}. \tag{3.21}$$

Now, using (3.18) and (2.7), we obtain

$$\begin{aligned} |g'(\beta K f(u_2)(x)) + \beta h)| &\leq k_3 \beta |K f(u_2(x))| + k_3 \beta h + k_4 \\ &\leq k_3 \beta \|J\|_q \|f(u_2)\|_{L^p(\Omega)} + k_3 \beta h + k_4 \\ &\leq k_3 \beta \|J\|_q \left(c_1 \|u_2\|_{L^p(\Omega)} + c_2 |\Omega|^{\frac{1}{p}} \right) + k_3 \beta h + k_4. \end{aligned}$$

Whence we obtain

$$II \leq [k_3 \beta \|J\|_q (c_1 \|u_2\|_{L^p(\Omega)} + c_2 |\Omega|^{\frac{1}{p}}) + k_3 \beta h + k_4] \beta \|K[f'(u_1) - f'(u_2)]\|_{L^p(\Omega)}.$$

Using (2.9) and Hölder inequality, we have

$$\begin{aligned} II &\leq \left[k_3 \beta \|J\|_q \left(c_1 \|u_2\|_{L^p(\Omega)} + c_2 |\Omega|^{\frac{1}{p}} \right) + k_3 \beta h + k_4 \right] \\ &\qquad \beta \|J\|_p \|[f'(u_1) - f'(u_2)]v\|_{L^1(\Omega)} \\ &\leq \left[k_3 \beta \|J\|_q \left(c_1 \|u_2\|_{L^p(\Omega)} + c_2 |\Omega|^{\frac{1}{p}} \right) + k_3 \beta h + k_4 \right] \\ &\qquad \beta \|J\|_p \|[f'(u_1) - f'(u_2)]v\|_{L^q(\Omega)} \|v\|_{L^p(\Omega)}. \end{aligned} \tag{3.22}$$

From (3.21) and (3.22), it follows that

$$
\begin{aligned}
&\|[DF(u_1) - DF(u_2)]v\|_{L^p(\Omega)} \leq \\
&\leq c\beta^2 \|J\|_q \|u_1 - u_2\|_{L^p(\Omega)} \|J\|_q \left(c_3 \|u_1\|_{L^p(\Omega)} + c_4 |\Omega|^{\frac{1}{p}} \right) \|v\|_{L^p(\Omega)} \\
&+ \left[k_3 \beta \|J\|_q (c_1 \|u_2\|_{L^p(\Omega)} + c_2 |\Omega|^{\frac{1}{p}}) + k_3 \beta h + k_4 \right] \\
&\qquad\qquad\qquad \beta \|J\|_p \|f'(u_1) - f'(u_2) v\|_{L^q(\Omega)} \|v\|_{L^p(\Omega)}.
\end{aligned}
$$

Thus, to prove continuity of the derivative, it is enough to show that

$$\|f'(u_1) - f'(u_2)\|_{L^q(\Omega)} \to 0$$

when

$$\|u_1 - u_2\|_{L^p(\Omega)} \to 0.$$

But, from the growth condition on f' it follows that

$$|f'(u_1)(x) - f'(u_2)(x)|^q \leq [c_3(|u_1(x)| + |u_2(x)|) + 2c_4]^q$$

and a simple computation show that the right-hand is in $L^1(\Omega)$. Then the result follows from Lebesgue's Convergence Theorem.

In the case $p = \infty$, from (2.8), we obtain

$$
\begin{aligned}
&\|[DF(u_1) - DF(u_2)]v\|_{L^\infty(\Omega)} \leq \\
&\leq c\beta \|K[f'(u_1) - f'(u_2)]\|_{L^\infty(\Omega)} \beta \|Kf'(u_1)v\|_\infty \\
&+(k_3 \beta \|Kf(u_2)\|_\infty + k_3 \beta h + k_4) \beta \|K[f'(u_1) - f'(u_2)]\|_{L^\infty(\Omega)} \|v\|_{L^\infty(\Omega)} \\
&\leq c\beta \|J\|_1 \|f'(u_1) - f'(u_2)\|_{L^\infty(\Omega)} \beta \|J\|_1 \|f'(u_1)\|_\infty \|v\|_\infty \\
&+(k_3 \beta \|J\|_1 \|f(u_2)\|_\infty + k_3 \beta h + k_4) \beta \|J\|_1 \|f'(u_1) - f'(u_2)\|_{L^\infty(\Omega)} \|v\|_{L^\infty(\Omega)} \\
&\leq c\beta \|f'(u_1) - f'(u_2)\|_{L^\infty(\Omega)} \beta (c_3 \|u\|_{L^\infty(\Omega)} + c_4) \|v\|_\infty \\
&+(k_3 \beta (c_1 \|u\|_{L^\infty(\Omega)} + c_2) + k_3 \beta h + k_4) \beta \|f'(u_1) - f'(u_2)\|_{L^\infty(\Omega)} \|v\|_{L^\infty(\Omega)}.
\end{aligned}
$$

And the continuity of DF follows from the continuity of f'. Therefore, it follows from Proposition 3.1 that F is Fréchet differentiable with continuous derivative in $L^p(\Omega)$. ■

Remark 3.3. *From Proposition 3.2, it follows that the flow generated by (2.5), given by $(T(t)u_0)(x) = u(x,t)$, where $u(x,t)$ is given in (2.17), is $\mathscr{C}^1$ with respect to initial condition (see [19]).*

4 Existence of a global attractor

We prove, in this section, the existence of a global maximal invariant compact set $\mathcal{A} \subset X \equiv L^p(\Omega)$ for the flow of (2.5), which attracts each bounded set of X (the global attractor, see [18] and [29]).

We recall that a set $\mathcal{B} \subset X$ is an absorbing set for the flow $T(t)$ if, for any bounded set $C \subset X$, there is a $t_1 > 0$ such that $T(t)C \subset \mathcal{B}$ for any $t \geq t_1$.

The following result was proven in [29].

Theorem 4.1. *Let X be a Banach space and $T(t)$ a semigroup on X. Assume that, for every t, $T(t) = T_1(t) + T_2(t)$, where the operators $T_1(\cdot)$ are uniformly compact for t sufficiently large, that is, for every bounded set B there exists t_0, which may depend on B, such that*

$$\bigcup_{t \geq t_0} T_1(t)B$$

is relatively compact in X and $T_2(t)$ is a continuous mapping from X into itself such that the following holds: For every bounded set $C \subset X$,

$$r_c(t) = \sup_{\varphi \in C} \|T_2(t)\varphi\|_X \to 0 \quad as \quad t \to \infty.$$

Assume also that there exists an open set $\mathcal{U}$ and bounded subset $\mathcal{B}$ of $\mathcal{U}$ such that $\mathcal{B}$ is absorbing in $\mathcal{U}$. Then the ω-limit set of $\mathcal{B}$, $\mathcal{A} = \omega(\mathcal{B})$, is a compact attractor which attracts the bounded sets of $\mathcal{U}$. It is the maximal bounded attractor in $\mathcal{U}$ (for the inclusion relation). Furthermore, if $\mathcal{U}$ is convex and connected, then $\mathcal{A}$ is connected.

Lemma 4.2. *Assume that (2.3) and (2.4) hold with $k_1\beta c_1 < 1$. Then, for any positive number σ, the ball of radius*

$$R = (1+\sigma)\left(\frac{k_1\beta c_2 + k_1\beta h + k_2}{1 - k_1\beta c_1}\right)$$

is an absorbing set for the flow $T(t)$ generated by (2.5).

Proof. If $u(\cdot,t)$ is a solution of (2.5) with initial condition $u(\cdot,0)$ then, for $1 \leq p < \infty$,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |u(x,t)|^p dx &= \int_\Omega p|u(x,t)|^{p-1}\mathrm{sgn}[u(x,t)]u_t(x,t)dx \\
&= -p\int_\Omega |u(x,t)|^p dx \\
&\quad + p\int_\Omega |u(x,t)|^{p-1}\mathrm{sgn}[u(x,t)]g(\beta Kf(u(x,t)) + \beta h)dx.
\end{aligned}$$

But, using Hölder inequality, (2.3) and (2.4), it follows that

$$
\begin{aligned}
&\int_\Omega |u(x,t)|^{p-1}\mathrm{sgn}[u(x,t)]g(\beta K f(u(x,t)) + \beta h)dx \leq \\
&\leq \left(\int_\Omega (|u(x,t)|^{p-1})^q dx\right)^{\frac{1}{q}} \left(\int_\Omega |g(\beta K f(u(x,t)) + \beta h)|^p dx\right)^{\frac{1}{p}} \\
&\leq \left(\int_\Omega |u(x,t)|^p dx\right)^{\frac{1}{q}} \left(\int_\Omega (k_1|\beta K f(u(x,t)) + \beta h| + k_2)^p dx\right)^{\frac{1}{p}} \\
&\leq \|u(\cdot,t)\|^{p-1}_{L^p(\Omega)} \left(k_1\beta\|K(f(u(\cdot,t)))\|_{L^p(\Omega)} + \|k_1\beta h + k_2\|_{L^p(\Omega)}\right) \\
&\leq \|u(\cdot,t)\|^{p-1}_{L^p(\Omega)} \left(k_1\beta\|J\|_1\|f(u(\cdot,t))\|_{L^p(\Omega)} + (k_1\beta h + k_2)|\Omega|^{\frac{1}{p}}\right) \\
&\leq \|u(\cdot,t)\|^{p-1}_{L^p(\Omega)} \left(k_1\beta\left(c_1\|u(\cdot,t)\|_{L^p(\Omega)} + c_2|\Omega|^{\frac{1}{p}}\right) + (k_1\beta h + k_2)|\Omega|^{\frac{1}{p}}\right) \\
&= k_1\beta c_1\|u(\cdot,t)\|^p_{L^p(\Omega)} + \left(k_1\beta c_2|\Omega|^{\frac{1}{p}} + (k_1\beta h + k_2)|\Omega|^{\frac{1}{p}}\right)\|u(\cdot,t)\|^{p-1}_{L^p(\Omega)}.
\end{aligned}
$$

Thus, we have that

$$
\begin{aligned}
\frac{d}{dt}\|u(\cdot,t)\|^p_{L^p(\Omega)} &\leq -p\|u(\cdot,t)\|^p_{L^p(\Omega)} + pk_1\beta c_1\|u(\cdot,t)\|^p_{L^p(\Omega)} \\
&\quad + p\left[k_1\beta c_2|\Omega|^{\frac{1}{p}} + (k_1\beta h + k_2)|\Omega|^{\frac{1}{p}}\right]\|u(\cdot,t)\|^{p-1}_{L^p(\Omega)} \\
&= p\|u(\cdot,t)\|^p_{L^p(\Omega)}\left[-1 + k_1\beta c_1 + \frac{[k_1\beta c_2 + k_1\beta h + k_2]\,|\Omega|^{\frac{1}{p}}}{\|u(\cdot,t)\|_{L^p(\Omega)}}\right].
\end{aligned}
$$

Letting $\varepsilon = 1 - k_1\beta c_1$, when

$$\|u(\cdot,t)\|_{L^p(\Omega)} \geq (1+\sigma)\frac{(k_1\beta c_2 + k_1\beta h + k_2)\,|\Omega|^{\frac{1}{p}}}{\varepsilon},$$

we have that

$$\frac{d}{dt}\|u(\cdot,t)\|^p_{L^p(\Omega)} \leq p\|u(\cdot,t)\|^p_{L^p(\Omega)}\left(-\varepsilon + \frac{\varepsilon}{1+\sigma}\right) = -p\frac{\sigma}{1+\sigma}\varepsilon\|u(\cdot,t)\|^p_{L^p(\Omega)}.$$

Therefore when $\|u(\cdot,t)\|_{L^p(\Omega)} \geq (1+\sigma)\frac{(k_1\beta c_2 + k_1\beta h + k_2)|\Omega|^{\frac{1}{p}}}{\varepsilon}$,

$$\|u(\cdot,t)\|^p_{L^p(\Omega)} \leq e^{-\frac{\varepsilon\sigma p}{1+\sigma}t}\|u(\cdot,0)\|_{L^p(\Omega)} \leq e^{-\frac{\sigma p(1-k_1\beta c_1)}{1+\sigma}t}\|u(\cdot,0)\|_{L^p(\Omega)},$$

what concludes the proof. ■

The next result is an extension for Theorem 3.3 of [25], Theorem 3.3 of [3] and Theorem 8 of [10].

Theorem 4.3. *In addition of the hypotheses assumed in Lemma 4.2, suppose that (3.18) holds and lets* $\|J_x\|_r = \sup_{x\in\Omega} \frac{\partial}{\partial x}\|J(x,\cdot)\|_{L^r(\Omega)}$. *Then there exists a global attractor* $\mathcal{A}$ *for the flow* $T(t)$ *generated by (2.5) in* $L^p(\Omega)$, *which is contained in the ball of radius R.*

Proof. If $u(\cdot,t)$ is the solution of (2.5) with initial condition $u(\cdot,0)$. For $x \in \Omega$ we have, by the variation of constants formula,

$$u(x,t) = e^{-t}u(x,0) + \int_0^t e^{s-t} g(\beta K f(u)(x,s) + \beta h) ds. \tag{4.23}$$

Consider

$$T_1(t)u(x) = e^{-t}u(x,0)$$

and

$$T_2(t)u(x) = \int_0^t e^{s-t} g(\beta K f(u)(x,s) + \beta h) ds.$$

Then, assuming that $u(\cdot,0) \in \mathcal{C}$, where $\mathcal{C}$ is a bounded set in $L^p(\Omega)$, (for example $B(0,\rho)$), it follows that

$$\|T_1(t)u\|_{L^2} \underset{t\to\infty}{\longrightarrow} 0 \text{ uniformly in } u.$$

Also, using (4.23), we have that $\|u(\cdot,t)\|_{L^p(\Omega)} \leq L$, for $t \geq 0$, where

$$L = \max\left\{\rho, \frac{2\,(k_1\beta c_2 + k_1\beta h + k_2)\,|\Omega|^{\frac{1}{p}}}{1 - k_1\beta c_1}\right\}.$$

Therefore, for $t \geq 0$, we have that

$$\begin{aligned}
\frac{\partial T_2(t)u(x)}{\partial x} &= \int_0^t e^{s-t} \frac{\partial}{\partial x} g(\beta K f(u)(x,s) + \beta h) ds \\
&= \beta \int_0^t e^{s-t} g'(\beta K f(u)(x,s) + \beta h) \frac{\partial K f(u)}{\partial x}(x,s) ds.
\end{aligned}$$

Thus, using (3.18) and (2.9), we obtain

$$\begin{aligned}
\left\|\frac{\partial T_2(t)u}{\partial x}\right\|_{L^p(\Omega)} &\leq \int_0^t e^{s-t}\|g'(\beta Kf(u)(\cdot,s)+\beta h)\beta\frac{\partial Kf(u)}{\partial x}(\cdot,s)\|_{L^p(\Omega)}ds \\
&\leq \int_0^t e^{s-t}[k_3\beta\|J\|_1\|f(u(\cdot,s))\|_{L^p(\Omega)} \\
&+ k_3\beta h+k_4]\beta\|J_x\|_1\|f(u(\cdot,s))\|_{L^p(\Omega)}ds \\
&\leq \int_0^t e^{s-t}[k_3\beta(c_1\|u(\cdot,s)\|_{L^p(\Omega)}+c_2|\Omega|^{\frac{1}{p}}) \\
&+ k_3\beta h+k_4]\beta\|J_x\|_1(c_1\|u(\cdot,s)\|_{L^p(\Omega)}+c_2|\Omega|^{\frac{1}{p}})ds \\
&\leq [k_3\beta(c_1\|u(\cdot,s)\|_{L^p(\Omega)}+c_2|\Omega|^{\frac{1}{p}}) \\
&+ k_3\beta h+k_4]\beta\|J_x\|_1(c_1\|u(\cdot,s)\|_{L^p(\Omega)}+c_2|\Omega|^{\frac{1}{p}}) \\
&\leq [k_3\beta(c_1L+c_2|\Omega|^{\frac{1}{p}})+k_3\beta h+k_4]\beta\|J_x\|_1(c_1L+c_2|\Omega|^{\frac{1}{p}}).
\end{aligned}$$

It follows that, for $t > 0$ and for any $u \in \mathcal{C}$, the value of $\left\|\frac{\partial T_2(t)u}{\partial x}\right\|_{L^p(\Omega)}$ is bounded by a constant (independent of t and u). Thus, for all $u \in \mathcal{C}$, we have that $T_2(t)u$ belongs to a ball of $W^{1,2}(\Omega)$. From Sobolev's Imbedding Theorem, it follows that

$$\bigcup_{t\geq 0} T_2(t)\mathcal{C}$$

is relatively compact. Therefore, the result follows from Theorem 4.1, with the attractor $\mathcal{A}$ being the set ω-limit of the ball $B(0,R)$. ∎

5 Comparison and boundedness results

In this section we prove a comparison result that extends the Theorem 2.7 of [24] (where $g \equiv \tanh$, $f(x) = x, \forall\, x \in \mathbb{R}$ and $h = 0$) and it extends Theorem 4.2 of [25] (where $f(x) = x$, $\forall\, x \in \mathbb{R}$).

Definition 5.1. *A function $v(x,t)$ is a subsolution of the Cauchy problem for (2.5) with initial condition $u(\cdot,0)$ if $v(x,0) \leq u(x,0)$ for almost all $x \in \Omega$, v is continuously differentiable with respect to t and satisfies*

$$\frac{\partial v(x,t)}{\partial t} \leq -v(x,t) + g(\beta Kf(v)(x,t)+\beta h), \tag{5.24}$$

almost everywhere (a.e.).

Analogously, a function $V(x,t)$ is a super solution if it has the same regularity properties as above, satisfies (5.24) with reversed inequality and $V(x,0) \geq u(x,0)$ for almost all $x \in \Omega$.

Theorem 5.2. *In addition to the hypotheses of Theorem 4.3, assume that the functions g and f are monotonic and Lipschitz continuous on bounded with Lipschitz's constants N and M, respectively. Let $v(w,t)$, $[V(w,t)]$ be a subsolution [super solution] of the Cauchy problem of (2.5) with initial condition $u(\cdot,0)$. Then*

$$v(x,t) \leq u(x,t) \leq V(x,t), \text{ a.e..}$$

Proof. Define the operator G on $L^\infty(\Omega \times [0,T])$ by

$$G(w)(x,t) = e^{-t}w(x,0) + \int_0^t e^{-(t-s)}g(\beta(Kf(w)(x,s)+h))ds.$$

Then $(G(w))(x,0) = w(x,0)$. Also, since f and g are monotonic, it follows that G is monotonic, that is, for any $w_1, w_2 \in L^\infty(\Omega \times [0,T])$ with $w_1 \geq w_2$ (a.e. in $\Omega \times [0,T]$), we have $G(w_1) \geq G(w_2)$ (a.e. in $\Omega \times [0,T]$).

From (2.7), we obtain

$$\begin{aligned}
|G(w)(x,t)| &\leq e^{-t}|w(x,0)| + \int_0^t e^{-(t-s)}|g(\beta Kf(w)(x,s)+\beta h)|ds \\
&\leq e^{-t}|w(x,0)| + \int_0^t e^{-(t-s)}[k_1|\beta Kf(w)(x,s)+\beta h| + k_2]ds \\
&\leq e^{-t}|w(x,0)| + \int_0^t e^{-(t-s)}k_1\beta|Kf(w)(x,s)|ds \\
&\qquad + \int_0^t e^{-(t-s)}(k_1\beta h + k_2)ds.
\end{aligned}$$

Since $|Kf(w)(x,s)| \leq \|J\|_1\|f(w)\|_\infty \leq k_1\|w\|_\infty + k_2$ a.e. in $\Omega \times [0,T]$, we obtain

$$\begin{aligned}
\|G(w)\|_\infty &\leq e^{-t}\|w(\cdot,0)\|_\infty + \int_0^t e^{-(t-s)}k_1\beta(k_1\|w\|_\infty + k_2)ds \\
&\qquad + \int_0^t e^{-(t-s)}(k_1\beta h + k_2)ds \\
&\leq \|w\|_\infty + k_1\beta(k_1\|w\|_\infty + k_2) + (k_1\beta h + k_2).
\end{aligned}$$

Therefore $G : L^\infty(\Omega \times [0,T]) \to L^\infty(\Omega \times [0,T])$.

Furthermore, if $\beta NMT < 1$, G is a contraction in any subset of functions of $L^\infty(\Omega \times [0,T])$ with the same values at $t = 0$. In fact

$$
\begin{aligned}
&|G(w_1)(x,t) - G(w_2)(x,t)| \\
&\quad = \left| \int_0^t e^{-(t-s)}[g(\beta(Kf(w_1)(x,s) + \beta h) - g(\beta(Kf(w_2)(x,s) + \beta h)]ds \right| \\
&\quad \leq \int_0^t e^{-(t-s)} N\beta |Kf(w_1)(x,s) - Kf(w_2)(x,s)| ds \\
&\quad \leq \int_0^t e^{-(t-s)} N\beta (K|f(w_1) - Kf(w_2)|(x,s)) ds \\
&\quad \leq \int_0^t e^{-(t-s)} N\beta K \|f(w_1) - f(w_2)\|_\infty ds \\
&\quad = N\beta T \|f(w_1) - f(w_2)\|_\infty \int_0^t e^{-(t-s)} ds \\
&\quad \leq N\beta MT \|w_1 - w_2\|_\infty,
\end{aligned}
$$

a.e. in $\Omega \times [0,T]$. Hence $\|G(w_1) - G(w_2)\|_\infty \leq \beta NMT \|w_1 - w_2\|_\infty$. Therefore, if $\beta NMT < 1$, G is a contraction. Thus, if $u(x,t)$ is a solution of (2.5) with $u^0 = u(x,0)$, we have

$$u = \lim_{n\to\infty} G^n(u^0)$$

on $L^\infty(\Omega \times [0,T])$. The same holds for a solution $\widetilde{u}$ with $\widetilde{u}^0 = \widetilde{u}(x,0)$. If $\widetilde{u}^0 \leq u^0$ a.e., with g and f monotonic, it follows that

$$G^n(\widetilde{u}^0) \leq G^n(u^0), \text{ a.e.}$$

Now, if v is a subsolution of (2.5), it's easy to see that

$$v(x,t) \leq e^{-t} v(x,0) + \int_0^t e^{-(t-s)} g(\beta(Kf(v)(x,s) + h)) ds, \text{ a.e.}$$

Therefore $v(x,t) \leq G(v)(x,t)$, a.e., and since g and f are monotonic, it follows that $v(w,t) \leq G^n(v)(x,t)$ a.e. Thus, $v(x,t) \leq z(x,t)$, a.e., where

$$z = \lim_{n\to\infty} G^{n+1}(v).$$

Now, from the continuity of G, it follows that

$$G(z) = G\left(\lim_{n\to\infty} G^n(v)\right) = \lim_{n\to\infty} G^{n+1}(v) = z.$$

Therefore z is a fixed point of G, that is, z is a solution of (2.5) in $\Omega \times [0,T]$ with initial condition $z(\cdot,0) = v(\cdot,0)$. Thus, if $z(\cdot,0) \leq u(\cdot,0)$, a.e., then

$$v \leq z \leq u, \text{ a.e. in } \Omega \times [0,T],$$

where u is the solution of (2.5) with initial condition $u(\cdot,0)$. If $V(x,t)$ is a super solution, we obtain, by the same arguments

$$u \leq \tilde{z} \leq V, \text{ a.e. in } \Omega \times [0,T].$$

Therefore

$$v(x,t) \leq u(x,t) \leq V(x,t), \text{ a.e.}$$

in $\Omega \times [0,T]$.

Since the estimates above do not depend on the initial condition, we may extend the result to $[T,2T]$ and, by iteration, we can complete the proof of the theorem. ∎

Remark 5.3. *If we add the hypothesis $g(x) < \rho$, the comparison result holds in the ball $\mathbb{B} = \{L^\infty(\Omega \times [0,T]), \|\cdot\|_\infty \leq \rho\}$.*

In fact, it is enough to prove that $G|_{\mathbb{B}} : \mathbb{B} \to \mathbb{B}$. But

$$|(G|_{\mathbb{B}}(w))(x,t)| \leq e^{-t}|w(x,0)| + \rho \int_0^t e^{-(t-s)}ds.$$

Hence

$$\|(G|_{\mathbb{B}}(w))\|_\infty \leq e^{-t}\|w\|_\infty + \rho \int_0^t e^{-(t-s)}ds \leq \rho e^{-t} + \rho \int_0^t e^{-(t-s)}ds = \rho.$$

Therefore, $G|_{\mathbb{B}}(w) \in \mathbb{B}$.

Theorem 5.4. *In the same conditions from Theorem 4.3, we have that the attractor $\mathcal{A}$ belongs to the ball $\|\cdot\|_\infty \leq \rho$ in $L^\infty(\Omega)$, where $\rho = k_1\beta\|J\|_q c_1 R + k_1\beta\|J\|_q c_2|\Omega|^{\frac{1}{p}} + k_1\beta h + k_2$.*

Proof. From Theorem 4.3 the attractor is contained in the ball $B[0,\rho]$ in $L^p(\Omega)$.

Let $u(x,t)$ be a solution of (2.5) in $\mathcal{A}$. Then, for $x \in \Omega$, by the variation of constants formula

$$u(x,t) = e^{-(t-t_0)}u(x,t_0) + \int_{t_0}^t e^{-(t-s)}g(\beta Kf(u)(x,s) + \beta h)ds.$$

Since $\|u(\cdot,t)\|_{L^p(\Omega)} \leq R$ for all $u \in \mathcal{A}$, we obtain for all $(x,t) \in \Omega \times \mathbb{R}^+$ letting $t_0 \to -\infty$

$$u(x,t) = \int_{-\infty}^t e^{-(t-s)}g(\beta Kf(u)(x,s) + \beta h)ds,$$

where the equality above is in the sense of $L^p(\Omega)$. Thus, using (2.3), we have

$$\begin{aligned} |u(x,t)| &\leq \int_{-\infty}^{t} e^{-(t-s)}|g(\beta Kf(u)(x,s)+\beta h)|ds \\ &\leq \int_{-\infty}^{t} e^{-(t-s)}[k_1\beta|Kf(u(x,t))+\beta h|+k_2]ds \\ &\leq \int_{-\infty}^{t} e^{-(t-s)}[k_1\beta\|J\|_q\|f(u(\cdot,t))\|_{L^p(\Omega)}+k_1\beta h+k_2]ds \\ &\leq \int_{-\infty}^{t} e^{-(t-s)}[k_1\beta\|J\|_q(c_1\|u(\cdot,t)\|_{L^p(\Omega)}+c_2|\Omega|^{\frac{1}{p}})+k_1\beta h+k_2]ds \\ &\leq \int_{-\infty}^{t} e^{-(t-s)}[k_1\beta\|J\|_q(c_1R+c_2|\Omega|^{\frac{1}{p}})+k_1\beta h+k_2]ds \\ &\leq \int_{-\infty}^{t} \rho e^{-(t-s)}ds. \end{aligned}$$

Therefore $\|u(\cdot,t)\|_\infty \leq \rho$, as claimed ∎

6 Existence of a Lyapunov's functional

In this section we exhibit a continuous "Lyapunov's functional" for the flow of (2.5), restricted to the ball of radius ρ in $L^\infty(\Omega)$, concluding that this flow is gradient, in the sense of [18].

Initially, we claim that $\{L^p(\Omega), \|\cdot\|_\infty \leq \rho\}$ is an invariant set for the flow generated by (2.5).

In fact, let

$$u(x,t) = e^{-t}u(x,0) + \int_0^t e^{-(t-s)}g(\beta Kf(u(x,s))+\beta h)ds$$

be the solution of (2.5) with initial condition $u(\cdot,0) \in \{L^p(\Omega), \|\cdot\|_\infty \leq \rho\}$. Then

$$\begin{aligned}
|u(x,t)| &\leq e^{-t}|u(x,0)| + \int_0^t e^{-(t-s)}|g(\beta K f(u(x,s)) + \beta h)|ds \\
&\leq e^{-t}|u(x,0)| + \int_0^t e^{-(t-s)}[k_1\beta|Kf(u(x,t)) + \beta h| + k_2]ds \\
&\leq e^{-t}|u(x,0)| + \int_0^t e^{-(t-s)}[k_1\beta\|J\|_q\|f(u(\cdot,t))\|_{L^p(\Omega)} + k_1\beta h + k_2]ds \\
&\leq e^{-t}|u(x,0)| + \int_0^t e^{-(t-s)}[k_1\beta\|J\|_q(c_1\|u(\cdot,t)\|_{L^p(\Omega)} + c_2|\Omega|^{\frac{1}{p}}) \\
&\qquad\qquad + k_1\beta h + k_2]ds \\
&\leq e^{-t}|u(x,0)| + \int_0^t e^{-(t-s)}\rho ds.
\end{aligned}$$

Whence,

$$\begin{aligned}
\|u(\cdot,t)\|_\infty &\leq e^{-t}\|u(\cdot,0)\|_\infty + \rho\int_0^t e^{-(t-s)}ds \\
&\leq e^{-t}\rho + \rho\int_0^t e^{-(t-s)}ds \\
&= \rho.
\end{aligned}$$

In order to exhibit a continuous "Lyapunov's functional" for the flow of (2.5), we assume that the functions f and g satisfy the following conditions:

$$0 < |g(x)| < \rho, \ \forall\, x \in \mathbb{R}, \tag{6.25}$$

the function g^{-1} is continuous in $]-\rho,\rho[$ and the function

$$\theta(m) = -\frac{1}{2}f(m)^2 - hf(m) - \beta^{-1}i(m), \ m \in [-\rho,\rho], \tag{6.26}$$

where i is defined by

$$i(m) = -\int_0^{f(m)} g^{-1}(f^{-1}(s))ds, \ m \in [-\rho,\rho],$$

has a global minimum $\overline{m}$ in $]-\rho,\rho[$.

Note that if (6.25) holds, it follows that (2.3) holds with $k_1 = 0$ and $k_2 = \rho$.

Motivated by functionals that appear in [25, 11, 13, 21] and [24], we define the functional $\mathcal{F} : \{L^p(\Omega),\ \|u\|_\infty \leq \rho\} \to \mathbb{R}$ by

$$\mathcal{F}(u) = \int_\Omega [\theta(u(x)) - \theta(\overline{m})]dx + \frac{1}{4}\int_\Omega\int_\Omega J(x,y)[f(u(x)) - f(u(y))]^2 dxdy, \quad (6.27)$$

where θ is given in (6.26), which has been adapted from functions considered in [24] and [25].

Note that the functional in (6.27) is defined in the whole space $\{L^p(\Omega),$ $\|u\|_\infty \leq \rho\}$. Furthermore, using the hypotheses on f and g and Lebesgue's Dominated Convergence Theorem, we obtain the following result:

Theorem 6.1. *In addition to the hypotheses of Theorem 4.3, assume that the hypotheses established in (6.25) and (6.26) hold. Then the functional given in (6.27) is continuous in the topology of $L^p(\Omega)$.*

Now, we are ready to prove the main result of this section.

Theorem 6.2. *In addition of the hypotheses from Theorem 4.3, assume that the hypotheses established in (6.25) and (6.26) hold and that f has positive derivative. Let $u(\cdot,t)$ be a solution of (2.5) with $\|u(\cdot,t)\|_\infty \leq \rho$. Then $\mathcal{F}(u(\cdot,t))$ is differentiable with respect to t for $t > 0$ and*

$$\frac{d}{dt}\mathcal{F}(u(\cdot,t)) = -\mathcal{I}(u(\cdot,t)) \leq 0,$$

where, for any $u \in L^p(\Omega)$ with $\|u\|_\infty \leq \rho$,

$$\begin{aligned}\mathcal{I}(u(\cdot)) = \int_\Omega &[K(f(u)(x)) \\ &+ h - \beta^{-1}g^{-1}(u(x))][g(\beta K(f(u)(x)) + \beta h) - u(x)]f'(u(x))dx.\end{aligned}$$

Furthermore, the integrand in $\mathcal{I}(u(\cdot))$ is a non negative function and, u is a critical point of $\mathcal{F}$ if only if u is an equilibrium of (2.5).

Proof. From hypotheses on g and f, it follows that $\mathcal{F}(u(\cdot,t))$ is well defined for all $t \geq 0$. We assume first that, given $t > 0$, there exists $\varepsilon > 0$ such that $\|u(\cdot,s)\|_\infty \leq \rho - \varepsilon$, for $s \in \Delta$ where Δ is a closed finite interval containing t. For $s \in \Delta$ we write

$$\mathcal{F}(u(\cdot,s)) = \int_\Omega \phi(x,s)dx \text{ and } \mathcal{I}(u(\cdot,s)) = \int_\Omega \iota(x,s)dx.$$

As

$$\begin{aligned}\frac{\partial\phi}{\partial s}(x,s) = &[-f(u(x,s)) - h + \beta^{-1}g^{-1}(u(x,s))]f'(u(x,s))\frac{\partial}{\partial s}u(x,s) \\ &+ \frac{1}{2}\int_\Omega J(x,y)[f(u(x,s)) - f(u(y,s))] \\ &\quad \star \left[f'(u(x,s))\frac{\partial u(x,s)}{\partial s} - f'(u(y,s))\frac{\partial u(y,s)}{\partial s}\right] dy,\end{aligned}$$

the hypotheses on g, f and f' imply that $\frac{\partial\phi(x,s)}{\partial s}$ is almost everywhere continuous and bounded in x for $s \in \Delta$. Thus

$$\sup_{s\in\Delta}\left\|\frac{\partial\phi(\cdot,s)}{\partial s}\right\|_{L^1} < \infty.$$

Therefore, we can derive under the integration sign getting

$$\frac{d}{ds}\mathcal{F}(u(\cdot,s)) = \int_\Omega [-f(u(x,s)) - h + \beta^{-1}g^{-1}(u(x,s))]f'(u(x,s))\frac{\partial u(x,s)}{\partial s}dx$$

$$+\frac{1}{2}\int_\Omega\int_\Omega J(x,y)[f(u(x,s)) - f(u(y,s))]$$

$$\star\left[f'(u(x,s))\frac{\partial u(x,s)}{\partial s} - f'(u(y,s))\frac{\partial u(y,s)}{\partial s}\right]dxdy.$$

But

$$\int_\Omega\int_\Omega J(x,y)[f(u(x,s)) - f(u(y,s))]$$

$$\star\left[f'(u(x,s))\frac{\partial u(x,s)}{\partial s} - f'(u(y,s))\frac{\partial u(y,s)}{\partial s}\right]dxdy$$

$$= \int_\Omega\int_\Omega J(x,y)f(u(x,s))f'(u(x,s))\frac{\partial u(x,s)}{\partial s}dxdy$$

$$-\int_\Omega\int_\Omega J(x,y)f(u(x,s))f'(u(y,s))\frac{\partial u(y,s)}{\partial s}dxdy$$

$$-\int_\Omega\int_\Omega J(x,y)f(u(y,s))f'(u(x,s))\frac{\partial u(x,s)}{\partial s}dxdy$$

$$+\int_\Omega\int_\Omega J(x,y)f(u(y,s))f'(u(y,s))\frac{\partial u(y,s)}{\partial s}dxdy$$

$$= 2\int_\Omega\int_\Omega J(x,y)f(u(x,s))f'(u(x,s))\frac{\partial u(x,s)}{\partial s}dxdy$$

$$-2\int_\Omega\int_\Omega J(x,y)f(u(y,s))f'(u(x,s))\frac{\partial u(x,s)}{\partial s}dxdy$$

$$= 2\int_\Omega\left(\int_\Omega J(x,y)dy\right)f(u(x,s))f'(u(x,s))\frac{\partial u(x,s)}{\partial s}dx$$

$$-2\int_\Omega\left(\int_\Omega J(x,y)f(u(y,s))dy\right)f'(u(x,s))\frac{\partial u(x,s)}{\partial s}dx.$$

Using the fact that

$$\int_\Omega J(x,y)dy = \int_\Omega J(x,y)dx = 1,$$

it follows that

$$\begin{aligned}
\frac{d}{ds}\mathcal{F}(u(\cdot,s)) &= \int_\Omega \Big[-f(u(x,s)) - h + \beta^{-1}g^{-1}(u(x,s))\Big] f'(u(x,s))\frac{\partial u(x,s)}{\partial s}dx \\
&+ \int_\Omega [f(u(x,s)) - Kf(u(x,s))] f'(u(x,s))\frac{\partial u(x,s)}{\partial s}dx \\
&= \int_\Omega \Big[-f(u(x,s)) - h + \beta^{-1}g^{-1}(u(x,s)) + f(u(x,s)) \\
&- Kf(u(x,s))\Big] f'(u(x,s))\frac{\partial u(x,s)}{\partial s}dx \\
&= -\int_\Omega \Big[Kf(u(x,s)) + h - \beta^{-1}g^{-1}(u(x,s))\Big] f'(u(x,s))\frac{\partial u(x,s)}{\partial s}dx \\
&= -\int_\Omega \Big[Kf(u(x,s)) + h - \beta^{-1}g^{-1}(u(x,s))\Big]\big[-u(x,s) \\
&+ g(\beta Kf(u(x,s)) + \beta h)\big] f'(u(x,s))dx \\
&= -\mathcal{I}(u(\cdot,s)).
\end{aligned}$$

This proves the first part of theorem with the additional hypothesis that $\|u(\cdot,s)\|_\infty \leq \rho - \varepsilon$, for $s \in \Delta$ and some $\varepsilon > 0$, where Δ is a closed finite interval containing t.

Proceeding as [25] it is easy to see that this hypothesis actually holds for all $t > 0$. In fact, let $\lambda(x,t)$ be the solution of (2.5) such that $\lambda(x,0) = \rho$ for any $x \in \Omega$. Then $\lambda(x,t) = \lambda(t)$, where

$$\frac{d\lambda}{dt} = -\lambda(t) + g(\beta(\lambda(t) + h)).$$

Since $|g(x)| < \rho$, $\forall\, x \in \mathbb{R}$, it follows easily that $\lambda(t) < \rho$ for any $t > 0$. As $u(x,0) \leq \rho$, we obtain by the Comparison Theorem

$$u(x,t) \leq \lambda(t) < \rho,$$

for almost every $x \in \Omega$ and $t > 0$. Repeating the same argument, starting from inequality $u(x,0) \geq -\rho$, for almost every $x \in \Omega$, we obtain $u(x,t) \geq -\lambda(t) > -\rho$, and thus

$$\|u(\cdot,t)\|_\infty \leq \lambda(t) < \rho,\ \forall\, t > 0$$

and the claim follows by continuity.

To conclude the proof, it is enough to show that u is a critical point of $\mathcal{F}$ if and only if u is an equilibrium of (2.5). For this, let $u(x)$ be a critical point of the functional $\mathcal{F}$, then $\mathcal{I}(u(\cdot)) = 0$. Since the integrand is non negative almost everywhere, it follows that

$$[(Kf(u)(x)) + h - \beta^{-1}g^{-1}(u(x))]f'(u(x))[g(\beta(Kf(u)(x) + h)) - u(x)] = 0$$

almost everywhere. Since $f'(u(x)) > 0$, for all $x \in \mathbb{R}^N$, we have that

$$[(Kf(u)(x)) + h - \beta^{-1}g^{-1}(u(x))][g(\beta(Kf(u)(x) + h)) - u(x)] = 0$$

almost everywhere. But the annihilation of any of these factors implies that

$$g(\beta Kf(u)(x) + \beta h) = u(x).$$

Reciprocally, if u is a equilibrium of (2.5), it is easy to see that $\mathcal{I}(u(\cdot)) = 0$. ■

As a immediate consequence of the existence of the functional $\mathcal{F}$, we obtain the following result.

Corollary 6.3. *Under the same hypotheses of Theorem 6.2, there are no non trivial recurrent points under the flow of (2.5).*

Remark 6.4. *The integrand in the functional $\mathcal{F}$ above is always non negative since J is positive and $\overline{m}$ is a global minim of θ. Thus, $\mathcal{F}$ is lower bounded.*

We recall that a C^r-semigroup, $T(t)$, is gradient if each bounded positive orbit is precompact and there exists a Lyapunov's Functional for $T(t)$ (see [18]).

Proposition 6.5. *Assume the same hypotheses of Theorem 6.2. Then the flow generated by equation (2.5) is gradient.*

Proof. The precompacity of the orbits follows from the existence of the global attractor (see Theorem 4.3). From Theorems 6.1 and 6.2, and Remark 6.4, we have the existence of a continuous Lyapunov's functional. ■

From Proposition 6.5, we have the following characterization of the attractor (see [18] - Theorem 3.8.5).

Theorem 6.6. *Assume the same assumptions of Proposition 6.5. Then the attractor $\mathcal{A}$ is the unstable set of the equilibrium point set of $T(t)$, that is, $\mathcal{A} = W^u(E)$.*

7 Acknowledgements

The authors would like to thank the anonymous referee for his/her reading of the manuscript and valuable suggestions. They also would like to thank the professor Daniel De Morais Filho by valuable suggestions the english usage this paper. Finally, the first author dedicates this work to your sons Arthur and Luana and your wife Michelli Barros, the second author thanks CAPES-Brasil and dedicates this work to your sons Raquel, Gelson, Inácio and Lara and your wife Maria do Carmo and; the third author thanks CAPES-Brasil and dedicates this work to your mother Élita.

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