Common Fixed Points Results of Multivalued Perov type Contractions on Cone Metric Spaces with a Directed Graph

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Abstract

In this paper, we establish the existence of common fixed points of multivalued Perov type contraction mappings on cone metric space endowed with a graph. An example is presented to support the results proved herein. Our results unify, generalize and complement various known comparable results in the literature.

1 Introduction

Order oriented fixed point theory has many applications in economics, computer science and other related disciplines. The interplay between the order structure of underlying mathematical structure and fixed point theory is very strong and fruitful.

This theory is studied in the framework of a partially ordered sets along with appropriate mappings satisfying certain order conditions. Existence of fixed points in partially ordered metric spaces was first investigated in 2004 by Ran and Reurings [35], and then by Nieto and Lopez [30]. Further results in this direction under different contractive conditions were proved in [2, 4, 7, 12, 32].

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Jachymski [24] introduced a new approach in metric fixed point theory by replacing order structure with a graph structure on a metric space. In this way, the results obtained in ordered metric spaces are generalized (see also [23] and the reference therein); in fact, Gwodzdz-lukawska and Jachymski [22] developed the Hutchinson-Barnsley theory for finite families of mappings on a metric space endowed with a directed graph.

Abbas and Nazir [3] obtained some fixed point results for power graphic contraction pair on a metric space equipped with a graph. Recently, Bojor [18] proved fixed point results for Reich type contractions on such spaces. For more results in this direction, we refer to [6, 17, 19, 31] and reference mentioned therein.

Huang and Zhang [21] generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. They called such space a cone metric space. It is worth mentioning that the notion of cone metric spaces was initially defined by Kantorovich (as cited in [25]). Following the results of Huang and Zhang, recently a lot of papers have been dedicated to show that results of fixed point or common fixed point known in the setting of metric spaces hold in the framework of cone metric space. Altun and Durmaz [9] and Altun, Damjanović and Djorić [8] obtained fixed point of mappings on partially ordered cone metric spaces. On the other hand, Perov [33] generalized the Banach contraction principle by replacing the contractive factor with a matrix convergent to zero. Cvetković and Rakočević [20] introduced Perov-type quasi-contractive mapping replacing contractive factor with bounded linear operator with spectral radius less than one and obtained some interesting fixed point results in the setup of cone metric spaces.

The study of fixed points for multivalued contractions and nonexpansive maps using the Hausdorff metric was initiated by Markin [29]. Theory of multivalued maps has rich applications in control theory, convex optimization, differential equations and economics.

The aim of this paper is to prove some common fixed point results for multivalued generalized graphic Perov type contraction mappings without exploiting the notion of a normality of a cone in the underlying cone metric space endowed with a graph. Our results extend and unify various comparable results in the existing literature ([1], [27], [28], and [37]).

In the sequel the letters $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}$ will denote the set of natural numbers, the set of positive real numbers and the set of real numbers, respectively.

2 Preliminaries

Consistent with Jachymski [23], let (X, d) be a metric space and Δ denotes the diagonal of $X \times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and $E(G)$ be the set of edges of the graph which contains all loops, that is, $\Delta \subseteq E(G)$. Let $E^*(G)$ denotes the set of all edges of G that are not loops i.e., $E^*(G) = E(G) - \Delta$. Also assume that the graph G has no parallel edges and, thus one can identify G with the pair $(V(G), E(G))$.

Definition 1.1. [23] An operator $f : X \to X$ is called a Banach G-contraction or simply a G-contraction if

(i) f preserves edges of G; for each $x, y \in X$ with $(x, y) \in E(G)$, we have $(f(x), f(y)) \in E(G)$,

(ii) f decreases weights of edges of G; there exists $\alpha \in (0,1)$ such that for all $x, y \in X$ with $(x, y) \in E(G)$, we have $d(f(x), f(y)) \leq \alpha d(x, y)$.

If x and y are vertices of G, then a (directed) path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $\{x_n\}$ ($n \in \{0,1,2,...,k\}$) of vertices such that $x_0 = x$, $x_k = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i \in \{1,2,...,k\}$.

Notice that a graph G is connected if there is a (directed) path between any two vertices and it is weakly connected if $\widetilde{G}$ is connected, where $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Denote by G^{-1} the graph obtained from G reversing the direction of edges. Thus,

$$E\left(G^{-1}\right) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

It is more convenient to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric, under this convention; we have that

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

If G is such that $E(G)$ is symmetric, then for $x \in V(G)$, $[x]_G$ denotes the equivalence class of the relation R defined on $V(G)$ by the rule:

$$yRz \text{ if there is a path in } G \text{ from } y \text{ to } z.$$

If $f : X \to X$ is an operator. Set

$$X_f := \{x \in X : (x, f(x)) \in E(G)\}.$$

Jachymski [24] used the following property:
(P) : for any sequence $\{x_n\}$ in X, if $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$, then $(x_n, x) \in E(G)$.

Theorem 1.2. [24] Let (X, d) be a complete metric space, G a directed graph such that $V(G) = X$ and $f : X \to X$ a G-contraction. Suppose that $E(G)$ and the triplet (X, d, G) has the property (P). Then the following statements hold:

(1) f has a fixed point if and only if $X_f \neq \emptyset$;

(2) if $X_f \neq \emptyset$ and G is weakly connected, then f is a Picard operator;

(3) for any $x \in X_f$, $f\mid_{[x]_{\widetilde{G}}}$ is a Picard operator;

(4) if $X_f \times X_f \subseteq E(G)$, then f is a weakly Picard operator.

For detailed discussion on Picard operators, we refer to Berinde [13, 14, 15, 16].

Now we present some review about the topological structure of cone.

Definition 1.3. Let E be a real Banach space. A subset P of E is called a cone if and only if:
(i) P is nonempty, closed and $P \neq \{\theta\}$ (where θ is the zero element of E);
(ii) $a, b \in \mathbb{R}, a, b \geq 0$ and $x, y \in P$ implies that $ax + by \in P$;
(iii) $P \cap (-P) = \{\theta\}$.

Partial ordering on E is defined with help of a cone P as follows: $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$ and $x \ll y$ stands for $y - x \in intP$, where $intP$ denotes the interior of P. A cone P is normal or semi monotone if

$$\inf\{\| x + y \| : x, y \in P \text{ and } \| x \| = \| y \| = 1\} > 0 \tag{1.1}$$

or equivalently, if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \leq x \leq y \text{ implies that } \|x\| \leq K \|y\| .$$

The least positive number satisfying the above inequality is called a normal constant of P. If $x = (x_1, ..., x_n)^T, y = (y_1, ..., y_n)^T \in \mathbb{R}^n$, then $a \leq b$ means that $a_i \leq b_i$, $i = 1, ..., n$. In this case, the set $P = \{x = (x_1, ..., x_n)^T \in \mathbb{R}^n : x_i \geq 0$ for $i = 1, 2, ..., n\}$ is a normal cone with the normal constant $K = 1$.

Example 1.4. [36] Consider the normed space $E = (C^2_{\mathbb{R}}([0, 1]), \| \, . \, \|)$ with

$$\|f\| = \max_{0 \leq t \leq 1} (|f(t)| + \left|f'(t)\right|).$$

and the cone $P = \{f \in E : f \geq 0\}$. For each $n \geq 1$, define self mappings f and g on E by $f(x) = x$ and $g(x) = x^{2n}$. Then, $0 \preceq g \preceq f$, $\| f \| = 2$ and $\| g \| = 2n + 1$. There is no $K > 0$ such that $\|g\| \leq K \|f\|$ holds for all $n \geq 1$. Therefore P is a non-normal cone [36].

A cone P is called regular if every bounded above increasing sequence in E is convergent, or equivalently a cone P is regular if every decreasing sequence which is bounded below is convergent.

A selfmapping f on E is said to be nonincreasing (a) if for any $x, y \in E$ with $x \preceq y$ we have $f(x) \succeq f(y)$ (b) nondecreasing if for any $x, y \in E$ with $x \preceq y$ implies that $f(x) \preceq f(y)$.

Unless or otherwise stated, it is assumed that E is a Banach space, P is a cone in E with $intP \neq \emptyset$ and $\preceq$ is partial ordering on E induced by P.

Definition 1.5. [21] Let X be a nonempty set. A mapping $d : X \times X \to E$ is said to be a cone metric on X if for any $x, y, z \in X$, the following conditions hold:

d$_1$ $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

d$_2$ $d(x, y) = d(y, x)$;

d$_3$ $d(x, y) \preceq d(x, z) + d(y, z)$.

The pair (X, d) is called a cone metric space.

If $E = \mathbb{R}^n$, then a nonempty set X with a vector-valued metric d is called a generalized metric.

The concept of a cone metric space is more general than that of a metric space.

Example 1.6 [36] Suppose that $E = \ell^1$ and $P = \{\{x_n\}_{n\in\mathbb{N}} \in E : x_n \geq 0 \text{ for all } n\}$ and (X, δ) is a metric space. Define a mapping $d : X \times X \to E$ by

$$d(x, y) = \{\frac{\delta(x, y)}{2^n}\}_{n\in\mathbb{N}}.$$

Then (X, d) is a cone metric space.

Example 1.7. If a generalized metric on $\mathbb{R}$ is given by $d(x, y) = (\mid x - y \mid, k_1 \mid x - y \mid, ..., k_{n-1} \mid x - y \mid)$, then it is a cone metric on X, where $k_i \geq 0$ for all $\{i = 1, 2, ...n - 1\}$.

Example 1.8. Let $E = C^1[0, 1]$ with $\| f \| = \| f \|_\infty + \| f^{'} \|_\infty$ on $P = \{x \in E : x(t) \geq 0 \text{ on } [0, 1]\}$, where $f^{'}$ denotes the derivative of f. This cone is not normal. Consider for example,

$$f_n(t) = \frac{1 - \sin nt}{n + 2} \text{ and } g_n(t) = \frac{1 + \sin nt}{n + 2}.$$

Since, $\| f_n \| = \| g_n \| = 1$ and $\| f_n + g_n \| = \frac{2}{n+2} \longrightarrow 0$, it follows from (1.1) that P is non-normal.

Definition 1.9. Let X be a cone metric space, $c \in E$ with $0 \ll c$. A sequence $\{x_n\}$ in X is called:

(i) Cauchy sequence if there is an $\mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m > \mathbb{N}$.

(ii) Convergent if there exist an $\mathbb{N}$ and $x \in X$ such that $d(x_n, x) \ll c$ for all $n > \mathbb{N}$.

The limit of a convergent sequence is unique.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

If the cone is normal then a sequence $\{x_n\}$ converges to a point $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$ ([21], [25], [26]).

A subset A of X is closed if and only if every convergent sequence in A has its limit in A. A set $V \subset E$ is said to be symmetric if $x \in V$ implies that $-x \in V$, that is, $-V = V$.

Definition 1.10. Let (X, d) be a cone metric space. We say that two sequences $\{x_n\}$ and $\{y_n\}$ in X are equivalent if for every $c \in E$ with $\theta \ll c$, there exists a natural number $\mathbb{N}$ such that $d(x_n, y_n) \ll c$ for all $n \geq \mathbb{N}$. Furthermore, if each of them is Cauchy sequence, then they are called Cauchy equivalent.

Remark 1.11. Let (X, d) be a cone metric space, $\{x_n\}$ and $\{y_n\}$ equivalent sequences in X. Then

(i) if $\{x_n\}$ converges to $x \in X$, then $\{y_n\}$ also converges to x and vice versa,

(ii) if $\{x_n\}$ is a Cauchy sequence, then $\{y_n\}$ is a Cauchy sequence and vice versa.

Let (X,d) be a cone metric space. Then we have the following properties:

(1) If $u \preceq v$ and $v \ll w$ then $u \ll w$.

(2) If $0 \preceq u \ll c$ for each $c \in intP$, then $u = 0$.

(3) If $a \preceq b + c$ for each $c \in intP$, then $a \preceq b$.

(4) If $0 \preceq x \preceq y$, and $a \geq 0$, then $0 \preceq ax \preceq ay$.

(5) If $a \preceq ha$ for all $a \in P$ and $h \in (0,1)$, then $a = 0$.

(6) If $0 \preceq x_n \preceq y_n$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, then $x \preceq y$.

(7) If $0 \preceq d(x_n, x_m) \preceq b_n$ for all $m > n$ and $b_n \to 0$ as $n \to \infty$, then $\{x_n\}$ is a Cauchy sequence. Also if $0 \preceq d(x_n, x) \preceq b_n$ and $b_n \to 0$, then $x_n \to x$.

(8) If $c \in intP$, $0 \preceq a_n$ and $a_n \to 0$, then there exists n_0 such that for all $n > n_0$ we have $a_n \ll c$.

From (7) it follows that the sequence $\{x_n\}$ converges to $x \in X$ if $d(x_n, x) \to 0$ as $n \to \infty$ and $\{x_n\}$ is a Cauchy sequence if $d(x_n, x_m) \to 0$ as $n, m \to \infty$. In the situation with a non-normal cone we have only one part of Lemmas 1 and 4 in [21]. Also, in this case the fact that $d(x_n, y_n) \to d(x, y)$ if $x_n \to x$ and $y_n \to y$ is not applicable.

For further details of these properties, we refer to [20].

Lemma 1.12. [36] Let (X,d) be a cone metric space over a cone P in E. Then one has the following.

(a) $Int(P) + Int(P) \subseteq Int(P)$ and $\lambda Int(P) \subseteq Int(P), \lambda > 0$.

(b) If $c \gg 0$, then there exists $\delta > 0$ such that $\| b \| < \delta$ implies $b \ll c$.

(c) For any given $c \gg 0$ and $c_0 \gg 0$ there exists $n_0 \in \mathbb{N}$ such that $\frac{c_0}{n_0} \ll c$.

(d) If a_n, b_n are sequences in E such that $a_n \to a$, $b_n \to b$ and $a_n \leq b_n$ for all $n \geq 1$, then $a \leq b$.

Let $M_{n\times n}(\mathbb{R}^+)$ be the set of all $n \times n$ matrices with non negative elements. It is well known that if A is any square matrix of order n, then $A(P) \subset P$ if and only if $A \in M_{n,n}(\mathbb{R}^+)$. A matrix $A \in M_{n,n}(\mathbb{R}^+)$ is said to be convergent to zero if $A^n \longrightarrow \Theta$ as $n \longrightarrow \infty$, where Θ is the null matrix of size n.

Regarding this class of matrices we have the following classical result in matrix analysis (see [34], [38] and [39]).

Theorem 1.13. Let $A \in M_{n,n}(\mathbb{R}^+)$. The following statements are equivalent:

i) $A^n \to \Theta$, as $n \to \infty$;

ii) the eigenvalues of A lies in the open unit disc, that is, $|\lambda| < 1$, for all $\lambda \in C$ with $\det(A - \lambda I_n) = 0$;

iii) the matrix $I_n - A$ is non-singular and $(I_n - A)^{-1} = I_n + A + A^2 + ... + A^m + ...$;

iv) the matrix $(I_n - A)$ is non-singular and $(I_n - A)^{-1}$ has nonnegative elements;

v) the Av and $v^t A$ converges to zero for each $v \in \mathbb{R}^+$.

Perov [33] obtained the following generalization of a Banach contraction principle.

Theorem 1.14. Let (X, d) be a complete generalized metric space, $f : X \to X$ and $A \in M_{n,n}(\mathbb{R}^+)$ a matrix convergent to zero. If for any $x, y \in X$, we have

$$d(f(x), f(y)) \leq A(d(x, y)).$$

Then the following statements hold:

1. f has a unique fixed point $x^* \in X$;

2. The Picard iterative sequence $x_n = f^n(x_0)$, $n \in \mathbb{N}$ converges to x^* for all $x_0 \in X$;

3. $d(x_n, x^*) \leq A^n(I_n - A)^{-1}(d(x_0, x_1))$, $n \in \mathbb{N}$;

4. if $g : X \to X$ satisfies the condition $d(f(x), g(x)) \leq c$ for all $x \in X$ and some $c \in \mathbb{R}^n$, then for the sequence $y_n = g^n(x_0)$, $n \in \mathbb{N}$, the following inequality

$$d(y_n, x^*) \leq (I_n - A)^{-1}(c) + A^n(I_n - A)^{-1}(d(x_0, x_1))$$

is valid for all $n \in \mathbb{N}$.

The role of vector valued norm is important in the study of semi linear operator systems. For details, we refer to [34].

We write $\mathcal{B}(E)$ for the set of all bounded linear operators on E and $L(E)$ for the set of all linear operators on E.

$\mathcal{B}(E)$ is a Banach algebra, and if $A \in \mathcal{B}(E)$ let

$$r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = \inf_n \|A^n\|^{\frac{1}{n}}$$

be the spectral radius of A. We write $\mathcal{B}(E)^{-1}$ for the set of all invertible elements in $\mathcal{B}(E)$. Let us remark that if $r(A) < 1$, then

1. Series $\sum\limits_{n=0}^{\infty} A^n$ is absolutely convergent;

2. $I - A$ is invertible in $\mathcal{B}(E)$.

$$\sum_{n=0}^{\infty} A^n = (I - A)^{-1}.$$

If $A, B \in \mathcal{B}(E)$ and $AB = BA$ then $r(AB) \leq r(A)r(B)$.
If $A \in \mathcal{B}(E)$ and $A^{-1} \in \mathcal{B}(E)$ exists, then $r(A^{-1}) = 1/r(A)$.
Furthermore,if $\| A \| < 1$, then $I - A$ is invertible and

$$\left\|(I - A)^{-1}\right\| \leq \frac{1}{1 - \|A\|}.$$

The above is known as Geometric series theorem. Note that $r(A) \leq \|A\|$.

Remark 1.15. [20] Let X be a cone metric space, $P \subseteq E$ cone in E and $A : E \to E$ a linear operator. The following conditions are equivalent:

1. A is increasing, that is, $x \preceq y$ implies that $A(x) \preceq A(y)$;
2. A is positive, that is, $A(P) \subset P$.

Remark 1.16. Let $P \subseteq E$ be a cone in E and $A : E \to E$ a linear operator with $\|A\| < 1$ and $A\,(P) \subset P$. If for
(a) for any u in P, we have

$$u \preceq A(u), \tag{1.2}$$

then $u = 0$.
(b) for any u, v in P, we have

$$u \preceq A(\frac{u+v}{2}) = \frac{1}{2}A\,(u) + \frac{1}{2}A\,(v)\,, \tag{1.3}$$

then $u \preceq A\,(v)$.

Proof. To prove (a), from equation (1.2), we have

$$u \preceq (I - A)^{-1}(0) = 0$$

implies $u = 0$.
To prove (b), assume on contrary that $u \succ A\,(v)$. Then from (1.3), we have

$$u \preceq \frac{1}{2}A\,(u) + \frac{1}{2}A\,(v) \prec \frac{1}{2}A\,(u) + \frac{1}{2}u$$

which further implies that

$$u \prec A\,(u)\,.$$

Using (a), we get that $u = 0$, a contradiction. ∎

Latif and Beg [28] introduced a notion of $K-$ multivalued mapping and extended fixed point results for Kannan mapping to multivalued mappings. Rus [37] coined the term $R-$ multivalued mapping which is a generalization of $K-$ multivalued mapping. Abbas and Rhoades [5] introduced the notion of a

generalized $R-$ multivalued mappings, which in turn generalizes $R-$ multivalued mappings and obtained common fixed point results for such mappings.
Let (X, d) be a cone metric space. Denote by $P(X)$ the family of all nonempty subsets of X, by $P_{cl}(X)$ the family of all nonempty closed subset of X. Throughout this paper, we assume that each vertex $x \in X$ is labelled with a zero vector $d(x, x) = 0$ and each edge having vertices x and y is labelled with a unique vector $d(x, y) \in E$ so that the graph is properly labeled.
A point x in X is a fixed point of a multivalued mapping $T : X \to P(X)$ iff $x \in Tx$. The set of all fixed points of multivalued mapping T is denoted by $Fix(T)$.

Suppose that $T_1, T_2 : X \to P_{cl}(X)$. Set

$$X_{T_1,T_2} := \{x \in X : (x, u_x) \in E(G) \text{ where } u_x \in T_1(x) \cap T_2(x)\}.$$

Now we give the following definition:
Definition 1.17. Let $T_1, T_2 : X \to P_{cl}(X)$ be two multivalued mappings. Suppose that for every vertex x in G and for every $u_x \in T_i(x), i \in \{1, 2\}$ we have $(x, u_x) \in E(G)$. A pair (T_1, T_2) is said to form:

(I) a cone graphic P_1-contraction pair if there exists a linear bounded operator $A : E \to E$ with $\|A\| < 1$ and $A(P) \subset P$ such that for any $x, y \in X$ with $(x, y) \in E(G)$ and $u_x \in T_i(x)$, there exists $u_y \in T_j(y)$ for $i, j \in \{1, 2\}$ with $i \neq j$ such that $(u_x, u_y) \in E(G)$ and

$$d(u_x, u_y) \preceq A(M_1(x, y; u_x, u_y)), \tag{1.4}$$

hold, where

$$\begin{aligned} M_1(x, y; u_x, u_y) \in \; & \{d(x, y), d(x, u_x), d(y, u_y), \\ & \frac{d(x, u_x) + d(y, u_y)}{2}, \frac{d(x, u_y) + d(y, u_x)}{2}\}. \end{aligned}$$

(II) a cone graphic P_2-contraction pair if there exist linear bounded operators $A_k : E \to E$ for $k = 1, 2, ..., 5$ with $\sum_{k=1}^{5} \|A_k\| < 1$, $A_k(P) \subset P$ for $k = 1, 2, ..., 5$ and $A_4(v) \leq A_5(v)$ for all $v \in P$ such that for any $x, y \in X$ with $(x, y) \in E(G)$ and $u_x \in T_i(x)$, there exists $u_y \in T_j(y)$ for $i, j \in \{1, 2\}$ with $i \neq j$ such that $(u_x, u_y) \in E(G)$ and

$$d(u_x, u_y) \preceq M_2(x, y; u_x, u_y), \tag{1.5}$$

hold, where

$$\begin{aligned} M_2(x, y; u_x, u_y) = \; & A_1(d(x, y)) + A_2(d(x, u_x)) + A_3(d(y, u_y)) \\ & + A_4(d(x, u_y)) + A_5(d(y, u_x)). \end{aligned}$$

A clique in an undirected graph $G = (V, E)$ is a subset of the vertex set $W \subset V$, such that for every two vertices in W, there exists an edge connecting the two. This is equivalent to saying that the subgraph induced by W is complete, that is, for every $x, y \in W(G)$, we have $(x, y) \in E(G)$.

3 Common fixed point results

In this section, we obtain several common fixed point results for two Perov type multivalued mappings on a cone metric space endowed with a directed graph. We start with the following result.

Theorem 2.1. Let (X, d) be a cone complete metric space endowed with a directed graph G such that $V(G) = X$ and $E(G) \supseteq \Delta$. If mappings $T_1, T_2 : X \to P_{cl}(X)$ form a cone graphic P_1-contraction pair, then following statements hold:

(i). $Fix(T_1) \neq \emptyset$ or $Fix(T_2) \neq \emptyset$ if and only if $Fix\,(T_1) = Fix\,(T_2) \neq \emptyset$.

(ii). $X_{T_1,T_2} \neq \emptyset$ provided that $Fix\,(T_1) \cap Fix\,(T_2) \neq \emptyset$.

(iii). If $X_{T_1,T_2} \neq \emptyset$ and G is weakly connected, then $Fix\,(T_1) = Fix\,(T_2) \neq \emptyset$ provided that graph G has property (P).

(iv). $Fix\,(T_1) \cap Fix\,(T_2)$ is a clique of $\widetilde{G}$ if and only if $Fix\,(T_1) \cap Fix\,(T_2)$ is a singleton.

Proof. To prove (i), let $x^* \in T_1(x^*)$. As (T_1, T_2) form a cone graphic P_1-contraction pair, there exists an $x \in T_2\,(x^*)$ with $(x^*, x) \in E\,(G)$ such that

$$d(x^*, x) \preceq A(M_1(x^*, x^*; x^*, x)),$$

where

$$\begin{aligned} M_1(x^*, x^*; x^*, x) &\in \{d(x^*, x^*), d(x^*, x^*), d(x, x^*), \\ &\quad \frac{d(x^*, x^*) + d(x, x^*)}{2}, \frac{d(x^*, x) + d(x^*, x^*)}{2}\} \\ &= \{0, d(x, x^*), \frac{d(x, x^*)}{2}\}. \end{aligned}$$

Now $M_1(x^*, x^*; x^*, x) = 0$ implies that $x^* = x$ and $M_1(x^*, x^*; x^*, x) = d(x^*, x)$ gives

$$d(x^*, x) \preceq A(d(x^*, x)),$$

which by Remark 1.16 (a) implies that $x^* = x$. Similarly, for $M_1(x^*, x^*; x^*, x) = \frac{d(x^*,x)}{2}$, we obtain that $x^* = x$. Hence $x^* \in T_2\,(x^*)$ and so $Fix(T_1) \subseteq Fix(T_2)$. Similarly, $Fix(T_2) \subseteq Fix(T_1)$ and therefore $Fix(T_1) = Fix(T_2)$. Also, if $x^* \in T_2(x^*)$, then we have $x^* \in T_1(x^*)$. The converse is straightforward.

To prove (ii), let $Fix\,(T_1) \cap Fix\,(T_2) \neq \emptyset$. Then there exists $x \in X$ such that $x \in T_1(x) \cap T_2(x)$. As $\Delta \subseteq E(G)$, we conclude that $X_{T_1,T_2} \neq \emptyset$.

To prove (iii), Suppose that x_0 is an arbitrary point of X. If $x_0 \in T_1\,(x_0)$ or $x_0 \in T_2\,(x_0)$, then by (i), the proof is finished. So we assume that $x_0 \notin T_i\,(x_0)$ for $i \in \{1, 2\}$. Now for $i, j \in \{1, 2\}$ with $i \neq j$, if $x_1 \in T_i(x_0)$, then there exists $x_2 \in T_j(x_1)$ with $(x_1, x_2) \in E(G)$ such that

$$d(x_1, x_2) \preceq A(M_1(x_0, x_1; x_1, x_2)),$$

where

$$\begin{aligned} M_1(x_0,x_1;x_1,x_2) &\in \{d(x_0,x_1), d(x_0,x_1), d(x_1,x_2), \\ &\quad \frac{d(x_0,x_1)+d(x_1,x_2)}{2}, \frac{d(x_0,x_2)+d(x_1,x_1)}{2}\} \\ &= \{d(x_0,x_1), d(x_1,x_2), \frac{d(x_0,x_1)+d(x_1,x_2)}{2}, \frac{d(x_0,x_2)}{2}\}. \end{aligned}$$

Now, $M_1(x_0,x_1;x_1,x_2) = d(x_0,x_1)$ implies that $d(x_1,x_2) \preceq A\left(d(x_0,x_1)\right)$. If $M_1(x_0,x_1;x_1,x_2) = d(x_1,x_2)$ then $d(x_1,x_2) \preceq A\left(d(x_1,x_2)\right)$, which by Remark 1.16 (a), implies that $x_1 = x_2$, that is, $x_1 \in T_j(x_1)$ and by (i), the prove is finished. If

$$M_1(x_0,x_1;x_1,x_2) = \frac{d(x_0,x_1)+d(x_1,x_2)}{2},$$

then we obtain

$$d(x_1,x_2) \preceq \frac{1}{2}A\left(d(x_0,x_1)\right) + \frac{1}{2}A\left(d(x_1,x_2)\right),$$

which by Remark 1.16 (b), implies that $d(x_1,x_2) \preceq A\left(d(x_0,x_1)\right)$. Finally, for $M_1(x_0,x_1;x_1,x_2) = \dfrac{d(x_0,x_2)}{2}$, we get

$$\begin{aligned} d(x_1,x_2) &\preceq \frac{1}{2}A\left(d(x_0,x_2)\right) \\ &\preceq \frac{1}{2}A\left(d(x_0,x_1)\right) + \frac{1}{2}A\left(d(x_1,x_2)\right) \end{aligned}$$

and again by Remark 1.16 (b), we have $d(x_1,x_2) \preceq A\left(d(x_0,x_1)\right)$.

Continuing this way, for $x_{2n} \in T_j(x_{2n-1})$, there exist $x_{2n+1} \in T_i\left(x_{2n}\right)$ with $(x_{2n}, x_{2n+1}) \in E\left(G\right)$ such that

$$d(x_{2n},x_{2n+1}) \preceq A\left(M_1(x_{2n-1},x_{2n};x_{2n},x_{2n+1})\right),$$

where

$$\begin{aligned} M_1(x_{2n-1},x_{2n};x_{2n},x_{2n+1}) &\in \{d(x_{2n-1},x_{2n}), d(x_{2n-1},x_{2n}), \\ &\quad d(x_{2n},x_{2n+1}), \frac{d(x_{2n-1},x_{2n})+d(x_{2n},x_{2n+1})}{2}, \\ &\quad \frac{d(x_{2n-1},x_{2n+1})+d(x_{2n},x_{2n})}{2}\} \\ &= \{d(x_{2n-1},x_{2n}), d(x_{2n},x_{2n+1}), \\ &\quad \frac{d(x_{2n-1},x_{2n})+d(x_{2n},x_{2n+1})}{2}, \frac{d(x_{2n-1},x_{2n+1})}{2}\}. \end{aligned}$$

If $M_1(x_{2n-1},x_{2n};x_{2n},x_{2n+1}) = d(x_{2n-1},x_{2n})$, then $d(x_{2n},\, x_{2n+1}) \preceq A(d(x_{2n-1},\, x_{2n}))$. For $M_1(x_{2n-1},x_{2n};x_{2n},x_{2n+1}) = d(x_{2n},x_{2n+1})$, $d(x_{2n},\, x_{2n+1}) \preceq A(d(x_{2n},x_{2n+1}))$, which by Remark 1.16 (a) gives $x_{2n} = x_{2n+1}$. When $M_1(x_{2n-1},x_{2n};x_{2n},x_{2n+1}) = \dfrac{d(x_{2n-1},x_{2n})+d(x_{2n},x_{2n+1})}{2}$, we obtain

$$\begin{aligned} d(x_{2n},x_{2n+1}) &\preceq \frac{1}{2}A(d(x_{2n-1},x_{2n})+d(x_{2n},x_{2n+1})) \\ &\preceq \frac{1}{2}A\left(d(x_{2n-1},x_{2n})\right) + \frac{1}{2}d(x_{2n},x_{2n+1}) \end{aligned}$$

and by Remark 1.16 (b), we have

$$d(x_{2n}, x_{2n+1}) \preceq A\left(d(x_{2n-1}, x_{2n})\right).$$

Finally $M_1(x_{2n-1}, x_{2n}; x_{2n}, x_{2n+1}) = d(x_{2n-1}, x_{2n+1})/2$ gives that

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &\preceq \frac{1}{2}A\left(d(x_{2n-1}, x_{2n+1})\right) \preceq \frac{1}{2}A(d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})) \\ &\preceq \frac{1}{2}A\left(d(x_{2n-1}, x_{2n})\right) + \frac{1}{2}d(x_{2n}, x_{2n+1}), \end{aligned}$$

which again by Remark 1.16 (b), implies that

$$d(x_{2n}, x_{2n+1}) \preceq A(d(x_{2n-1}, x_{2n})).$$

In a similar manner, for $x_{2n+1} \in T_j(x_{2n})$, there exists $x_{2n+2} \in T_i\left(x_{2n+1}\right)$ such that for $(x_{2n+1}, x_{2n+2}) \in E\left(G\right)$ implies

$$d(x_{2n+1}, x_{2n+2}) \preceq A\left(d(x_{2n}, x_{2n+1})\right).$$

Hence, we obtain a sequence $\{x_n\}$ in X such that for $x_n \in T_j(x_{n-1})$, there exists $x_{n+1} \in T_i\left(x_n\right)$ with $(x_n, x_{n+1}) \in E\left(G\right)$ and it satisfies

$$d(x_n, x_{n+1}) \preceq A\left(d(x_{n-1}, x_n)\right).$$

Therefore

$$\begin{aligned} d(x_n, x_{n+1}) &\preceq A\left(d(x_{n-1}, x_n)\right) \preceq A^2\left(d(x_{n-2}, x_{n-2})\right) \\ &\preceq .... \preceq A^n(d(x_0, x_1)) \end{aligned}$$

for all $n \geq 1$. Now for $m, n \in \mathbb{N}$ with $m > n$, we obtain that

$$\begin{aligned} d(x_n, x_m) &\preceq d(x_n, x_{n+1}) + \ldots + d(x_{m-1}, x_m) \\ &\preceq [A^n + A^{n+1} + \ldots + A^{m-1}](d(x_0, x_1)) \\ &\preceq A^n\left(I - A\right)^{-1}\left(d(x_0, x_1)\right). \end{aligned}$$

Let $c \gg 0$. Choose $\delta > 0$ such that $c + N_\delta(\theta) \subseteq P$, where $N_\delta(\theta) = \{x \in E : \|x\| < \delta\}$. Also, choose $N_1 \in \mathbb{N}$ such that $A^n(I - A)^{-1}(d(x_0, x_1)) \in N_\delta(\theta)$ for all $n > N_1$. Thus for all $m > n > N_1$,

$$d(x_n, x_m) \preceq A^n(I - A)^{-1}(d(x_0, x_1)) \ll c$$

implies that $\{x_n\}$ is a Cauchy sequence. By completeness of X, there exists an element $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.
Let $0 \ll c$ be given. Choose a natural number N such that $d(x_m, x^*) \ll c$ for all $m \geq N$.
Since $\{x_{2n}\}$ converges to x^* as $n \to \infty$ and $(x_{2n}, x_{2n+1}) \in E\left(G\right)$, we have $(x_{2n}, x^*) \in E\left(G\right)$. For $x_{2n} \in T_j\left(x_{2n-1}\right)$, there exists $u_n \in T_i\left(x^*\right)$ such that $(x_{2n}, u_n) \in E\left(G\right)$. Since (T_1, T_2) form a graphic P_1-contraction,

$$d(x_{2n}, u_n) \preceq A(M_1(x_{2n-1}, x^*; x_{2n}, u_n)),$$

where

$$\begin{aligned} M_1(x_{2n-1}, x^*; x_{2n}, u_n) \in & \{d(x_{2n-1}, x^*), d(x_{2n-1}, x_{2n}), d(x^*, u_n), \\ & \frac{d(x_{2n-1}, x_{2n}) + d(x^*, u_n)}{2}, \\ & \frac{d(x_{2n-1}, u_n) + d(x^*, x_{2n})}{2}\}. \end{aligned}$$

Note that

$$\begin{aligned} d(u_n, x^*) & \preceq d(u_n, x_{2n}) + d(x_{2n}, x^*) \\ & \preceq A(M_1(x_{2n}, x^*; x_{2n+1}, u_n)) + d(x_{2n}, x^*). \end{aligned}$$

Now, $M_1(x_{2n}, x^*; x_{2n+1}, u_n) = d(x_{2n-1}, x^*)$ implies that

$$\begin{aligned} d(u_n, x^*) & \preceq A(d(x_{2n-1}, x^*)) + d(x_{2n}, x^*) \\ & \ll A(c) + c. \end{aligned}$$

As $c \gg 0$ is arbitrary, for $m \geq 1$

$$\begin{aligned} d(u_n, x^*) & \preceq A(\frac{c}{m}) + \frac{c}{m} \\ & = \frac{A(c)}{m} + \frac{c}{m} \to 0 \end{aligned}$$

as $m \to \infty$. If $M_1(x_{2n}, x^*; x_{2n+1}, u_n) = d(x_{2n-1}, x_{2n})$, then

$$\begin{aligned} d(u_n, x^*) & \preceq A\left(d(x_{2n-1}, x_{2n})\right) + d(x_{2n}, x^*) \\ & \preceq A\left(d(x_{2n-1}, x^*)\right) + A\left(d(x^*, x_{2n})\right) + d(x_{2n}, x^*) \\ & \preceq A(c) + A(c) + c, \end{aligned}$$

where $c \gg 0$ is arbitrary. For $m \geq 1$

$$\begin{aligned} d(u_n, x^*) & \preceq A(\frac{c}{m}) + A(\frac{c}{m}) + \frac{c}{m} \\ & = \frac{A(c)}{m} + \frac{A(c)}{m} + \frac{c}{m} \to 0 \end{aligned}$$

as $m \to \infty$. In case $M_1(x_{2n}, x^*; x_{2n+1}, u_n) = d(x^*, u_n)$, we have

$$d(u_n, x^*) \preceq A\left(d(x^*, u_n)\right) + d(x_{2n}, x^*)$$

and so

$$\begin{aligned} d(u_n, x^*) & \preceq (I - A)^{-1} A\left(d(x_{2n}, x^*)\right) \\ & \preceq (I - A)^{-1} A(c), \end{aligned}$$

where $c \gg 0$ is arbitrary. For $m \geq 1$

$$\begin{aligned} d(u_n, x^*) & \preceq (I - A)^{-1} A(\frac{c}{m}) \\ & = \frac{1}{m} (I - A)^{-1} A(c) \to 0 \end{aligned}$$

as $m \to \infty$. If $M_1(x_{2n}, x^*; x_{2n+1}, u_n) = \dfrac{d(x_{2n-1}, x_{2n}) + d(x^*, u_n)}{2}$, we get

$$\begin{aligned} d(u_n, x^*) &\preceq \frac{1}{2}A\left(d(x_{2n-1}, x_{2n}) + d(x^*, u_n)\right) + d(x_{2n}, x^*) \\ &\preceq \frac{1}{2}A\left(d(x_{2n-1}, x^*) + d(x^*, x_{2n})\right) + \frac{1}{2}d(x^*, u_n) + d(x_{2n}, x^*) \end{aligned}$$

and so

$$\begin{aligned} d(u_n, x^*) &\preceq A\left(d(x_{2n-1}, x^*) + d(x^*, x_{2n})\right) + 2d(x^*, x_{2n}) \\ &\preceq A\left(c + c\right) + 2c. \end{aligned}$$

As $c \gg 0$ is arbitrary, for $m \geq 1$

$$\begin{aligned} d(u_n, x^*) &\preceq A(\frac{2c}{m}) + \frac{2c}{m} \\ &= \frac{2}{m}A\left(c\right) + \frac{2c}{m} \to 0 \end{aligned}$$

as $m \to \infty$. Finally, if $M_1(x_{2n}, x^*; x_{2n+1}, u_n) = \dfrac{d(x_{2n-1}, u_n) + d(x^*, x_{2n})}{2}$, then

$$\begin{aligned} d(u_n, x^*) &\preceq \frac{1}{2}A(d(x_{2n-1}, u_n) + d(x^*, x_{2n})) + d(x_{2n}, x^*) \\ &\preceq \frac{1}{2}A(d(x_{2n-1}, x^*) + d(x^*, u_n)) + \frac{1}{2}A\left(d(x^*, x_{2n})\right) + d(x_{2n}, x^*), \end{aligned}$$

that is,

$$\begin{aligned} d(u_n, x^*) &\preceq \frac{1}{2}(I - \frac{1}{2}A)^{-1}[\frac{1}{2}A(d(x_{2n-1}, x^*) + \frac{1}{2}A\left(d(x^*, x_{2n})\right) + d(x_{2n}, x^*)] \\ &\preceq \frac{1}{2}(I - \frac{1}{2}A)^{-1}[\frac{1}{2}A(c) + \frac{1}{2}A(c) + c], \end{aligned}$$

where $c \gg 0$ is arbitrary. For $m \geq 1$

$$\begin{aligned} d(u_n, x^*) &\preceq \frac{1}{2}(I - \frac{1}{2}A)^{-1}[\frac{1}{2}A(\frac{c}{m}) + \frac{1}{2}A(\frac{c}{m}) + \frac{c}{m}] \\ &= \frac{1}{2m}(I - \frac{1}{2}A)^{-1}[\frac{1}{2}A(c) + \frac{1}{2}A(c) + c] \to 0 \end{aligned}$$

as $m \to \infty$. Thus $u_n \to x^*$ as $n \to \infty$. Since $T_i(x^*)$ is closed, $x^* \in F(T_j) = F(T_i)$.
Finally to prove (iv), suppose the set $Fix\,(T_1) \cap Fix\,(T_2)$ is a clique of $\widetilde{G}$. We are to show that $Fix\,(T_1) \cap Fix\,(T_2)$ is singleton. Suppose that there exist u and v such that $u, v \in Fix\,(T_1) \cap Fix\,(T_2)$. As $(u, v) \in E(G)$ and T_1 and T_2 form a graphic P_1-contraction, so for $(u, v) \in E\,(G)$ implies

$$d(u, v) \preceq A(M_1(u, v; u, v)),$$

where

$$\begin{aligned} M_1(u, v; u, v) &\in \{d(u, v), d(u, u), d(v, v), \frac{d\,(u, u) + d\,(v, v)}{2}, \frac{d\,(u, v) + d\,(v, u)}{2}\} \\ &= \{d(u, v), 0\}. \end{aligned}$$

If $M_1(u,v;u,v) = d(u,v)$, then by Remark 1.16 (a) we have $u = v$. Similarly, for $M_1(u,v;u,v) = 0$, we obtain $u = v$. Conversely, if $Fix(T_1) \cap Fix(T_2)$ is singleton, then it follows that $Fix(T_1) \cap Fix(T_2)$ is a clique of $\widetilde{G}$. ∎

Example 2.2. Let $E = \mathbb{R}^2$, $P = \{(x,y) \in \mathbb{R}^2 : x, y \geq 0\}$, and $\|x\| = \max\{|x_1|, |x_2|\}$, where $x = (x_1, x_2) \in E$. Suppose that $X = \{(x,0) \in \mathbb{R}^2 : x \geq 0\} \cup \{(0,x) \in \mathbb{R}^2 : x \geq 0\}$ and define $d : X \times X \to E$ by:

$$\begin{aligned}
d((x,0),(y,0)) &= (\frac{4}{3}|x-y|, |x-y|), \\
d((0,x),(0,y)) &= (|x-y|, \frac{2}{3}|x-y|), \text{ and} \\
d((x,0),(0,y)) &= d((0,y),(x,0)) = (\frac{4}{3}x + y, x + \frac{2}{3}y)
\end{aligned}$$

Note that (X,d) is a complete cone metric space [21]. Consider a graph G with $V(G) = X$ and

$$\begin{aligned}
E(G) = {} & \{((0,0),(0,0))\} \cup \left\{((0,\frac{1}{2}),(0,0))\right\} \cup \left\{((0,\frac{1}{2}),(0,\frac{1}{4}))\right\} \\
& \cup \left\{((\frac{1}{2},0),(0,0))\right\} \cup \left\{((\frac{1}{2},0),(\frac{1}{4},0))\right\}.
\end{aligned}$$

Define a mapping $T_1, T_2 : X \to P_{cl}(X)$ by

$$T_1(x,y) = \begin{cases} \{(0,x)\} & \text{if } y = 0, \\ \{(\frac{x}{2},0) : x \geq 0\} & \text{if } y \neq 0 \end{cases}$$

$$T_2(x,y) = \begin{cases} \{(0,x)\} & \text{if } y = 0, \\ \{(\frac{x}{4},0) : x \geq 0\} & \text{if } y \neq 0. \end{cases}$$

First, we show that for $x, y \in X$ with $(x,y) \in E(G)$ and $u_x \in T_1(x)$, there exists $u_y \in T_2(y)$ such that (1.4) is satisfied. We consider the following cases:

(i) If $x = y = (0,0)$, then (1.4) is satisfied obviously as $u_x = u_y = (0,0)$.

(ii) For $x = (0,\frac{1}{2})$, $y = (0,0)$ and $u_x = (0,0) \in T_1(x)$, take $u_y = (0,0) \in T_2(y)$.

(iii) When $x = (0,\frac{1}{2})$, $y = \left(0,\frac{1}{4}\right)$ and $u_x = (0,0) \in T_1(x)$, take $u_y = (0,0) \in T_2(y)$.

(iv) In case $x = (\frac{1}{2},0)$, $y = (0,0)$ and $u_x = \left(0,\frac{1}{2}\right) \in T_1(x)$, take $u_y = (0,0) \in T_2(x)$ with $(u_x, u_y) \in E(G)$, we have

$$d(u_x, u_y) = d((0,\frac{1}{2}),(0,0)) = (\frac{1}{2},\frac{1}{3})$$

$$d(x,y) = d((\frac{1}{2},0),(0,0)) = (\frac{2}{3},\frac{1}{2}).$$

Now

$$\begin{aligned} d\left(u_x, u_y\right) &= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}^t \\ &= \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{2} \end{bmatrix}^t = A(d\left(x, y\right)), \end{aligned}$$

where $d\left(x, u_x\right) \in M_1(x, y; u_x, u_y)$.

(v) For $x = (\frac{1}{2}, 0)$, $y = \left(\frac{1}{4}, 0\right)$ and $u_x = \left(0, \frac{1}{2}\right) \in T_1\left(x\right)$, take $u_y = \left(0, \frac{1}{4}\right) \in T_2\left(x\right)$ with $\left(u_x, u_y\right) \in E\left(G\right)$, we have

$$d\left(u_x, u_y\right) = d((0, \frac{1}{2}), (0, \frac{1}{4})) = (\frac{1}{4}, \frac{1}{6})$$

$$d\left(x, u_x\right) = d((\frac{1}{2}, 0), (0, \frac{1}{2})) = (\frac{11}{12}, \frac{2}{3}).$$

Now

$$\begin{aligned} d\left(u_x, u_y\right) &= \begin{bmatrix} \frac{1}{4} \\ \frac{1}{6} \end{bmatrix}^t \\ &\leq \begin{bmatrix} \frac{11}{16} \\ \frac{4}{9} \end{bmatrix}^t = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{11}{12} \\ \frac{2}{3} \end{bmatrix}^t = A(d\left(x, u_x\right)), \end{aligned}$$

where $d\left(x, u_x\right) \in M_1(x, y; u_x, u_y)$.

Now we show that for $x, y \in X$ with $(x, y) \in E\left(G\right)$, such that $u_x \in T_2\left(x\right)$, there exists $u_y \in T_1\left(y\right)$ such that (1.4) is satisfied. We consider the following cases.

(i) If $x = y = (0, 0)$, then (1.4) is satisfied obviously as $u_x = u_y = (0, 0)$.

(ii) For $x = (0, \frac{1}{2})$, $y = (0, 0)$ and $u_x = (0, 0) \in T_2\left(x\right)$, take $u_y = (0, 0)$.

(iii) When $x = (0, \frac{1}{2})$, $y = \left(0, \frac{1}{4}\right)$ and $u_x = (0, 0) \in T_2\left(x\right)$, take $u_y = (0, 0)$.

(iv) In case $x = (\frac{1}{2}, 0)$, $y = (0, 0)$ and $u_x = \left(0, \frac{1}{2}\right) \in T_2\left(x\right)$, take $u_y = (0, 0) \in T_1\left(x\right)$ with $\left(u_x, u_y\right) \in E\left(G\right)$, we have

$$d\left(u_x, u_y\right) = d((0, \frac{1}{2}), (0, 0)) = (\frac{1}{2}, \frac{1}{3})$$

$$d\left(x, u_x\right) = d((\frac{1}{2}, 0), (0, \frac{1}{2})) = (\frac{7}{6}, \frac{5}{6}).$$

Now

$$\begin{aligned} d\left(u_x, u_y\right) &= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}^t \\ &\leq \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}^t = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{2} \end{bmatrix}^t = A(d\left(x, u_x\right)), \end{aligned}$$

where $d\left(x, u_x\right) \in M_1(x, y; u_x, u_y)$.

(v) For $x = (\frac{1}{2}, 0)$, $y = \left(\frac{1}{4}, 0\right)$ and $u_x = \left(0, \frac{1}{2}\right) \in T_2\left(x\right)$, take $u_y = \left(0, \frac{1}{4}\right) \in T_1\left(x\right)$ with $\left(u_x, u_y\right) \in E\left(G\right)$, we have

$$d\left(u_x, u_y\right) = d((0, \frac{1}{2}), (0, \frac{1}{4})) = (\frac{1}{4}, \frac{1}{6})$$

$$\begin{aligned} d\left(x, u_x\right) &= d((\frac{1}{2}, 0), (0, \frac{1}{2})) \\ &= (\frac{7}{6}, \frac{5}{6}). \end{aligned}$$

Now

$$\begin{aligned} d\left(u_x, u_y\right) &= \begin{bmatrix} \frac{1}{4} \\ \frac{1}{6} \end{bmatrix}^t \\ &\leq \begin{bmatrix} \frac{7}{8} \\ \frac{5}{9} \end{bmatrix}^t = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{7}{6} \\ \frac{5}{6} \end{bmatrix}^t = A(d\left(x, u_x\right)), \end{aligned}$$

where $d\left(x, u_x\right) \in M_1(x, y; u_x, u_y)$.

Thus the pair (T_1, T_2) is form a cone graphic P_1-contraction with operator $A = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}$. Indeed $A^n \to \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\|A\| < 1$. So all the conditions of Theorem 2.1 are satisfied. Moreover, $(0, 0)$ is the fixed point of T_1 and T_2. ■

The following results generalizes Theorem 3.4 in [37].

Theorem 2.3. Let (X, d) be a complete cone metric space endowed with a directed graph G such that $V(G) = X$ and $E(G) \supseteq \Delta$. If $T_1, T_2 : X \to P_{cl}(X)$ form a cone graphic P_2-contraction pair, then following statements hold:

(i). $Fix(T_1) \neq \emptyset$ or $Fix(T_2) \neq \emptyset$ if and only if $Fix\left(T_1\right) = Fix\left(T_2\right) \neq \emptyset$.

(ii). $X_{T_1,T_2} \neq \emptyset$ provided that $Fix\left(T_1\right) \cap Fix\left(T_2\right) \neq \emptyset$.

(iii). If $X_{T_1,T_2} \neq \emptyset$ and G is weakly connected, then $Fix\left(T_1\right) = Fix\left(T_2\right) \neq \emptyset$ provided that G has property (P).

(iv). $Fix\left(T_1\right) \cap Fix\left(T_2\right)$ is a clique of $\widetilde{G}$ if and only if $Fix\left(T_1\right) \cap Fix\left(T_2\right)$ is a singleton.

Proof. To prove (i), let $x^* \in T_1(x^*)$. Assume $x^* \notin T_2(x^*)$, then since (T_1, T_2) form a cone graphic P_2-contraction pair, there exists an $x \in T_2(x^*)$ with $(x^*, x) \in E(G)$ such that

$$d(x^*, x) \preceq M_2(x^*, x^*; x^*, x),$$

where

$$\begin{aligned} M_2(x^*, x^*; x^*, x) &= A_1(d(x^*, x^*)) + A_2(d(x^*, x^*)) + A_3(d(x, x^*)) \\ &\quad + A_4(d(x^*, x)) + A_5(d(x^*, x^*)) \\ &= (A_3 + A_4)(d(x, x^*)). \end{aligned}$$

Thus we have

$$\begin{aligned} d(x^*, x) &\preceq (A_3 + A_4)(d(x, x^*)) \\ &\preceq A(d(x^*, x)), \end{aligned}$$

where $A = A_1 + A_2 + A_3 + A_4 + A_5$. By using Remark 1.16 (a), we obtain $x^* \in T_2(x^*)$ and so $Fix(T_1) \subseteq Fix(T_2)$. Similarly, $Fix(T_2) \subseteq Fix(T_1)$ and therefore $Fix(T_1) = Fix(T_2)$. Also, if $x^* \in T_2(x^*)$, then we have $x^* \in T_1(x^*)$. The converse is straightforward.
To prove (ii), let $Fix(T_1) \cap Fix(T_2) \neq \emptyset$. Then there exists $x \in X$ such that $x \in T_1(x) \cap T_2(x)$. Since $\Delta \subseteq E(G)$, we conclude that $X_{T_1,T_2} \neq \emptyset$.

To prove (iii), suppose that x_0 is an arbitrary point of X. For $i, j \in \{1, 2\}$, with $i \neq j$, take $x_1 \in T_i(x_0)$, there exists $x_2 \in T_j(x_1)$ with $(x_1, x_2) \in E(G)$ such that

$$d(x_1, x_2) \preceq M_2(x_0, x_1; x_1, x_2),$$

where

$$\begin{aligned} M_2(x_0, x_1; x_1, x_2) &= A_1(d(x_0, x_1)) + A_2(d(x_0, x_1)) + A_3(d(x_1, x_2)) \\ &\quad + A_4(d(x_0, x_2)) + A_5(d(x_1, x_1)) \\ &\preceq (A_1 + A_2 + A_4)(d(x_0, x_1)) + (A_3 + A_4)(d(x_1, x_2)). \end{aligned}$$

If $d(x_0, x_1) \preceq d(x_1, x_2)$, then we have

$$\begin{aligned} d(x_1, x_2) &\preceq (A_1 + A_2 + A_3 + 2A_4)(d(x_1, x_2)) \\ &\preceq A(d(x_1, x_2)), \end{aligned}$$

where $A = A_1 + A_2 + A_3 + A_4 + A_5$ and by Remark 1.16 (a) implies $x_1 = x_2$. Therefore

$$d(x_1, x_2) \preceq A\left(d(x_0, x_1)\right).$$

Continuing this process, for $x_{2n} \in T_j(x_{2n-1})$, there exists $x_{2n+1} \in T_i(x_{2n})$ such that for $(x_{2n}, x_{2n+1}) \in E(G)$, we have

$$d(x_{2n}, x_{2n+1}) \preceq M_2(x_{2n-1}, x_{2n}; x_{2n}, x_{2n+1}),$$

where

$$\begin{aligned} & M_2(x_{2n-1}, x_{2n}; x_{2n}, x_{2n+1}) \\ = \; & A_1\left(d(x_{2n-1}, x_{2n})\right) + A_2\left(d(x_{2n-1}, x_{2n})\right) + A_3\left(d(x_{2n}, x_{2n+1})\right) \\ & + A_4\left(d(x_{2n-1}, x_{2n+1})\right) + A_5\left(d(x_{2n}, x_{2n})\right) \\ \preceq \; & (A_1 + A_2 + A_4)\left(d(x_{2n-1}, x_{2n})\right) + (A_3 + A_4)\left(d(x_{2n}, x_{2n+1})\right). \end{aligned}$$

If $d(x_{2n-1}, x_{2n}) \preceq d(x_{2n}, x_{2n+1})$, then

$$\begin{aligned} d(x_{2n}, x_{2n+1}) & \preceq (A_1 + A_2 + A_3 + 2A_4)\left(d(x_{2n}, x_{2n+1})\right) \\ & \preceq A\left(d(x_{2n}, x_{2n+1})\right), \end{aligned}$$

which gives $x_{2n} = x_{2n+1}$. Therefore

$$d(x_{2n}, x_{2n+1}) \preceq A\left(d(x_{2n-1}, x_{2n})\right).$$

In a similar way, for $x_{2n+1} \in T_j(x_{2n})$, there exists $x_{2n+2} \in T_i\left(x_{2n+1}\right)$ with $(x_{2n+1}, x_{2n+2}) \in E\left(G\right)$ such that

$$d(x_{2n+1}, x_{2n+2}) \preceq A\left(d(x_{2n}, x_{2n+1})\right).$$

Hence, we obtain a sequence $\{x_n\}$ in X such that for $x_n \in T_j(x_{n-1})$, there exists $x_{n+1} \in T_i\left(x_n\right)$ with $(x_n, x_{n+1}) \in E\left(G\right)$ such that

$$d(x_n, x_{n+1}) \preceq A\left(d(x_{n-1}, x_n)\right).$$

Following arguments similar to those in proof of Theorem 2.1, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists an element $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Let $0 \ll c$ be given. Choose a natural number N such that $d(x_m, x^*) \ll c$ for all $m \geq N$.

Since (T_1, T_2) form a cone graphic P_2-contraction,

$$d(x_{2n}, u_n) \preceq M_2(x_{2n-1}, x^*; x_{2n}, u_n)),$$

where

$$\begin{aligned} M_2(x_{2n-1}, x^*; x_{2n}, u_n) = \; & A_1(d(x_{2n-1}, x^*)) + A_2(d(x_{2n-1}, x_{2n})) + A_3(d(x^*, u_n)) \\ & + A_4(d(x_{2n-1}, u_n)) + A_5(d(x^*, x_{2n})). \end{aligned}$$

It follows that

$$\begin{aligned} d(x^*, u_n) & \preceq d(x_{2n}, x^*) + d(x_{2n}, u_n) \\ & \preceq A_1(d(x_{2n-1}, x^*)) + A_2(d(x_{2n-1}, x_{2n})) + A_3(d(x^*, u_n)) \\ & \quad + A_4(d(x_{2n-1}, u_n)) + A_5(d(x^*, x_{2n})) \\ & \preceq A_1(d(x_{2n-1}, x^*)) + A_2(d(x_{2n-1}, x^*) + d(x^*, x_{2n})) + A_3(d(x^*, u_n)) \\ & \quad + A_4(d(x_{2n-1}, x^*) + d(x^*, u_n)) + A_5(d(x^*, x_{2n})) \end{aligned}$$

that is,

$$\begin{aligned} d(x^*,u_n) &\preceq (I-A_3-A_4)^{-1}\left(A_1(d(x_{2n-1},x^*)) + A_2(d(x_{2n-1},x^*)+d(x^*,x_{2n}))\right. \\ &\quad \left. +A_4(d(x_{2n-1},x^*)) + A_5(d(x^*,x_{2n}))\right) \\ &\preceq (I-A_3-A_4)^{-1}\left(A_1(c)+A_2(2c)+A_4(c)+A_5(c)\right)). \end{aligned}$$

As $c \gg 0$ is arbitrary, for $m \geq 1$

$$\begin{aligned} d(x^*,u_n) &\preceq (I-A_3-A_4)^{-1}\left(A_1(\frac{c}{m})+A_2(\frac{2c}{m})+A_4(\frac{c}{m})+A_5(\frac{c}{m})\right)) \\ &= \frac{1}{m}(I-A_3-A_4)^{-1}\left(A_1(c)+A_2(2c)+A_4(c)+A_5(c)\right)) \to 0 \end{aligned}$$

as $m \to \infty$. Thus $u_n \to x^*$ as $n \to \infty$. Since $T_i(x^*)$ is closed, $x^* \in F(T_1) = F(T_2)$. Finally to Prove (iv), suppose the set $Fix\,(T_1) \cap Fix\,(T_2)$ is a clique of $\widetilde{G}$. We are to show that $Fix\,(T_1) \cap Fix\,(T_2)$ is singleton. Assume that there exist u and v such that $u, v \in Fix\,(T_1) \cap Fix\,(T_2)$. As $(u,v) \in E(G)$ and T_1 and T_2 form a cone graphic P_2-contraction, so for $(u,v) \in E\,(G)$, we have

$$\begin{aligned} d(u,v) &\preceq F(M_2(u,v;u,v)) \\ &= A_1(d(u,v)) + A_2(d(u,u)) + A_3(d(v,v)) + A_4(d\,(u,v)) + A_5(d\,(v,u)) \\ &= (A_1+A_3+A_5)(d\,(u,v)). \end{aligned}$$

Hence by Remark 1.16 (a) implies that $u = v$. Conversely, if $Fix(T_1) \cap Fix(T_2)$ is singleton, then it follows that $Fix(T_1) \cap Fix(T_2)$ is a clique of $\widetilde{G}$. ∎

Remark 2.4. Let (X,d) be a complete cone metric space endowed with a directed graph G such that $V(G) = X$ and $E(G) \supseteq \Delta$. For maps $T_1, T_2 : X \to P_{cl}(X)$, if we replace (1.4) by either of the following three conditions:

1. there exist linear bounded operators $A_1, A_2, A_3 : E \to E$ with $\|A_1\| + \|A_2\| + \|A_3\| < 1$ and $A_1(P), A_2(P), A_3(P) \subset P$ such that for any $x, y \in X$ with $(x,y) \in E\,(G)$ and $u_x \in T_i(x)$, there exists $u_y \in T_j(y)$ for $i, j \in \{1,2\}$ with $i \neq j$ such that $(u_x, u_y) \in E\,(G)$ and
$$d(u_x,u_y) \leq A_1(d(x,y)) + A_2(d(x,u_x)) + A_3(d(y,u_y)).$$
2. there exists a linear bounded operators $A^* : E \to E$ with $\|A^*\| < 1/2$, $A^*(P) \subset P$ such that for any $x, y \in X$ with $(x,y) \in E\,(G)$ and $u_x \in T_i(x)$, there exists $u_y \in T_j(y)$ for $i, j \in \{1,2\}$ with $i \neq j$ such that $(u_x, u_y) \in E\,(G)$ and
$$d(u_x,u_y) \preceq A^*(d(x,u_x) + d(y,u_y)).$$
3. there exists a linear bounded operators $A^{**} : E \to E$ with $\|A^{**}\| < 1$, $A^{**}(P) \subset P$ such that for any $x, y \in X$ with $(x,y) \in E\,(G)$ and $u_x \in T_i(x)$, there exists $u_y \in T_j(y)$ for $i, j \in \{1,2\}$ with $i \neq j$ such that $(u_x, u_y) \in E\,(G)$ and
$$d(u_x,u_y) \preceq A^{**}(d(x,y)).$$

Then the conclusions obtained in Theorem 2.1 remain true.

Remarks 2.5.

(1) If $E(G) := X \times X$, then clearly G is connected and our Theorem 2.1 improves and generalizes:

(i) Theorem 1 in [1], (ii) Theorem 1.9 in [5], (iii) Theorem 4.1 in [28], (iv) Theorem 3.4 of [37].

(2) If $E(G) := X \times X$, then Theorem 2.3 improves and extends:

(i) Theorem 2 in [1], Theorem 3.4 in [37].

(3) If $E(G) := X \times X$, then our Remark 2.4 extends and generalizes (i) Corollary 2, Corollary 3 and Corollary 4 in [1], (ii) Theorem 3.4 in [37], (iii) Theorem 4.1 of [28].

(4) If $E(G) := X \times X$, then our Remark 2.4 improves and generalizes Theorem 4.1 in [28].

(5) If we take $T_1 = T_2$ in cone graphic P_1-contraction pair and cone graphic P_2-contraction pair, then we obtain the fixed point results for cone graphic P_1-contraction and cone graphic P_2-contraction of a single multivalued map.

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