Value distribution and uniqueness results of L-functions concerning certain linear differential polynomials*

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Abstract

In this paper, we study a uniqueness question of meromorphic functions concerning certain linear differential polynomials that share a nonzero finite value with the same of L-functions. The results in this paper extend the corresponding results from Li[6] and Li & Li[7].

1 Introduction and main results

In this paper, by L-functions we always mean L-functions that are Dirichlet series with the Riemann zeta function $\zeta(s) = \sum\limits_{n=1}^{\infty} n^{-s}$ as the prototype and are important objects in number theory. The Selberg class S of L-functions is the set of all Dirichlet series $L(s) = \sum\limits_{n=1}^{\infty} a(n)n^{-s}$ of a complex variable $s = \sigma + it$ with $a(1) = 1$, satisfying the following axioms (cf.[10, 11]):

(i) Ramanujan hypothesis: $a(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$.

(ii) Analytic continuation: There is a nonnegative integer k such that $(s-1)^k L(s)$ is an entire function of finite order.

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(iii) Functional equation: L satisfies a functional equation of type $\Lambda_L(s) = \omega\overline{\Lambda_L(1-\bar{s})}$, where $\Lambda_L(s) = L(s)Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + \nu_j)$ with positive real numbers Q, λ_j and complex numbers ν_j, ω with $Re\nu_j \geq 0$ and $|\omega| = 1$.

(iv) Euler product hypothesis: $L(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right)$ with suitable coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$, where the product is taken over all prime numbers p.

In the last few years, value distribution of L-functions has been studied extensively, which can be found, for example in Steuding [11]. Value distribution of L-functions concerns the distribution of zeros of an L-function L and, more generally, the c-points of L, i. e., the roots of the equation $L(s) = c$, or the points in the pre-image $L^{-1} = \{s \in \mathbb{C} : L(s) = c\}$, here and throughout the paper, s denotes the complex variable in the complex plane $\mathbb{C}$ and c denotes a value in the extended complex plane $\mathbb{C} \cup \{\infty\}$. L-functions can be analytically continued as meromorphic functions in $\mathbb{C}$. Two meromorphic functions f and g in the complex plane are said to share a value $c \in \mathbb{C} \cup \{\infty\}$ IM (ignoring multiplicities) if $f^{-1}(c) = g^{-1}(c)$ as two sets in $\mathbb{C}$. Moreover, f and g are said to share a value c CM (counting multiplicities) if they share the value c and if the roots of the equations $f(s) = c$ and $g(s) = c$ have the same multiplicities. In terms of sharing values, two nonconstant meromorphic functions in the complex plane must be identically equal if they share five values IM, and one must be a Möbius transformation of the other if they share four values CM. The numbers "five" and "four" are the best possible, as shown by Nevanlinna (cf.[2, 9, 13, 14]), which are famous theorems due to Nevanlinna and often referred to as Nevanlinnas uniqueness theorems.

Throughout this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. To prove the main results in the present paper, we will apply Nevanlinna's theory and adopt the standard notations of the Nevanlinna's theory. We assume that the readers are familiar with the standard notations which are used in the Nevanlinna's theory such as the characteristic function $T(r, f)$, the proximity function $m(r, f)$, the counting function $N(r, f)$ and the reduced counting function $\overline{N}(r, f)$ that are explained in [2, 5, 13, 14]. Here f is a meromorphic function. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. In addition, we will use the lower order $\mu(f)$ and the order $\rho(f)$ of a meromorphic function f, which can be found, for example in [2, 5, 13, 14], and are in turn defined as follows:

$$\mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r, f)}{\log r}, \qquad \rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r}.$$

For a nonconstant meromorphic function h, we denote by $S(r, h)$ any quantity satisfying $S(r, h) = o(T(r, h))$, as $r \to \infty$ and $r \notin E$. We say that a meromorphic function a is a small function with respect to h, if $T(r, a) = S(r, h)$ (cf.[13]). We also need the following two definitions:

Definition 1.1 ([13] and [4, Definition 1]). Let F and G be two nonconstant meromorphic functions in the complex plane such that F and G share 1 IM. Next we denote by $\overline{N}_L\left(r, \frac{1}{F-1}\right)$ the reduced counting function of those common zeros of $F-1$ and $G-1$ in $|z|<r$, where the multiplicity of each such common zero of $F-1$ and $G-1$ as the zero of $F-1$ is greater than its multiplicity as the zero of $G-1$. We denote by $\overline{N}_E^{1)}\left(r, \frac{1}{F-1}\right)$ the reduced counting function of the common simple zeros of $F-1$ and $G-1$ in $|z|<r$, and denote by $\overline{N}_E^{(2}\left(r, \frac{1}{F-1}\right)$ the reduced counting function of the common multiple zeros of $F-1$ and $G-1$ in $|z|<r$, where each such common multiple zero of $F-1$ and $G-1$ has the same multiplicities. Similarly we can define $\overline{N}_L\left(r, \frac{1}{G-1}\right)$, $\overline{N}_E^{1)}\left(r, \frac{1}{G-1}\right)$ and $\overline{N}_E^{(2}\left(r, \frac{1}{G-1}\right)$. Also we denote by $\overline{N}_{1)}\left(r, \frac{1}{F}\right)$ the reduced counting function of the simple zeros of F in $|z|<r$, and denote by $\overline{N}_{(2}\left(r, \frac{1}{F}\right)$ the reduced counting function of the multiple zeros of F in $|z|<r$.

Definition 1.2 ([2, 13, 14]) Let a be a value in the extended complex plane. We call

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f-a}\right)}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r,f)}$$

the deficiency of a with respect to f. If $\delta(a,f)>0$, a is called a deficient value of f. Here $m\left(r, \frac{1}{f-\infty}\right)$ means $m(r,f)$.

We first recall the following result due to Steuding[11], which actually holds without the Euler product hypothesis:

Theorem A ([11, p. 152]). If two L-functions L_1 and L_2 with $a(1)=1$ share a complex value $c \neq \infty$ CM, then $L_1 = L_2$.

Remark 1.1 Recently Hu & Li pointed out that Theorem A is false when $c=1$. A counter example was given by Hu & Li, see [3].

About in 2010, Chung-Chun Yang posed the question that can be found in [6]:

Question A ([6]). If f is a meromorphic function in $\mathbb{C}$ that shares three distinct values a, b CM and c IM with the Riemann zeta function ζ, where $c \notin \{a, b, 0, \infty\}$, is f equal to ζ ?

In this direction, Li[6] proved the following result:

Theorem B ([6]). Let $a, b \in \mathbb{C}$ be two distinct values and let f be a meromorphic function in $\mathbb{C}$ with finitely many poles. If f and a nonconstant L-function L share a CM and b IM, then $L=f$.

Remark 1.2. In 2012, Gao and Li completely solved Question A, see [1].

Recently, Li & Li[7] proved the following two theorems:

Theorem C ([7, Theorem 1.3]). Let f and g be two nonconstant entire functions. Suppose that f and g share the value 0 CM, $P(f)$ and $P(g)$ share the value 1 CM and $\delta(0,f) > \frac{1}{2}$. If $\rho(f) \neq 1$, then $f=g$ unless $P(f)P(g)=1$.

Theorem D ([7, Theorem 1.4]). Let f and g be two nonconstant entire functions. Suppose that f and g share the value 0 CM, $P(f)$ and $P(g)$ share the value 1 IM and $\delta(0,f) > \frac{4}{5}$. If $\rho(f) \neq 1$, then $f=g$ unless $P(f)P(g)=1$.

Let h be a nonconstant meromorphic function. Next we denote by

$$P(h) = h^{(k)} + a_1 h^{(k-1)} + \cdots + a_{k-1}h' + a_k h. \tag{1.1}$$

the linear differential polynomial of h, where $a_1, a_2 \dots a_k$ are finite complex numbers and $k \geq 1$ is a positive integer.

Regarding Theorems C and D, one may ask, what can be said about the relationship between a meromorphic function f with finitely many poles and an L-function L, if $P(f)$ and $P(L)$ share 1 CM or IM. In this paper, we will prove the following general results which extend Theorems C and D respectively:

Theorem 1.1. Let f be a nonconstant meromorphic function with at most finitely many poles in the complex plane and let L be an L-function such that f and L share 0 CM, if $\delta(0, f) > \frac{4}{5}$ and that $P(f)$ and $P(L)$ share 1 IM, then $f = L$.

Proceeding as in the proof of Theorem 1.1, we can get the following result:

Theorem 1.2. Let f be a nonconstant meromorphic function with at most finitely many poles in the complex plane and let L be an L-function such that f and L share 0 CM, if $\delta(0, f) > \frac{1}{2}$ and that $P(f)$ and $P(L)$ share 1 CM, then $f = L$.

The following example shows that the L-function L in Theorems 1.1 and 1.2 can not be replaced with an entire function g that is not an L-function:

Example 1.1 (cf.[7]). Let $f(z) = \frac{1}{2}e^{-2z}$ and $g(z) = e^{-2z}$. Then f and g share 0 CM. $f'' + 2f'$ and $g'' + 2g'$ share 1 CM. Moreover, we can verify that $\delta(0, f) > \frac{1}{2}$, but $f \not\equiv g$.

2 Preliminaries

In this section, we will give the following lemmas that play an important role in proving the main results in this paper:

Lemma 2.1 ([8]). Let f be a nonconstant meromorphic function, and let $P(h)$ be defined as in (1.1). Then

$$T(r, P(f)) \leq T(r, f) + k\overline{N}(r, f) + S(r, f). \tag{2.1}$$

Lemma 2.2 ([7, Lemma 2.2]). Suppose that f is a nonconstant meromorphic function in the complex plane and that a is a small function of f. If f is not a polynomial, then

$$N\left(r, \frac{1}{P(f) - P(a)}\right) \leq T(r, P(f)) - T(r, f) + N\left(r, \frac{1}{f - a}\right) + S(r, f) \tag{2.2}$$

and

$$N\left(r, \frac{1}{P(f) - P(a)}\right) \leq N\left(r, \frac{1}{f - a}\right) + \overline{N}(r, f) + S(r, f). \tag{2.3}$$

Lemma 2.3 ([7, Lemma 2.3]). Let F and G be nonconstant meromorphic functions such that F and G share 1 IM. Set

$$H = \frac{F''}{F'} - \frac{2F'}{F - 1} - \frac{G''}{G'} + \frac{2G'}{G - 1}. \tag{2.4}$$

If $H \not\equiv 0$, then

$$\begin{aligned} T(r,F) \leq & N\left(r,\frac{1}{F}\right) + 2\overline{N}(r,F) + 2\overline{N}_L\left(r,\frac{1}{F-1}\right) + N\left(r,\frac{1}{G}\right) \\ & + 2\overline{N}(r,G) + \overline{N}_L\left(r,\frac{1}{G-1}\right) + S(r,F) + S(r,G). \end{aligned} \tag{2.5}$$

Remark 2.1. By the context of the proof of Lemma 2.3 from [7] and Nevanlinna's second fundamental theorem (cf.[Theorem 2.1, 2]), we can find that the quantities $S(r,F)$ and $S(r,G)$ in (2.5) are such that

$$S(r,F) = 2m\left(r,\frac{F'}{F}\right) + 2m\left(r,\frac{F'}{F-1}\right) + m\left(r,\frac{F''}{F'}\right) + O(1) \tag{2.6}$$

and

$$S(r,G) = 2m\left(r,\frac{G'}{G}\right) + 2m\left(r,\frac{G'}{G-1}\right) + m\left(r,\frac{G''}{G'}\right) + O(1) \tag{2.7}$$

respectively. Moreover, by Theorem 2.2 from [2] we have that the quantity $S(r,F)$ in (2.6) and the quantity $S(r,G)$ in (2.7) are respectively such that if F and G are of finite order, then

$$S(r,F) = O(\log r) \quad \text{and} \quad S(r,G) = O(\log r),$$

as $r \to \infty$, and that if F and G are of infinite order, then

$$S(r,F) = O(\log(rT(r,F))) \quad \text{and} \quad S(r,G) = O(\log(rT(r,G))),$$

as $r \to \infty$, possibly out of an exceptional subset $E \subset (0,+\infty)$ of finite linear measure.

Remark 2.2. If a is a finite complex value, by (1.1) and the context of the proof of Lemma 2.2 from [7] and the context of the proof of Lemma 2.1 from [8] we can see that the quantity $S(r,f)$ in Lemmas 2.2 and 2.1 is such that

$$S(r,f) = m\left(r,\frac{P(f)}{f}\right) + O(1) \leq \sum_{j=1}^{k} m\left(r,\frac{f^{(j)}}{f}\right) + O(1). \tag{2.8}$$

Next, in the same manner as in Remark 2.1 we can see that if f is of finite order, then the quantity $S(r,f)$ in (2.8) is such that

$$S(r,f) = O(\log r),$$

as $r \to \infty$, and that if f is of infinite order, then the quantity $S(r,f)$ in (2.8) is such that

$$S(r,f) = O(\log(rT(r,f))),$$

as $r \to \infty$, possibly out of an exceptional subset $E \subset (0,+\infty)$ of finite linear measure.

Lemma 2.4 ([12, p.106]). Let f be a nonconstant meromorphic function, and let $k \geq 1$ be a positive integer. Suppose that f is a solution of the differential equation $a_0\omega^{(k)} + a_1\omega^{(k-1)} + \cdots + a_k\omega = 0$, where $a_0, a_1, \ldots, a_k$ are constants and $a_0 \neq 0$. Then $T(r,f) = O(r)$. Additionally, if f is a transcendental meromorphic function, then $r = O(T(r,f))$.

Lemma 2.5 ([15, Lemma 6]). Let f_1 and f_2 be two nonconstant meromorphic functions such that $\overline{N}(r,f_j) + \overline{N}\left(r,\frac{1}{f_j}\right) = S(r), (j = 1,2)$. Then, either $\overline{N}_0(r,1;f_1,f_2) = S(r)$ or there exist two integers p and q satisfying $|p| + |q| > 0$, such that $f_1^p f_2^q = 1$, where $\overline{N}_0(r,1;f_1,f_2)$ denotes the reduced counting function of the common 1-points of f_1 and f_2 in $|z| < r$, $T(r) = T(r,f_1) + T(r,f_2)$ and $S(r) = o(T(r))$, as $r \notin E$ and $r \to \infty$. Here $E \subset (0,+\infty)$ is a subset of finite linear measure.

Lemma 2.6 ([13, Theorem 1.5]). Suppose that f is a transcendental meromorphic function. Then

$$\lim_{r\to\infty} \frac{T(r,f)}{\log r} = \infty.$$

3 Proof of theorems

Proof of Theorem 1.1. First of all, we denote by d the degree of L. Then $d = 2\sum_{j=1}^{K} \lambda_j > 0$ (cf.[11, p.113]), where K and λ_j are respectively the positive integer and the positive real number in the functional equation of the axiom (iii) of the definition of L-function. Therefore, by Steuding [11, p.150] we have

$$T(r,L) = \frac{d}{\pi} r \log r + O(r). \tag{3.1}$$

Next we let $P(h)$ be defined as in (1.1). We consider the following two cases:

Case 1. Suppose that $P(f)$ and $P(L)$ are not constants. Then, by noting that f and L share 0 CM and that $P(f)$ and $P(L)$ share 1 IM, we have by Milloux' inequality (cf.[2, Theorem 3.2]) that

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + \overline{N}\left(\frac{1}{P(f)-1}\right) + S(r,f) \\ &= N\left(r,\frac{1}{L}\right) + \overline{N}\left(\frac{1}{P(L)-1}\right) + S(r,f) + O(\log r) \\ &\leq T(r,L) + T(r,P(L)) + S(r,f) + O(\log r). \end{aligned} \tag{3.2}$$

By (1.1), (3.2), Lemma 2.1 and the assumption that an L-function has at most one pole $z = 1$ in the complex plane, we get

$$\begin{aligned} T(r,f) &\leq T(r,L) + T(r,L) + k\overline{N}(r,L) + S(r,f) + O(\log r) \\ &\leq 2T(r,L) + S(r,f) + O(\log r). \end{aligned} \tag{3.3}$$

Similarly we have by (1.1) and the assumptions of Theorem 1.1 that

$$T(r,L) \leq 2T(r,f) + S(r,f) + O(\log r). \tag{3.4}$$

Then, by (3.3), the definition of the order of meromorphic functions and the standard reasoning of removing an exceptional set (cf.[5, Lemma 1.1.1]), we deduce

$$\mu(f) \leq \mu(L), \qquad \rho(f) \leq \rho(L). \tag{3.5}$$

Similarly we have by (3.4) that

$$\mu(L) \leq \mu(f), \qquad \rho(L) \leq \rho(f). \tag{3.6}$$

Therefore, by (3.1), (3.5) and (3.6) we have

$$\mu(L) = \mu(f) = \rho(f) = \rho(L) = 1. \tag{3.7}$$

Set

$$F = P(f), \qquad G = P(L) \tag{3.8}$$

and let H be defined as in (2.4). Then, by (3.7) and the assumption of Theorem 1.1 we know that F and G share 1 IM.

Suppose that $H \not\equiv 0$. Then, by (3.7), Lemma 2.3 and Remark 2.1 we have

$$T(r,F) \leq N\left(r,\frac{1}{F}\right) + N\left(r,\frac{1}{G}\right) + 2\overline{N}_L\left(r,\frac{1}{F-1}\right) + \overline{N}_L\left(r,\frac{1}{G-1}\right) + O(\log r). \tag{3.9}$$

By Definition 1.1 we know that each point in $\overline{N}_L\left(r,\frac{1}{F-1}\right)$ and $\overline{N}_L\left(r,\frac{1}{G-1}\right)$ is of multiplicity not less than 2. This together with (2.3), (3.7) and Remark 2.2 gives

$$\overline{N}_L\left(r,\frac{1}{F-1}\right) \leq N\left(r,\frac{1}{F'}\right) \leq N\left(r,\frac{1}{F}\right) + \overline{N}(r,F) + O(\log r) \tag{3.10}$$

and

$$\overline{N}_L\left(r,\frac{1}{G-1}\right) \leq N\left(r,\frac{1}{G'}\right) \leq N\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + O(\log r). \tag{3.11}$$

By substituting (3.10) and (3.11) into (3.9) we have

$$T(r,F) \leq 3N\left(r,\frac{1}{F}\right) + 2N\left(r,\frac{1}{G}\right) + O(\log r). \tag{3.12}$$

By taking $a = 0$ in (2.2) and (2.3) we have by (1.1), (3.7) and Remark 2.2 that

$$N\left(r,\frac{1}{P(f)}\right) \leq T(r,P(f)) - T(r,f) + N\left(r,\frac{1}{f}\right) + O(\log r) \tag{3.13}$$

and

$$N\left(r,\frac{1}{P(f)}\right) \leq k\overline{N}\left(r,f\right) + N\left(r,\frac{1}{f}\right) + O(\log r). \tag{3.14}$$

Similarly we have

$$N\left(r,\frac{1}{P(L)}\right) \leq T(r,P(L)) - T(r,L) + N\left(r,\frac{1}{L}\right) + O(\log r) \tag{3.15}$$

and

$$N\left(r, \frac{1}{P(L)}\right) \leq k\overline{N}\left(r, L\right) + N\left(r, \frac{1}{L}\right) + O(\log r). \tag{3.16}$$

By noting that f has finitely many poles in the complex plane, and that L has at most one pole $z = 1$ in the complex plane, we have by (3.8), (3.12)-(3.14) and (3.16) that

$$T(r, f) \leq T(r, P(f)) - N\left(r, \frac{1}{P(f)}\right) + N\left(r, \frac{1}{f}\right) + O(\log r) \tag{3.17}$$

$$\leq 2N\left(r, \frac{1}{P(f)}\right) + 2N\left(r, \frac{1}{P(L)}\right) + N\left(r, \frac{1}{f}\right) + O(\log r)$$

$$\leq 3N\left(r, \frac{1}{f}\right) + 2N\left(r, \frac{1}{L}\right) + O(\log r). \tag{3.18}$$

By noting that f and L share 0 CM, we have by (3.18) that

$$T(r, f) \leq 5N\left(r, \frac{1}{f}\right) + O(\log r), \tag{3.19}$$

which contradicts the assumption $\delta(0, f) > \frac{4}{5}$. Therefore $H = 0$, and so it follows by (2.4) that

$$F = \frac{AG + B}{CG + D}, \tag{3.20}$$

where A, B, C and D are finite complex constants satisfying $AD - BC \neq 0$. Next we consider the following three subcases:

Subcase 1.1 Suppose that $AC \neq 0$. By (3.20), we know that $\frac{A}{C}$ is a Picard exceptional value of F. This together with (3.7), Remark 2.2 and Nevanlinna's second fundamental theorem gives

$$T(r, F) \leq N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{F - \frac{A}{C}}\right) + O(\log r) = N\left(r, \frac{1}{F}\right) + O(\log r). \tag{3.21}$$

By (3.7), (3.8), (3.13), (3.14) and Remark 2.2 we deduce (3.17). By (3.8), (3.17) and (3.21) we have

$$T(r, f) \leq N\left(r, \frac{1}{f}\right) + O(\log r), \tag{3.22}$$

which contradicts the assumption $\delta(0, f) > \frac{4}{5}$.

Subcase 1.2 Suppose that $A \neq 0$ and $C = 0$. Then $F = \frac{AG}{D} + \frac{B}{D}$. If $B \neq 0$, then $N\left(r, \frac{1}{F - \frac{B}{D}}\right) = N\left(r, \frac{1}{G}\right)$. Combining this with (3.7), (3.8) and Nevanlinna's second fundamental theorem, we have

$$T(r, F) \leq N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{F - \frac{B}{D}}\right) + N(r, F) + O(\log r)$$

$$\leq N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{G}\right) + O(\log r). \tag{3.23}$$

By (3.7), (3.8), (3.13), (3.14) and Remark 2.2 we deduce (3.17). By (3.8), (3.14), (3.16), (3.17), (3.23), the fact that L has at most one pole $z = 1$ in the complex plane and the assumption that f and L share 0 CM we have

$$T(r, f) \leq N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{L}\right) + O(\log r) = 2N\left(r, \frac{1}{f}\right) + O(\log r), \tag{3.24}$$

which contradicts the assumption $\delta(0, f) > \frac{4}{5}$. Thus $B = 0$, and so $F = \frac{A}{D}G$.

Suppose that 1 is a Picard exceptional value of F and G. Then, by the assumption that F and G share 1 IM, we see that $\frac{A}{D}$ is also a Picard exceptional value of F and G. If $\frac{A}{D} \neq 1$, then we have

$$\delta(1, F) + \delta\left(\frac{A}{D}, F\right) + \delta(0, F) > 2,$$

which is impossible. Hence $\frac{A}{D} = 1$, and so $F = G$ and

$$P(f) = P(L). \tag{3.25}$$

Suppose that 1 is not a Picard exceptional value of F and G, then there is a complex number z_0 such that $F(z_0) = G(z_0) = 1$. Therefore, $\frac{A}{D} = 1$ and so $F = G$. Hence (3.25) is also valid.

By rewriting (3.25), we have $P(f - L) = 0$. This together with (1.1) gives

$$(f - L)^{(k)} + a_1(f - L)^{(k-1)} + \cdots + a_{k-1}(f - L)' + a_k(f - L) = 0. \tag{3.26}$$

Then, by (3.26) and Lemma 2.4, we have

$$T(r, f - L) = O(r). \tag{3.27}$$

Now we set

$$f - L = \alpha. \tag{3.28}$$

Then, α is a meromorphic function. Moreover, by (3.1), (3.27) and (3.28) we have

$$T(r, \alpha) = o(T(r, L)). \tag{3.29}$$

Suppose that $\alpha \not\equiv 0$. Then, by (3.28) and the assumption that f and L share the value 0 CM, we have

$$\overline{N}\left(r, \frac{1}{f}\right) = \overline{N}\left(r, \frac{1}{L}\right) \leq N(r, \frac{1}{\alpha}) \leq T(r, \alpha) + O(1) = o(T(r, L)), \tag{3.30}$$

Now we let

$$\tilde{L} = -\frac{L}{\alpha}. \tag{3.31}$$

Then, by (3.29) and (3.31) we have

$$T(r, L) = T(r, \tilde{L}) + o(T(r, L)), \quad \overline{N}\left(r, \frac{1}{\tilde{L} - 1}\right) = \overline{N}\left(r, \frac{1}{L + \alpha}\right) + o(T(r, L)) \tag{3.32}$$

and

$$\overline{N}(r,\tilde{L}) = \overline{N}(r,L) + o(T(r,L)), \quad \overline{N}\left(r,\frac{1}{\tilde{L}}\right) = \overline{N}\left(r,\frac{1}{L}\right) + o(T(r,L)). \tag{3.33}$$

By (3.7) and (3.31) we deduce $\rho(\tilde{L}) = \rho(L) = 1$. This together with Nevanlinna's second fundamental theorem gives

$$T(r,\tilde{L}) \leq \overline{N}(r,\tilde{L}) + \overline{N}\left(r,\frac{1}{\tilde{L}}\right) + \overline{N}\left(r,\frac{1}{\tilde{L}-1}\right) + O(\log r). \tag{3.34}$$

By (3.28)-(3.34) we have

$$\begin{aligned} T(r,L) &\leq \overline{N}(r,L) + \overline{N}\left(r,\frac{1}{L}\right) + \overline{N}\left(r,\frac{1}{L+\alpha}\right) + O(\log r) + o(T(r,L)) \\ &\leq 2\overline{N}\left(r,\frac{1}{f}\right) + o(T(r,L)) \leq o(T(r,L)), \end{aligned} \tag{3.35}$$

which is impossible. Therefore $\alpha = 0$. Combining this with (3.28), we get the conclusion of Theorem 1.1.

Subcase 1.3 Suppose that $A = 0$ and $C \neq 0$. Then, in the same manner as in the proof of Subcase 1.2 we can get $FG = 1$, and so it follows by (3.8) that

$$P(f)P(L) = 1. \tag{3.36}$$

We consider the following two subcases:

Subcase 1.3.1 Suppose that $P(f)$ and $P(L)$ are transcendental meromorphic functions. Then, by (3.36) and the assumption that $P(f)$ has at most finitely many poles in the complex plane, we deduce that $P(L)$ has at most finitely many zeros in the complex plane. Since L has at most one pole $z = 1$ in the complex plane, we can see by (1.1) that $P(L)$ has at most one pole $z = 1$ in the complex plane. Therefore, by (3.36) we see that $P(f)$ has at most one zero $z = 1$ in the complex plane. Therefore,

$$\overline{N}\left(r,\frac{1}{P(L)}\right) + \overline{N}\left(r,P(L)\right) = O(\log r) \tag{3.37}$$

and

$$\overline{N}\left(r,\frac{1}{P(f)}\right) + \overline{N}\left(r,P(f)\right) = O(\log r). \tag{3.38}$$

Next we prove that ∞ is a Picard exceptional value of $P(f)$ and $P(L)$. For this purpose, now we set

$$f_1 = \frac{P(f)}{P(L)}, \quad f_2 = \frac{P(f)-1}{P(L)-1}. \tag{3.39}$$

By (3.39) and the assumption that $P(f)$ and $P(L)$ are transcendental meromorphic functions, we have $f_1 \not\equiv 0$ and $f_2 \not\equiv 0$. Suppose that one of f_1 and f_2 is a nonzero constant. Then, by (3.39) we see that $P(f)$ and $P(L)$ share ∞ CM. Combining this with (3.36) we deduce that ∞ is a Picard exceptional value of f and L.

Next we suppose that f_1 and f_2 are nonconstant meromorphic functions. Then, by (3.8) and (3.39), we have

$$F = \frac{f_1(1-f_2)}{f_1 - f_2}, \quad G = \frac{1-f_2}{f_1-f_2}. \tag{3.40}$$

By (3.40) we can find that there exists a subset $I \subset (0, +\infty)$ with infinite linear measure such that $S(r) = o(T(r))$ and

$$T(r,F) \le 2(T(r,f_1) + T(r,f_2)) + S(r) \le 8T(r,F) + S(r) \tag{3.41}$$

or

$$T(r,G) \le 2(T(r,f_1) + T(r,f_2)) + S(r) \le 8T(r,G) + S(r), \tag{3.42}$$

as $r \in I$ and $r \to \infty$, where $T(r) = T(r,f_1) + T(r,f_2)$. Without loss of generality, we suppose that (3.41) holds. Then we have $S(r) = S(r,F)$, as $r \in I$ and $r \to \infty$. By (3.36) we see that $P(f)$ and $P(L)$ share 1 and -1 CM. By noting that $P(f)$ and $P(L)$ are transcendental meromorphic functions such that $P(f)$ and $P(L)$ share 1 CM, by (3.37)-(3.39) and (3.41), we deduce

$$\overline{N}\left(r, \frac{1}{f_j}\right) + \overline{N}\left(r, f_j\right) = o(T(r)),\ j = 1, 2, \tag{3.43}$$

as $r \in I$ and $r \to \infty$. By noting that $P(f)$ and $P(L)$ share -1 CM, we deduce by (3.8), (3.38), (3.40), (3.41) and Nevanlinna's second fundamental theorem that

$$\begin{aligned} T(r,F) &\le \overline{N}(r,F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F+1}\right) + o(T(r,F)) \\ &\le \overline{N}\left(r, \frac{1}{F+1}\right) + O(\log r) + o(T(r,F)) \\ &\le \overline{N}_0(r, 1; f_1, f_2) + o(T(r,F)), \end{aligned} \tag{3.44}$$

as $r \in I$ and $r \to \infty$. By (3.41) and (3.44) we have

$$T(r,f_1) + T(r,f_2) \le 4\overline{N}_0(r,1;f_1,f_2) + o(T(r)), \tag{3.45}$$

By (3.39), (3.43), (3.45) and Lemma 2.5 we find that there exist two relatively prime integers s and t satisfying $|s| + |t| > 0$, such that $f_1^s f_2^t = 1$. Combining this with (3.8) and (3.39), we have

$$\left(\frac{F}{G}\right)^s \left(\frac{F-1}{G-1}\right)^t = 1. \tag{3.46}$$

We consider the following three cases:

Suppose that $st < 0$, say $s > 0$ and $t < 0$, say $t = -t_1$, where t_1 is some positive integer. Then, (3.46) can be rewritten as

$$\left(\frac{F}{G}\right)^s = \left(\frac{F-1}{G-1}\right)^{t_1}. \tag{3.47}$$

Let $z_1 \in \mathbb{C}$ be a pole of F of multiplicity $p_1 \ge 1$. Then, by $FG = 1$ we can see that z_1 be a zero of G of multiplicity p_1. Therefore, by (3.47) we deduce that

$2s = t_1 = -t$. Combining this with the assumption that s and t are two relatively prime integers we have $s = 1$ and $t = -2$, and so (3.47) can be rewritten as $F(G-1)^2 = G(F-1)^2$, which is equivalent to $FG = 1$. Combining this with the assumption that f has at most finitely many poles in the complex plane, and the fact that an L-function has at most one pole $z = 1$ in the complex plane, we have by (3.8) that

$$P(L(z)) = \frac{P_1(z)}{(z-1)^s} e^{\alpha_1(z)}, \qquad P(f(z)) = \frac{(z-1)^s}{P_1(z)} e^{-\alpha_1(z)}, \tag{3.48}$$

where P_1 is a nonzero polynomial, $s_1 \geq 0$ is some integer and that α_1 is a nonconstant entire function. Moreover, by (3.1) and the definition of the order of meromorphic functions we deduce

$$\rho(e^{\alpha_1}) = \rho(P(L)) \leq 1,$$

where implies that α_1 is a polynomial of degree equal to 1, say

$$\alpha_1(z) = A_1 z + B_1, \tag{3.49}$$

where $A_1 \neq 0$ and B_1 are constants. By Hayman [2, p.7] we have

$$T(r, e^{A_1 z + B_1}) = \frac{|A_1| r}{\pi} (1 + o(1)). \tag{3.50}$$

By (1.1), (3.4), (3.7) and Remark 2.2 we have

$$T(r, L) \leq 2T(r, f) + O(\log r). \tag{3.51}$$

By (3.1), (3.49), (3.50) and (3.51) we have

$$T\left(r, \frac{(z-1)^{s_1}}{P_1(z)} e^{-\alpha_1(z)}\right) = \frac{|A_1| r}{\pi} (1 + o(1)) + O(\log r) = o(T(r, f)). \tag{3.52}$$

By (1.1) and (3.48) we have

$$\frac{1}{f} = \frac{1}{\frac{(z-1)^{s_1}}{P_1(z)} e^{-\alpha_1(z)}} \left(a_k + a_{k-1}\frac{f'}{f} + \ldots + a_1 \frac{f^{(k-1)}}{f} + \frac{f^{(k)}}{f} \right). \tag{3.53}$$

By (3.7), (3.52), (3.53) and Remark 2.2 we deduce

$$m\left(r, \frac{1}{f}\right) \leq O(\log r) + o(T(r, f)). \tag{3.54}$$

By noting that f is a transcendental meromorphic function, we have by (3.54) and Lemma 2.6 that

$$\delta(0, f) = \liminf_{r \to \infty} \frac{m\left(r, \frac{1}{f}\right)}{T(r, f)} = 0, \tag{3.55}$$

which contradicts the assumption $\delta(0, f) > \frac{4}{5}$.

Suppose that one of s and t is equal to zero. Then, by (3.46) we can see that F and G share ∞ CM, this together with the supposition $FG = 1$ implies that ∞ is a Picard exceptional value of F and G.

Suppose that $st > 0$, say $s > 0$ and $t > 0$. Then, by (3.46) we can see that F and G share ∞ CM. This together with the assumption $FG = 1$ implies that ∞ is a Picard exceptional value of F and G.

Combining this with (3.8) and Hadamard's product (cf.[2, Theorem 1.9]), we have

$$G(z) = P(L(z)) = e^{A_2 z + B_2}, \tag{3.56}$$

where $A_2 \neq 0$ and B_2 are constants. By (1.1), (3.8) and (3.56) we have

$$P(L(z)) = L^{(k)}(z) + a_1 L^{(k-1)}(z) + \cdots + a_{k-1}L'(z) + a_k L(z) = e^{A_2 z + B_2}. \tag{3.57}$$

By (3.57) we have

$$A_2 L^{(k)}(z) + A_2 a_1 L^{(k-1)}(z) + \cdots + A_2 a_{k-1} L'(z) + A_2 a_k L(z) = A_2 e^{A_2 z + B_2} \tag{3.58}$$

and

$$L^{(k+1)}(z) + a_1 L^{(k)}(z) + \cdots + a_{k-1} L''(z) + a_k L'(z) = A_2 e^{A_2 z + B_2}. \tag{3.59}$$

By (3.58) and (3.59), we have

$$L^{(k+1)} + (a_1 - A_2) L^{(k)} + \cdots + (a_k - A_2 a_{k-1}) L' - A_2 a_k L = 0. \tag{3.60}$$

By (3.60) and Lemma 2.4 we have $T(r, L) = O(r)$, which contradicts (3.1).

Subcase 1.3.2 Suppose that F and G are nonzero rational functions. Then, by (3.8), the supposition $FG = 1$ and the fact that an L-function has at most one pole $z = 1$ in the complex plane, we have

$$P(L(z)) = \frac{P_2(z)}{(z-1)^{s_2}}, \qquad P(f(z)) = \frac{(z-1)^{s_2}}{P_2(z)}, \tag{3.61}$$

where P_2 is a nonzero polynomial and that $s_2 \geq 0$ is some integer. By (1.1) and (3.61) we have

$$P(f) = f^{(k)} + a_1 f^{(k-1)} + \cdots + a_{k-1} f' + a_k f = \frac{(z-1)^{s_2}}{P_1}, \tag{3.62}$$

and so

$$\frac{f^{(k)}}{f} + \frac{a_1 f^{(k-1)}}{f} + \cdots + \frac{a_{k-1} f'}{f} + a_k = \frac{(z-1)^{s_2}}{P_1 f}. \tag{3.63}$$

By (3.7), (3.63) and Remark 2.2 we have

$$m\left(r, \frac{1}{f}\right) + O(\log r) = m\left(r, \frac{(z-1)^{s_2}}{P_1 f}\right) \leq \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + O(1) \leq O(\log r). \tag{3.64}$$

By noting that f is a transcendental meromorphic function we deduce by (3.64) and Lemma 2.6 that (3.55) holds, which also contradicts the assumption $\delta(0, f) > \frac{4}{5}$.

Case 2 Suppose that one of $P(f)$ and $P(L)$ is a constant, say

$$P(f) = c, \tag{3.65}$$

where c is a finite complex constant. Then, by (3.65), we have

$$f(z) = c_1 + \sum_{j=1}^{m} q_j(z)e^{\beta_j z}, \tag{3.66}$$

where c_1 is finite complex constant, $q_j (1 \le j \le m)$ are nonzero polynomials and $\beta_j (1 \le j \le m)$ are distinct finite nonzero complex constants. Here $m \ge 1$ is a positive integer. Also by (1.1) and (3.65) we have

$$f^{(k+1)} + a_1 f^{(k)} + \cdots + a_{k-1} f'' + a_k f' = 0. \tag{3.67}$$

By (3.67) and Lemma 2.4 we have

$$T(r, f) = O(r). \tag{3.68}$$

By (3.7), (3.68), Hadamard's product (cf.[2, Theorem 1.9]), the assumption that f has at most finitely many poles in the complex plane, the assumption that f and L share the value 0 CM and the fact that L has at most one pole $z = 1$ in the complex plane, we have

$$L(z) = \frac{P_3(z)}{(z-1)^{s_3}} e^{A_3(z)+B_3} f(z), \tag{3.69}$$

where P_3 is a nonzero polynomial, $s_3 \ge 0$ is some integer, $A_3 \ne 0$ and B_3 are constants. By substituting (3.66) into (3.69) we have

$$L(z) = \frac{P_3(z)}{(z-1)^{s_3}} e^{A_3(z)+B_3} \left(c_1 + \sum_{j=1}^{m} q_j(z) e^{\beta_j z} \right). \tag{3.70}$$

By (3.70) and Hayman [2, p. 7] we deduce

$$T(r, L) \le O(r) + O(\log r),$$

which contradicts (3.1). This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. First of all, in the same manner as in the proof of Theorem 1.1 we have (3.1). Next we let $P(h)$ be defined as in (1.1). We consider the following two cases:

Case 1. Suppose that $P(f)$ and $P(L)$ are not constants. Then, by noting that f and L share 0 CM and that $P(f)$ and $P(L)$ share 1 CM, we have by Milloux' inequality (cf.[2, Theorem 3.2]) that (3.2)-(3.7) still hold. Again we set (3.8) and let H be defined as in (2.4). Then, by (3.8) and the assumption of Theorem 1.2 we know that F and G share 1 CM.

Suppose that $H \not\equiv 0$. Then, by (3.7), Lemma 2.3 and Remark 2.1 we know that (3.9) is rewritten as

$$T(r,F) \le N\left(r,\frac{1}{F}\right) + N\left(r,\frac{1}{G}\right) + O(\log r). \tag{3.71}$$

By taking $a = 0$ in (2.2) and (2.3) we have by (1.1), (3.7) and Remark 2.2 that (3.13)-(3.16) still hold. By noting that f has at most finitely many poles in the complex plane, and that L has at most one pole $z = 1$ in the complex plane, we have by (3.8), (3.13), (3.14), (3.16) and (3.71) that

$$\begin{aligned} T(r,f) &\le T(r,P(f)) - N\left(r,\frac{1}{P(f)}\right) + N\left(r,\frac{1}{f}\right) + O(\log r) \\ &\le N\left(r,\frac{1}{P(L)}\right) + N\left(r,\frac{1}{f}\right) + O(\log r) \\ &\le N\left(r,\frac{1}{L}\right) + N\left(r,\frac{1}{f}\right) + O(\log r). \end{aligned} \tag{3.72}$$

By noting that f and L share 0 CM, we have by (3.72) that

$$T(r,f) \le 2N\left(r,\frac{1}{f}\right) + O(\log r), \tag{3.73}$$

which contradicts the assumption $\delta(0,f) > \frac{1}{2}$. Therefore $H = 0$, and so we have (3.20). Next, in the same manner as in Subcases 1.1-1.3 in the proof of Theorem 1.1 we get a contradiction.

Case 2 Suppose that one of $P(f)$ and $P(L)$ is a constant, say (3.65) holds. Then, in the same manner as in Case 2 of the proof of Theorem 1.1 we can get a contradiction. This completes the proof of Theorem 1.2.

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