Pointwise amenability for dual Banach algebras

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Abstract

We shall develop two notions of pointwise amenability, namely pointwise Connes amenability and pointwise w^*-approximate Connes amenability, for dual Banach algebras which take the w^*-topology into account. We shall study these concepts for the Banach sequence algebras $\ell^1(\omega)$ and the weighted semigroup algebras $\ell^1(\mathbb{N}_\wedge, \omega)$. For a weight ω on a discrete semigroup S, we shall investigate pointwise amenability/Connes amenability of $\ell^1(S, \omega)$ in terms of diagonals.

1 Introduction

The key concept of amenability for Banach algebras introduced by B. E. Johnson [7]. The pointwise variant of amenability introduced by H. G. Dales, F. Ghahramani and R. J. Loy, however this appeared formally in [3].

Let $\mathcal{A}$ be a Banach algebra. Then the projective tensor product $\mathcal{A}\hat{\otimes}\mathcal{A}$ naturally is a Banach $\mathcal{A}$-bimodule and the map $\pi : \mathcal{A}\hat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$ defined by $\pi(a \otimes b) = ab$, $a, b \in \mathcal{A}$ is a linear continuous $\mathcal{A}$-bimodule homomorphism. Let E be a Banach $\mathcal{A}$-bimodule. A *derivation* is a bounded linear map $D : \mathcal{A} \longrightarrow E$ satisfying $D(ab) = Da \cdot b + a \cdot Db \quad (a, b \in \mathcal{A})$. A Banach algebra $\mathcal{A}$ is *pointwise amenable* at $a_0 \in \mathcal{A}$ if, for each Banach $\mathcal{A}$-bimodule E, every derivation $D : \mathcal{A} \to E^*$ is *pointwise inner* at a_0, that is, there exist $\phi \in E^*$ such that $D(a_0) = a_0 \cdot \phi - \phi \cdot a_0$ [3].

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Let $\mathcal{A}$ be a Banach algebra. A Banach $\mathcal{A}$-bimodule E is *dual* if there is a closed submodule E_* of E^* such that $E = (E_*)^*$. We call E_* the *predual* of E. A Banach algebra $\mathcal{A}$ is *dual* if it is dual as a Banach $\mathcal{A}$-bimodule. Equivalently, a Banach algebra $\mathcal{A}$ is dual if it is a dual Banach space such that multiplication is separately continuous in the w^*-topology. We write $\mathcal{A} = (\mathcal{A}_*)^*$ if we wish to stress that $\mathcal{A}$ is a dual Banach algebra with predual $\mathcal{A}_*$. For a dual Banach algebra $\mathcal{A}$, a dual Banach $\mathcal{A}$-bimodule E is *normal* if the module actions of $\mathcal{A}$ on E are w^*-continuous. The notion of Connes amenability for dual Banach algebras, which is another modification of the notion of amenability systematically introduced by V. Runde [12]. A dual Banach algebra $\mathcal{A}$ is *Connes amenable* if every w^*-continuous derivation from $\mathcal{A}$ into a normal, dual Banach $\mathcal{A}$-bimodule is inner.

The concept of w^*-approximately Connes amenability introduced by the second author in [10]. One may see also [5, 6, 9, 11, 13, 14], for more information on Connes amenability and other related notions.

The purpose of this paper is to study pointwise Connes amenability of dual Banach algebras, as well as their pointwise w^*-approximate Connes amenability. The organization of the paper is as follows. In section 2, some basic properties are given. It is shown that every commutative pointwise Connes amenable dual Banach algebras must be unital.

In section 3, it is proved that the Banach sequence algebra $\ell^1(\omega)$ is pointwise w^*-approximately Connes amenable while it is not pointwise Connes amenable, where ω is a weight function. It is also shown that the same is true for the class of weighted semigroup algebras of the form $\ell^1(\mathbb{N}_\wedge, \omega)$, provided $\lim_n \omega(n) = \infty$.

In section 4, the relation between pointwise amenability/Connes amenability of weighted semigroup algebras $\ell^1(S, \omega)$ and the existence of some specified diagonals is studied. For a discrete group G, the special case $\ell^1(G, \omega)$ is also considered.

2 Connes amenability; pointwise versions

From [10], we recall that a dual Banach algebra $\mathcal{A}$ is *w^*-approximately Connes amenable* if, for every normal, dual Banach $\mathcal{A}$-bimodule E, every w^*-continuous derivation $D : \mathcal{A} \to E$ is *w^*-approximately inner*, that is, there exists a net $(x_i) \subseteq E$ such that $D(a)$=w^*-$\lim_i (a \cdot \phi_i - \phi_i \cdot a)$.

We first introduce the pointwise versions of (w^*-approximate) Connes amenability.

Definition 2.1. Let $\mathcal{A}$ be a dual Banach algebra. Then:

(i) $\mathcal{A}$ is *pointwise Connes amenable* at $a_0 \in \mathcal{A}$ if for every normal, dual Banach $\mathcal{A}$-bimodule E, every w^*-continuous derivation $D : \mathcal{A} \to E$ is pointwise inner at a_0.

(ii) $\mathcal{A}$ is *pointwise w^*-approximately Connes amenable* at $a_0 \in \mathcal{A}$ if for every normal, dual Banach $\mathcal{A}$-bimodule E, every w^*-continuous derivation $D : \mathcal{A} \to E$ is *pointwise w^*-approximately inner* at a_0, that is, there exists a net $(x_i) \subseteq E$ such that $D(a_0) = w^* - \lim_i (a_0 \cdot x_i - x_i \cdot a_0)$.

(iii) $\mathcal{A}$ is *pointwise (w^*-approximately) Connes amenable* if $\mathcal{A}$ is pointwise (w^*-approximately) Connes amenable at each $a \in A$.

Let $\mathcal{A}$ be a Banach algebra. From [8], we recall that $\mathcal{A}$ has *left (right) approximate units* if, for each $a \in \mathcal{A}$ and $\varepsilon > 0$ there exists $u \in \mathcal{A}$ such that $\|a - ua\| < \epsilon$ ($\|a - au\| < \epsilon$), and $\mathcal{A}$ has approximate units if, for each $a \in \mathcal{A}$ and $\varepsilon > 0$, there exists $u \in \mathcal{A}$ such that $\|a - ua\| + \|a - au\| < \epsilon$. The appropriate approximate units have bound m if the element u can be chosen such that $\|u\| \leq m$. The algebra $\mathcal{A}$ has a *bounded* (left or right) approximate units if it has (left or right) approximate units of bound m for some $m \geq 1$.

Definition 2.2. A dual Banach algebra $\mathcal{A} = (\mathcal{A}_*)^*$ has *left (right) w^*-approximate units* if, for each $a \in \mathcal{A}$ and $\varepsilon > 0$, and for each finite subset $\mathcal{K} \subseteq \mathcal{A}_*$, there is $u \in \mathcal{A}$ such that $|\langle \psi, a - ua\rangle| < \varepsilon$ ($|\langle \psi, au - a\rangle| < \varepsilon$) for $\psi \in \mathcal{K}$. We say $\mathcal{A}$ has w^*-approximate units if, for each $a \in \mathcal{A}$ and $\varepsilon > 0$ and for each finite subset $\mathcal{K} \subseteq \mathcal{A}_*$, there is $u \in \mathcal{A}$ such that

$$| < \psi, a - au > | + | < \psi, a - ua > | < \epsilon \, , \ \ (\psi \in \mathcal{K}) \, .$$

The appropriate w^*-approximate units have bound m if the element u can be chosen such that $\|u\| \leq m$. The dual Banach algebra $\mathcal{A}$ has *bounded* (left or right) w^*-approximate units if it has (left or right) w^*-approximate units of bound m for some $m \geq 1$.

The following lemma is useful in considerations of identities.

Lemma 2.3. *Let $\mathcal{A}$ be a Banach algebra and take $m \geq 1$. Suppose that $\mathcal{A}$ has pointwise left identity of bound m (i.e. for every $a \in \mathcal{A}$ there exists $u \in \mathcal{A}$ with $\|u\| \leq m$ such that $ua = a$). Then, for each $a_1, ..., a_n \in \mathcal{A}$ there exists $u \in \mathcal{A}$ with $\|u\| \leq m$ such that $ua_i = a_i, 1 \leq i \leq n$.*

Proof. Take $a_1, ..., a_n \in \mathcal{A}$. Successively choose $u_1, ..., u_n \in \mathcal{A}$ with $\|u_i\| \leq m$, $1 \leq i \leq n$, and $(e - u_i)...(e - u_1)a_i = 0$ $(1 \leq i \leq n)$. Here, notice that we use e as a symbol. For instance, by $(e - u)a$ we mean $a - au$. Define $u \in \mathcal{A}$ by $e - u = (e - u_n)...(e - u_1)$. Then for each $1 \leq i \leq n$ we have

$$a_i - ua_i = (e - u_n)...(e - u_{i+1})((e - u_i)...(e - u_1)a_i) = 0 \, ,$$

as required. ∎

Theorem 2.4. *Let $\mathcal{A}$ be a dual Banach algebra. Suppose that for each $a_1, ..., a_n \in \mathcal{A}$, there exists $u \in \mathcal{A}$ with $\|u\| \leq m$ and $ua_i = a_i, 1 \leq i \leq n$. Then $\mathcal{A}$ has left identity.*

Proof. Let $\mathcal{F}$ be the family of all non-empty, finite subset of $\mathcal{A}$, ordered by inclusion. Therefore $\mathcal{F}$ is a directed set. For each $F \in \mathcal{F}$ choose $e_F \in \mathcal{A}$ with $\|e_F\| \leq m$ and $a = e_F a$, $a \in F$. Since $(e_F)_{F \in \mathcal{F}}$ is a bounded net in a dual Banach algebra, there is a subnet (e_{F_i}) and an element $e \in \mathcal{A}$ such that $e = \lim_i e_{F_i}$. Now, it is clear that e is a left identity for $\mathcal{A}$. ∎

We shall need the following pointwise version of [10, Proposition 4.2].

Lemma 2.5. *A dual Banach algebra $\mathcal{A}$ is pointwise Connes amenable if and only if for every $a \in \mathcal{A}$ there is a bounded net $(m_\alpha)_\alpha \in \mathcal{A}\hat{\otimes}\mathcal{A}$ such that $a \cdot m_\alpha - m_\alpha \cdot a \xrightarrow{w^*} 0$ in $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$, and $\pi_{\sigma wc}(m_\alpha)a \xrightarrow{w^*} a$ in $\mathcal{A}$.*

Theorem 2.6. *Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a commutative pointwise Connes amenable dual Banach algebra . Then $\mathcal{A}$ has an identity.*

Proof. By Lemma 2.5, for every $a \in \mathcal{A}$ there is a bounded net $(u_i) \subseteq \mathcal{A}$ such that $u_i a \xrightarrow{w^*} a$ in $\mathcal{A}$. Because $\mathcal{A}$ is a dual space, passing to a subnet, we may suppose that there is $u \in \mathcal{A}$ such that $u_i \longrightarrow u$. Hence, for every $a \in \mathcal{A}$, there is $u \in \mathcal{A}$ such that $ua = a$. For each $n = 1, 2, ...,$ we define

$$A_n = \{a \in \mathcal{A} : \ ua = a \text{ for some } u \in \mathcal{A} \text{ with } \|u\| \leq n\} .$$

We claim that A_n is a closed subset of $\mathcal{A}$. For if (a_k) is a sequence in A_n with $\lim_k a_k = a$, then there exists a sequence (u_k) in A_n such that $\|u_k\| \leq n$ and $u_k a_k = a_k$. From

$$||a - u_k a|| \leq ||a - a_k|| + ||a_k - u_k a_k|| + ||u_k|| \, ||a_k - a|| \leq (1 + n) \, ||a - a_k||$$

it follows that $\lim_k u_k a = a$. On the other hand, (u_k) has a w^*-limit point say u. Then, $w^* - \lim_\lambda u_\lambda = u$, where (u_λ) is a subnet of (u_k). Because of separately w^*-continuity of the multiplication, $w^* - \lim_\lambda u_\lambda a = ua$. As (u_λ) is a subnet of (u_k), $\lim_\lambda u_\lambda a = a$. From these two latest facts, we obtain $ua = a$ so that $a \in A_n$.

An argument similar to [8, Theorem 9.7], shows that $\mathcal{A}$ has pointwise identity of bound m for some $m \geq 1$. Note the commutativity of $\mathcal{A}$ is required at this point.

Now, Lemma 2.3 and Theorem 2.4 yield that $\mathcal{A}$ possesses an identity. ∎

3 Examples

We recall that a *weight* on a discrete semigroup S is a function $\omega : S \longrightarrow [1, \infty)$ such that $\omega(gh) \leq \omega(g)\omega(h)$, for all $g, h \in S$. Then $\ell^1(S, \omega)$ is the Banach space of all complex functions $f = (a_g)_{g \in S}$ on S with the norm $||(a_g)_g|| = \sum_{g \in S} |a_g|\omega(g) < \infty$.

It is known that $\ell^1(\omega) := \ell^1(\mathbb{N}, \omega)$ is a dual Banach algebra under pointwise multiplication with the predual $c_0(\omega^{-1})$ and without identity element, see for instance [2].

Theorem 3.1. *Let ω be a weight on $\mathbb{N}$. Then:*
(i) $\ell^1(\omega)$ is not Connes amenable;
(ii) $\ell^1(\omega)$ is not pointwise Connes amenable;
(iii) $\ell^1(\omega)$ is not w^*-approximately Connes amenable;
(iv) $\ell^1(\omega)$ is pointwise w^*-approximately Connes amenable.

Proof. Since $\ell^1(\omega)$ does not have an identity, the clause (i) is immediate by [12, Proposition 4.1]. The commutativity of $\ell^1(\omega)$ together with Theorem 2.6 imply (ii). The clause (iii) is exactly [10, Theorem 3.3]. Finally, the argument of [3, Corollary 1.8.5] shows that $\ell^1(\omega)$ is pointwise-approximately amenable. Therefore, automatically it is pointwise w^*-approximately Connes amenable. ∎

Next, we consider the semigroup $\mathbb{N}_\wedge$ which is $\mathbb{N}$ with the semigroup operation $m \wedge n := min\{m,n\}$, $(m,n \in \mathbb{N})$. It is clear that any function $\omega : \mathbb{N} \longrightarrow [1,\infty)$ is a weight on the semigroup $\mathbb{N}_\wedge$. Then $\ell^1(\mathbb{N}_\wedge,\omega)$ is a commutative Banach algebra with the *convolution* product $\delta_m \star \delta_n = \delta_{m\wedge n}$, where δ_n stands for the characteristic function of $\{n\}$ for $n \in \mathbb{N}$. It is well known that $\ell^1(\mathbb{N}_\wedge,\omega)$ is a dual Banach algebra, whenever $\lim_n \omega(n) = \infty$ and with the predual $c_0(\omega^{-1})$ [3, Proposition 3.1.1].

Suppose that $\lim_n \omega(n) = \infty$. Then, because of the lack of identity [3, Propositions 3.3.1 and 3.3.2], $\ell^1(\mathbb{N}_\wedge,\omega)$ is not Connes amenable. The same reason, using Theorem 2.6, implies that $\ell^1(\mathbb{N}_\wedge,\omega)$ is not pointwise Connes amenable. It was shown in [3, Theorem 3.7.1] that $\ell^1(\mathbb{N}_\wedge,\omega)$ is pointwise approximately amenable, and therefore it is pointwise w^*-approximately Connes amenable as well. We summarize these facts as follows.

Theorem 3.2. *Let ω be a function on $\mathbb{N}$ such that $\lim_n \omega(n) = \infty$. Then:*
(i) $\ell^1(\mathbb{N}_\wedge,\omega)$ is not Connes amenable;
(ii) $\ell^1(\mathbb{N}_\wedge,\omega)$ is not pointwise Connes amenable;
(iii) $\ell^1(\mathbb{N}_\wedge,\omega)$ is pointwise w^*-approximately Connes amenable.

Note that the w^*-approximate Connes amenability of $\ell^1(\mathbb{N}_\wedge,\omega)$ is unresolved.

4 Relations with diagonals

Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra and let E be a Banach $\mathcal{A}$-bimodule. We write $\sigma wc(E)$ for the set of all elements $x \in E$ such that the map

$$\mathcal{A} \longrightarrow E, \quad a \longmapsto \begin{cases} a \cdot x \\ x \cdot a \end{cases},$$

are w^*-weak continuous. It was shown that $\pi^*(\mathcal{A}_*) \subseteq \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$ [14, Corollary 4.6]. Taking adjoints, we can extend π to an $\mathcal{A}$-bimodule homomorphism $\pi_{\sigma wc}$ from $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ to $\mathcal{A}$.

Let E be a Banach space. We then have the canonical map $\imath_E : E \longrightarrow E^{**}$ defined by $\langle \mu, \imath_E(x)\rangle = \langle x, \mu\rangle$ for $\mu \in E^*$, $x \in E$. For Banach spaces E and F, we write $\mathcal{L}(E,F)$ for the Banach space of bounded linear maps between E and F. It is standard that $(E\hat{\otimes}F)^* = \mathcal{L}(F,E^*)$. For a Banach algebra $\mathcal{A}$, then we obtain a bimodule structure on $\mathcal{L}(\mathcal{A},\mathcal{A}^*) = (\mathcal{A}\hat{\otimes}\mathcal{A})^*$ through $(a\ .\ T)(b) = T(ba)$, $(T\ .\ a)(b) = T(b)\ .\ a$, for $a,b \in \mathcal{A}$, and for $T \in \mathcal{L}(\mathcal{A},\mathcal{A}^*)$.

Throughout, we use the term *unital* for a semigroup (or an algebra) X with an identity element e_X.

The following characterizations will be needed in the sequel.

Theorem 4.1. *Let $\mathcal{A}$ be a unital Banach algebra. Then $\mathcal{A}$ is pointwise amenable if and only if for each $a \in \mathcal{A}$ there exists $M \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that $a\ .\ M = M\ .\ a$, and $\pi^{**}(M) = e_{\mathcal{A}}$.*

Proof. The proof is a small variation of the standard argument in [1, Theorem 43.9]. ∎

Theorem 4.2. *Let $\mathcal{A}$ be a unital dual Banach algebra. Then:*

(i) *$\mathcal{A}$ is pointwise Connes amenable if and only if for each $a \in \mathcal{A}$ there exists $M \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that $a \,.\, M = M \,.\, a$, and $\pi_{\sigma wc}(M) = e_{\mathcal{A}}$.*

(ii) *$\mathcal{A}$ is pointwise Connes amenable if and only if for each $a \in \mathcal{A}$ there exists $M \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that $\langle T, a \,.\, M - M \,.\, a\rangle = 0$ for each $T \in \sigma wc(\mathcal{L}(\mathcal{A}, \mathcal{A}^*))$, and $\iota^*_{\mathcal{A}_*}\pi^{**}(M) = e_{\mathcal{A}}$.*

Proof. The clause (i) is analogous to [14, Theorem 4.8]. Because $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ is a quotient of $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$, the clause (ii) is just a re-statement of (i). ■

For a discrete semigroup S, we recall that $\ell^1(S)\hat{\otimes}\ell^1(S) = \ell^1(S \times S)$, where $\delta_g \otimes \delta_h$ is identified with $\delta_{(g,h)}$ for $g, h \in S$. Thus we have $\mathcal{L}(\ell^1(S), \ell^\infty(S)) = (\ell^1(S)\hat{\otimes}\ell^1(S))^* = \ell^1(S \times S)^* = \ell^\infty(S \times S)$, where $T \in \mathcal{L}(\ell^1(S), \ell^\infty(S))$ is identified with $(T_{(g,h)})_{(g,h)\in S\times S} \in \ell^\infty(S \times S)$, while $T_{(g,h)} := \langle\delta_h, T(\delta_g)\rangle$. Let ω be a weight on S. If S is unital then, without loss of generality, we put $\omega(e_S) = 1$. The Banach space $\ell^1(S, \omega)$ with the convolution product is a Banach algebra, called a *Beurling* algebra. We consider $\ell^1(S, \omega)$ as the Banach space $\ell^1(S)$ with the product $\delta_g \star_\omega \delta_h := \delta_{gh}\Omega(g, h)$, where $\Omega(g, h) := \frac{\omega(gh)}{\omega(g)\omega(h)}$, $(g, h \in S)$, and extend $\star_\omega$ to $\ell^1(S)$ by linearity and continuity. A semigroup S is *weakly cancellative* if, for each $s \in S$, the maps L_s and R_s, defined by $L_s(t) = st$ and $R_s(t) = ts$, are finite-to-one. In this case, $\ell^1(S, \omega)$ is a dual Banach algebra with predual $c_0(S)$ [4, Proposition 5.1]

Theorem 4.3. *Let S be a discrete unital semigroup, let ω be a weight on S and let $\mathcal{A} := \ell^1(S, \omega)$. Consider the following statements:*

(1) *$\mathcal{A}$ is pointwise amenable.*

(2) *For each $k \in S$ there exists $M \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**} = \ell^\infty(S \times S)^*$ such that:*

(i) *$\langle (f(hk, g)\Omega(h, k) - f(h, kg)\Omega(k, g))_{(g,h)\in S\times S}, M\rangle = 0$ for each bounded function $f : S \times S \longrightarrow \mathbb{C}$;*

(ii) *$\langle (f_{gh}\Omega(g, h))_{(g,h)\in S\times S}, M\rangle = f_{e_S}$ for each bounded family $(f_g)_{g\in S}$.*

(3) *$\mathcal{A}$ is pointwise amenable at δ_k for each $k \in S$.*

Then we have $(1) \Longrightarrow (2) \Longrightarrow (3)$.

Proof. $(1) \Longrightarrow (2)$ Suppose that $\mathcal{A}$ is pointwise amenable and that $k \in S$. Take $M \in \ell^\infty(S \times S)^*$ as in Theorem 4.1. For each bounded family $(f_g)_{g\in S}$, we have

$$\pi^*(f) = ((\langle\delta_{gh}, f\rangle\Omega(g, h))_{(g,h)\in S\times S} \in \ell^\infty(S \times S)\ .$$

Therefore

$$\langle (f_{gh}\Omega(g, h))_{(g,h)\in S\times S}, M\rangle = \langle f, \pi^{**}(M)\rangle = \langle f, e_{\mathcal{A}}\rangle = f_{e_S}.$$

Next, for every $T \in \mathcal{L}(\mathcal{A}, \mathcal{A}^*) = \ell^\infty(S \times S)$, we see that

$$\langle \delta_g \otimes \delta_h, \delta_k \,.\, T - T \,.\, \delta_k\rangle = \langle\delta_g, T(\delta_{hk})\rangle\Omega(h, k) - \langle\delta_{kg}, T(\delta_h)\rangle\Omega(k, g)\ .$$

Let $f : S \times S \longrightarrow \mathbb{C}$ be a bounded function and take $T \in \mathcal{L}(\mathcal{A}, \mathcal{A}^*) = \ell^\infty(S \times S)$ defined by $\langle\delta_h, T(\delta_g)\rangle = f(g, h)$. Hence

$$\begin{aligned}
&\langle (f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g))_{(g,h)\in S\times S}, M\rangle \\
&\qquad = \langle ((\langle \delta_g \otimes \delta_h, \delta_k \,.\, T - T \,.\, \delta_k\rangle))_{(g,h)\in S\times S}, M\rangle \\
&\qquad = \langle \delta_k \,.\, T - T \,.\, \delta_k, M\rangle \\
&\qquad = \langle T, \delta_k \,.\, M - M \,.\, \delta_k\rangle = 0,
\end{aligned}$$

as required. Likewise for the implication (2) $\Longrightarrow$ (3). ∎

The following is a part of [4, Proposition 5.5], in which $\mathcal{W}(\mathcal{A},\mathcal{A}^*)$ stands for the collection of weakly compact operators in $\mathcal{L}(\mathcal{A},\mathcal{A}^*)$.

Theorem 4.4. *Let S be a discrete, weakly cancellative semigroup, let ω be a weight on S, and let $\mathcal{A} := \ell^1(S,\omega)$. Let $T \in \mathcal{L}(\mathcal{A},\mathcal{A}^*)$ be such that $T(\mathcal{A}) \subseteq \imath_{c_0(S)}(c_0(S))$ and $T^*(\imath_{\mathcal{A}}(\mathcal{A})) \subseteq \imath_{c_0(S)}(c_0(S))$, then $T \in \mathcal{W}(\mathcal{A},\mathcal{A}^*)$.*

Theorem 4.5. *Let S be a discrete, weakly cancellative semigroup, let ω be a weight on S and let $\mathcal{A} := \ell^1(S,\omega)$ be unital. Consider the following statements:*

(1) $\mathcal{A}$ is pointwise Connes amenable.

(2) For each $k \in S$ there exists $M \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**} = \ell^\infty(S\times S)^*$ such that:

(i) $\langle (f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g))_{(g,h)\in S\times S}, M\rangle = 0$ for each bounded function $f : S\times S \longrightarrow \mathbb{C}$ which is such that the map $T \in \mathcal{L}(\mathcal{A},\mathcal{A}^*)$, defined by $\langle \delta_h, T(\delta_g)\rangle = f(g,h)$, for $g,h \in S$, satisfies the conclusions of Theorem 4.4;

(ii) $\langle (f_{gh}\Omega(g,h))_{(g,h)\in S\times S}, M\rangle = \langle f, e_{\mathcal{A}}\rangle$ for each family $(f_g)_{g\in S} \in c_0(S)$.

(3) $\mathcal{A}$ is pointwise Connes amenable at δ_k for each $k \in S$.

Then we have (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).

Proof. Using Theorem 4.2 (ii) in place of Theorem 4.1, this follows as Theorem 4.3. For the sake of convenience, we include the proof of (1) $\Longrightarrow$ (2). Suppose that $\mathcal{A}$ is pointwise Connes amenable and that $k \in S$. Take $M \in \ell^\infty(S\times S)^*$ as in Theorem 4.2 (ii). For each family $(f_g)_{g\in S} \in c_0(S)$, we have

$$\langle \delta_g \otimes \delta_h, \pi^*\imath_{c_0(S)}(f)\rangle = \langle \delta_{gh}\Omega(g,h), \imath_{c_0(S)}(f)\rangle$$

so that

$$\pi^*\imath_{c_0(S)}(f) = (\langle \delta_{gh}, \imath_{c_0(S)}(f)\rangle\Omega(g,h))_{(g,h)\in S\times S} \in \ell^\infty(S\times S)\,.$$

Hence

$$\langle (f_{gh}\Omega(g,h))_{(g,h)\in S\times S}, M\rangle = \langle \pi^*\imath_{c_0(S)}(f), M\rangle = \langle f, \imath^*_{c_0(S)}\pi^{**}(M)\rangle = \langle f, e_{\mathcal{A}}\rangle.$$

Next, let $f : S\times S \longrightarrow \mathbb{C}$ be a bounded function such that the map $T \in \mathcal{L}(\mathcal{A},\mathcal{A}^*) = \ell^\infty(S\times S)$ defined by $\langle \delta_h, T(\delta_g)\rangle = f(g,h)$, $(g,h \in S)$, satisfies the conclusions of Theorem 4.4. Then by [4, Corollary 3.5], $T \in \sigma wc(\mathcal{L}(\mathcal{A},\mathcal{A}^*))$. Therefore

$$\begin{aligned}
&\langle (f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g))_{(g,h)\in S\times S}, M\rangle \\
&\qquad = \langle \delta_g, T(\delta_{hk})\rangle\Omega(h,k) - \langle \delta_{kg}, T(\delta_h)\rangle\Omega(k,g) \\
&\qquad\quad \langle ((\langle \delta_g \otimes \delta_h, \delta_k \,.\, T - T \,.\, \delta_k\rangle))_{(g,h)\in S\times S}, M\rangle \\
&\qquad = \langle \delta_k \,.\, T - T \,.\, \delta_k, M\rangle \\
&\qquad = \langle T, \delta_k \,.\, M - M \,.\, \delta_k\rangle = 0,
\end{aligned}$$

as required. ∎

Let G be a discrete group and let $h \in G$. Define $J_h : \ell^\infty(G) \longrightarrow \ell^\infty(G)$ by

$$J_h(f) := (f_{hg}\Omega(h,g)\omega(h)\Omega(g^{-1},h^{-1})\omega(h^{-1}))_{g\in G} \quad (f = (f_g)_g \in \ell^\infty(G)) .$$

It is clear that $||J_h(f)|| \leq ||f||\omega(h)\omega(h^{-1})$, so that J_h is bounded.

The following is the pointwise variant of [4, Definition 5.10].

Definition 4.6. Let ω be a weight on a discrete group G, and let $h_0 \in G$. Then G is *pointwise ω-amenable at h_0* if there exists $N \in \ell^\infty(G)^*$ such that $\langle(\Omega(g,g^{-1})_{g\in G}, N\rangle = 1$ and $J^*_{h_0}(N) = N$. We say that G is *pointwise ω-amenable* if it is pointwise ω-amenable at each $h \in G$.

Theorem 4.7. *Let G be a discrete group, let ω be a weight on G and let $\mathcal{A} = \ell^1(G,\omega)$. Consider the following statements:*

(1) *$\mathcal{A}$ is pointwise amenable.*
(2) *$\mathcal{A}$ is pointwise Connes-amenable.*
(3) *G is pointwise ω-amenable.*
(4) *$\mathcal{A}$ is pointwise amenable at δ_k for each $k \in G$.*

Then we have $(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4)$.

Proof. The implication $(1) \Longrightarrow (2)$ is trivial, and $(2) \Longrightarrow (3)$ is a more or less verbatim of the proof of [4, Theorem 5.11 $(1) \Longrightarrow (3)$] .

$(3) \Longrightarrow (4)$: Let $k \in G$, and let $N \in \ell^\infty(G)^*$ be given as in (3). Define $\psi : \ell^\infty(G \times G) \longrightarrow \ell^\infty(G)$ by $\langle\delta_g, \psi(F)\rangle := F(g,g^{-1})$, for each $F \in \ell^\infty(G \times G)$ and $g \in G$. Put $M := \psi^*(N)$. Then it suffices to show that M has desired properties in Theorem 4.3 (2). First, for a bounded family $(f_g)_{g\in G}$, we see that

$$\langle(f_{gh}\Omega(g,h))_{(g,h)}, M\rangle = \langle(f_{e_G}\Omega(g,g^{-1}))_g, N\rangle = \langle(\Omega(g,g^{-1}))_g, N\rangle = 1 .$$

Next, for an arbitrary bounded function $f : G \times G \longrightarrow \mathbb{C}$, it is clear that

$$\psi((f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g))_{(g,h)}) = \\ (f(g^{-1}k,g)\Omega(g^{-1},k) - f(g^{-1},kg)\Omega(k,g))_g .$$

Define $F : G \times G \longrightarrow \mathbb{C}$, by $F(g,h) := f(hk,g)\Omega(h,k)$, for each $g,h \in G$. Hence, it is readily seen that F is bounded and $||F||_\infty \leq ||f||_\infty$. Therefore

$$\begin{aligned}\langle(f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g))_{(g,h)}, M\rangle \\ &= \langle(f(g^{-1}k,g)\Omega(g^{-1},k) - f(g^{-1},kg)\Omega(k,g))_g, N\rangle \\ &= \langle\psi(F) - J_k(\psi(F)), N\rangle \\ &= \langle\psi(F), N - J^*_k(N)\rangle = 0,\end{aligned}$$

as required. ∎

Remark 4.8. It seems to be a *right* conjecture that pointwise amenability of $\ell^1(G,\omega)$ coincides with its pointwise Connes amenability, however we are not able to prove (or disprove) it. In fact, we can not establish the implication $(4) \Longrightarrow (1)$ in Theorem 4.7, because we can see no reason that pointwise amenability at elements a and b gives any information about $a + b$.

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