Remarks on star-K-Menger spaces

Yan-Kui Song

Abstract

A space X is *star-K-Menger* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(K_n : n \in N)$ of compact subsets of X such that $\{St(K_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an open cover of X. In this paper, we construct an example of a Hausdorff star-Menger space that is not star-K-Menger which gives an answer to a question of Song [12], and continue to investigate topological properties of star-K-Menger spaces.

1   Introduction

By a space, we mean a topological space. We give definitions of terms which are used in this paper. Let $\mathbb{N}$ denote the set of positive integers. Let X be a space and $\mathcal{U}$ a collection of subsets of X. For $A \subseteq X$, let $St(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \varnothing\}$. As usual, we write $St(x, \mathcal{U})$ instead of $St(\{x\}, \mathcal{U})$.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a space X. Then the symbol $S_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$. The symbol $S_{fin}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ is an element of $\mathcal{B}$ (see [6,11]).

Kočinac [7,8] introduced star selection hypothesis similar to the previous ones. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a space X. Then:

(A) The symbol $S_{fin}^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such

that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n \in \mathbb{N}} \{St(V, \mathcal{U}_n) : V \in \mathcal{V}_n\}$ is an element of $\mathcal{B}$.

(B) The symbol $SS^*_{comp}(\mathcal{A}, \mathcal{B})$ $(SS^*_{fin}(\mathcal{A}, \mathcal{B}))$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(K_n : n \in N)$ of compact (resp., finite) subsets of X such that $\{St(K_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

Let $\mathcal{O}$ denote the collection of all open covers of X.

Definition 1.1. ([7,8]) A space X is said to be *star-Menger* if it satisfies the selection hypothesis $S^*_{fin}(\mathcal{O}, \mathcal{O})$.

Definition 1.2. ([7,8,12]) A space X is said to be *star-K-Menger* (*strongly star-Menger*) if it satisfies the selection hypothesis $SS^*_{comp}(\mathcal{O}, \mathcal{O})$ (resp., $SS^*_{fin}(\mathcal{O}, \mathcal{O})$).

Definition 1.3. ([3,9]) A space X is said to be *K-starcompact* if for every open cover $\mathcal{U}$ of X there exists a compact subset F of X such that $St(F, \mathcal{U}) = X$.

From the definitions, it is clear that every K-starcompact space is star-K-Menger, every strongly star-Menger space is star-K-Menger and every star-K-Menger space is star-Menger.

The author [12] studied the star-K-Menger and related spaces, and investigated topological properties of star-K-Menger spaces. The author [12] constructed a T_1 star-Menger space which is not star-K-Menger and asked if there exists Hausdorff or Tychonoff example. The purpose of this paper is to show the statement stated in the abstract, which gives an answer to the question above of Song in class of Hausdorff spaces, and continue to investigate topological properties of star-K-Menger spaces.

Throughout this paper, let ω denote the first infinite cardinal, ω_1 the first uncountable cardinal, $\mathfrak{c}$ the cardinality of the set of all real numbers. For a cardinal κ, let κ^+ be the smallest cardinal greater than κ. For each pair of ordinals α, β with $\alpha < \beta$, we write $[\alpha, \beta) = \{\gamma : \alpha \leq \gamma < \beta\}$, $(\alpha, \beta] = \{\gamma : \alpha < \gamma \leq \beta\}$, $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$ and $[\alpha, \beta] = \{\gamma : \alpha \leq \gamma \leq \beta\}$. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. A cardinal is often viewed as a space with the usual order topology. Other terms and symbols that we do not define follow [5].

2   Star-K-Menger spaces and related spaces

In this section, first we construct the example stated in the abstract.

Lemma 2.1. *If $X = \bigcup\{X_n : n \in \mathbb{N}\}$ and each X_n is star-Menger, then X is star-Menger.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X and let $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2 \cup \cdots$ be a partition of $\mathbb{N}$ into infinitely many infinite pairwise disjoint subsets. For each $n \in \mathbb{N}$, since X_n is star-Menger, thus for the sequence $(\mathcal{U}_m : m \in \mathbb{N}_n)$ of open

covers of X, there exists a sequence $\{\mathcal{V}_m : m \in \mathbb{N}_n\}$ such that for each $m \in \mathbb{N}_n$, $\mathcal{V}_m$ is a finite subset of $\mathcal{U}_m$ and

$$X_n \subseteq \bigcup_{m \in \mathbb{N}_n} St(\cup \mathcal{V}_m, \mathcal{U}_m).$$

Hence the sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ witnesses for $\{\mathcal{U}_n : n \in \mathbb{N}\}$ that X is star-Menger. Therefore we complete the proof. ∎

It is clear that every compact space is star-Menger. Thus we have the following corollary by Lemma 2.1.

Corollary 2.2. *If X is a σ-compact space, then X is star-Menger.*

Example 2.3. *There exists a Hausdorff star-Menger space which is not star-K-Menger.*

Proof. Let

$$A = \{a_\alpha : \alpha < \mathfrak{c}\}, B = \{b_n : n \in \omega\}$$
$$\text{and } Y = \{\langle a_\alpha, b_n \rangle : \alpha < \mathfrak{c}, n \in \omega\},$$

and let

$$X = Y \cup A \cup \{a\} \text{ where } a \notin Y \cup A.$$

We topologize X as follows: every point of Y is isolated; a basic neighborhood of a point $a_\alpha \in A$ for each $\alpha < \mathfrak{c}$ takes the form

$$U_{a_\alpha}(n) = \{a_\alpha\} \cup \{\langle a_\alpha, b_m \rangle : m > n\} \text{ for } n \in \omega$$

and a basic neighborhood of a point a takes the form

$$U_a(F) = \{a\} \cup \cup \{\langle a_\alpha, b_n \rangle : a_\alpha \in A \setminus F, n \in \omega\} \text{ for a countable subset } F \text{ of } A.$$

Clearly, X is a Hausdorff space by the construction of the topology of X. However, X is not regular, since the point a can not be separated from the closed subset A by disjoint open subsets of X.

Now we show that X is star-Menger. To this end, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Without loss of generality, we assume that $\mathcal{U}_n$ consists of basic open sets of X for each $n \in \mathbb{N}$. Since $\mathcal{U}_1$ is an open cover of X, there exists $U_a \in \mathcal{U}_1$ such that $a \in U_a$. By assumption, there exists a countable subset F of A such that

$$U_a = U_a(F) = \{a\} \cup \cup \{\langle a_\alpha, b_n \rangle : a_\alpha \in A \setminus F, n \in \omega\}$$

by the definition of the topology of X, thus we have

$$F \cup U_a \subseteq St(U_a, \mathcal{U}).$$

For each $a_\alpha \in F$, let
$$B_{a_\alpha} = \{a_\alpha\} \cup \{\langle a_\alpha, b_n \rangle : n \in \omega\}.$$

Then B_{a_α} is a compact subset of X by the definition of the topology of X. Let $A = \bigcup_{a_\alpha \in F} B_{a_\alpha}$. Then A is σ-compact and $X = A \cup F \cup U_a$. Thus A is star-Menger

by Corollary 2.2. For the sequence $(\mathcal{U}_n : n > 1)$ of open covers of X, there exists a sequence $(\mathcal{V}_n : n > 1)$ such that for each $n > 1$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and

$$A \subseteq \bigcup_{n>1} St(\cup \mathcal{V}_n, \mathcal{U}_n).$$

Let $\mathcal{V}_1 = \{U_a\}$. Then the sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ witnesses for $\{\mathcal{U}_n : n \in \mathbb{N}\}$ that X is star-Menger.

Next we show that X is not star-K-Menger. For each $\alpha < \mathfrak{c}$, let

$$U_\alpha = \{a_\alpha\} \cup \{\langle a_\alpha, b_n \rangle : n \in \omega\} \text{ and } U = U_a(\varnothing).$$

Then U_α is open and closed in X by the construction of the topology of X and

$$U_\alpha \cap U_{\alpha'} = \varnothing \text{ for } \alpha \neq \alpha'.$$

For $n \in \mathbb{N}$, let

$$\mathcal{U}_n = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{U\}.$$

Let us consider the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X. We only show that $\bigcup_{n \in \mathbb{N}} st(K_n, \mathcal{U}_n) \neq X$ for any sequence $(K_n : n \in \mathbb{N})$ of compact subsets of X. Let $(K_n : n \in \mathbb{N})$ be any sequence of compact subsets of X. For each $n \in \mathbb{N}$, since K_n is compact, then there exists $\alpha_n < \mathfrak{c}$ such that $K_n \cap U_\alpha = \varnothing$ for each $\alpha > \alpha_n$. Let $\alpha' = \sup\{\alpha_n : n \in \mathbb{N}\}$. If we pick $\beta > \alpha'$. Then $U_\beta \cap K_n = \varnothing$ for each $n \in \mathbb{N}$. Hence $a_\beta \notin St(K_n, \mathcal{U}_n)$ for each $n \in \mathbb{N}$, since U_β is the only element of $\mathcal{U}_n$ containing the point a_β for each $n \in \mathbb{N}$, which shows that X is not star-K-Menger. Thus we complete the proof. ∎

Question 2.1. *The author does not know if there exists a regular (or Tychonoff) star-Menger space which is not star-K-Menger.*

The author [12] showed that a regular-closed subset of a Tychonoff star-K-Menger space X need not be star-K-Menger. Next we construct an example showing that assuming $\mathfrak{d} = \mathfrak{c}$, there exists a Tychonoff star-K-Menger space having a regular-closed G_δ-subspace which is not star-K-Menger by using the following example from [1, 10]. We make use of two of the cardinals defined in [3,4]. Define $^\omega \omega$ as the set of all functions from ω to itself. For all $f, g \in {}^\omega \omega$, we say $f \leq^* g$ if and only if $f(n) \leq g(n)$ for all but finitely many n. The unbounding number, denoted by $\mathfrak{b}$, is the smallest cardinality of an unbounded subset of $(^\omega \omega, \leq^*)$. The dominating number, denoted by $\mathfrak{d}$, is the smallest cardinality of a cofinal subset of $(^\omega \omega, \leq^*)$. It is not difficult to show that $\omega_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$ and it is known that $\omega_1 < \mathfrak{b} = \mathfrak{c}$, $\omega_1 < \mathfrak{d} = \mathfrak{c}$ and $\omega_1 \leq \mathfrak{b} < \mathfrak{d} = \mathfrak{c}$ are all consistent with the axioms of ZFC (see [3] for details).

Example 2.4. ([1,9]) *Let $\mathcal{A}$ be an almost disjoint family of infinite subsets of ω (i.e., the intersection of every two distinct elements of $\mathcal{A}$ is finite) and Let $X = \omega \cup \mathcal{A}$ be the Isbell-Mrówka space constructed from $\mathcal{A}([4,5])$. Then X is strongly star-Menger if and only if $|\mathcal{A}| < \mathfrak{d}$.*

For the next example, we need the following Lemma. For a Tychonoff space X, let βX denote the Čech-Stone compactification of X.

Lemma 2.5. *Let κ be infinite cardinal and $D = \{d_\alpha : \alpha < \kappa\}$ be a discrete space of cardinality κ. Then the subspace $X = (\beta D \times \kappa) \cup (D \times \{\kappa\})$ of the product space $\beta D \times (\kappa + 1)$ is not star-K-Menger.*

Proof. We show that X is not star-K-Menger. For each $\alpha < \kappa$, let $U_\alpha = \{d_\alpha\} \times (\alpha, \kappa]$. Then U_α is open in X and

$$U_\alpha \cap U_{\alpha'} = \varnothing \text{ for each } \alpha \neq \alpha'$$

For each $n \in \mathbb{N}$, let
$$\mathcal{U}_n = \{U_\alpha : \alpha < \kappa\} \cup \{\beta D \times \kappa\}.$$

Then $\mathcal{U}_n$ is an open cover of X. Let us consider the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X. It suffices to show that $\bigcup_{n \in \mathbb{N}} St(K_n, \mathcal{U}_n) \neq X$ for any sequence $(K_n : n \in \mathbb{N})$ of compact subsets of X. Let $(K_n : n \in \mathbb{N})$ be any sequence of compact subsets of X. For each $n \in \mathbb{N}$, since K_n is compact and $\{\langle d_\alpha, \kappa \rangle : \alpha < \kappa\}$ is a discrete closed subset of X, the set $K_n \cap \{\langle d_\alpha, \omega \rangle : \alpha < \mathfrak{c}\}$ is finite. Then there exists $\alpha_n < \kappa$ such that

$$K_n \cap \{\langle d_\alpha, \kappa \rangle : \alpha > \alpha_n\} = \varnothing.$$

Let $\alpha' = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then $\alpha' < \kappa$ and

$$(\bigcup_{n \in \mathbb{N}} K_n) \cap \{\langle d_\alpha, \kappa \rangle : \alpha > \alpha'\} = \varnothing.$$

On the other hand, for each $n \in \mathbb{N}$, let $A_n = \{\alpha : \langle d_\alpha, \kappa \rangle \in K_n\}$. Then A_n is finite, since K_n is compact and $\{\langle d_\alpha, \kappa \rangle : \alpha < \kappa\}$ is discrete and closed in X. Let $K'_n = K_n \setminus \bigcup\{U_\alpha : \alpha \in A_n\}$. Then K'_n is closed in K_n and $K'_n \subseteq \beta D \times \kappa$. Hence $\pi(K'_n)$ is a compact subset of the countably compact space κ, where $\pi : \beta D \times \kappa \to \kappa$ is the projection, thus there exists $\alpha'_n < \kappa$ such that $\pi(K'_n) \cap (\alpha'_n, \kappa) = \varnothing$. Let $\alpha'' = \sup\{\alpha'_n : n \in \mathbb{N}\}$. Then $\alpha'' < \kappa$. If we pick $\beta > \max\{\alpha', \alpha''\}$. Then $U_\beta \cap K_n = \varnothing$ for each $n \in \mathbb{N}$. Hence $\langle d_\beta, \kappa \rangle \notin St(K_n, \mathcal{U}_n)$ for each $n \in \mathbb{N}$, since U_β is the only element of $\mathcal{U}_n$ containing the point $\langle d_\beta, \kappa \rangle$ for each $n \in \mathbb{N}$, which shows that X is not star-K-Menger. ∎

Recall from [3,9] that a space X is said to be *strongly starcompact* if for every open cover $\mathcal{U}$ of X there exists a finite subset F of X such that $St(F, \mathcal{U}) = X$. It is well known that strongly starcompactness is equivalent to countably compactness for Hausdorff spaces (see [3,9]).

Example 2.6. *Assuming $\mathfrak{d} = \mathfrak{c}$, there exists a Tychonoff star-K-Menger space having a regular-closed G_δ-subspace which is not star-K-Menger.*

Proof. Assuming $\mathfrak{d} = \mathfrak{c}$, let $S_1 = \omega \cup \mathcal{A}$ be the same space X as in the construction of Example 2.4 with $|\mathcal{A}| = \omega_1$. Then S_1 is strongly star-Menger by Example 2.4. Hence S_1 is star-K-Menger, since every strongly star-Menger space is star-K-Menger.

Let $D = \{d_\alpha : \alpha < \omega_1\}$ be a discrete space of cardinality ω_1 and let

$$S_2 = (\beta D \times \omega_1) \cup (D \times \{\omega_1\})$$

be the subspace of the product space $\beta D \times (\omega_1 + 1)$. Then S_2 is not star-K-Menger by Lemma 2.5.

Let $\pi : \mathcal{A} \to D \times \{\omega_1\}$ be a bijection and let X be the quotient image of the disjoint sum $S_1 \oplus S_2$ by identifying A of S_1 with $\pi(A)$ of S_2 for every $A \in \mathcal{A}$. Let $\varphi : S_1 \oplus S_2 \to X$ be the quotient map. Then $\varphi(S_2)$ is a regular-closed subspace of X. For each $n \in \omega$, let $F_n = \{m \in \omega : m \leq n\}$. For each $n \in \omega$, let

$$U_n = \varphi(\{\{A\} \cup (A \setminus F_n) : A \in \mathcal{A}\}) \cup \varphi(\beta D \times \omega_1).$$

Then U_n is open in X and $\varphi(S_2) = \bigcap_{n \in \omega} U_n$. Thus $\varphi(S_2)$ is a regular-closed G_δ-subspace of X. However $\varphi(S_2)$ is not star-K-Menger, since it is homeomorphic to S_2.

Finally we show that X is star-K-Menger. To this end, let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of X. Since $\varphi(\beta D \times \omega_1)$ is homeomorphic to $\beta D \times \omega_1$, and consequently $\varphi(\beta D \times \omega_1)$ is countably compact, we can find a finite subset K_1 of $\varphi(\beta D \times \omega_1)$ such that

$$\varphi(\beta D \times \omega_1) \subseteq St(K_1, \mathcal{U}_1).$$

On the other hand, Since $\varphi(S_1)$ is homeomorphic to S_1 and consequently $\varphi(S_1)$ is star-K-Menger. Thus for the sequence $(\mathcal{U}_n : n > 1)$ of open covers of X, there exists a sequence $\{K_n : n > 1\}$ of compact subsets of $\varphi(S_1)$ such that

$$\varphi(S_1) \subseteq \bigcup_{n>1} St(K_n, \mathcal{U}_n).$$

Hence the sequence $\{K_n : n \in \mathbb{N}\}$ of compact subsets of X witnesses for $\{\mathcal{U}_n : n \in \mathbb{N}\}$ that X is star-K-Menger. Therefore we complete the proof. ∎

Question 2.2. *The author does not know if there exists an example in ZFC showing that a regular-closed G_δ-subspace of a star-K-Menger space is not star-K-Menger.*

The following example gives an answer to Question 2.2 for Hausdorff spaces.

Example 2.7. *There exists a Hausdorff star-K-Menger space having a regular-closed G_δ-subspace which is not star-K-Menger.*

Proof. Let $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ be a discrete space of cardinality $\mathfrak{c}$ and let

$$S_1 = (\beta D \times (\mathfrak{c} + 1)) \cup (D \times \{\mathfrak{c}\})$$

be the subspace of the product space $\beta D \times (\mathfrak{c} + 1)$. Then S_2 is not star-K-Menger by Lemma 2.5.

Let $S_2 = Y \cup A \cup \{a\}$ where $a \notin Y \cup A$ be the same space X as the proof of Example 2.3. We topologize S_2 as follows: every point of $Y \cup A$ has the same neighborhood as the proof of Example 2.3 and a basic neighborhood of a point a takes the form

$$U_a(F) = \{a\} \cup \bigcup\{\langle a_\alpha, b_n \rangle : a_\alpha \in A \setminus F, n \in \omega\} \text{ for a finite subset } F \text{ of } A.$$

Clearly, S_2 is a Hausdorff space, but is not regular.

Now we show that S_2 is star-K-Menger. We only show that S_2 is K-starcompact, since every K-starcompact space is star-K-Menger. To this end, let $\mathcal{U}$ be an open cover of S_2. Without loss of generality, we assume that $\mathcal{U}$ consists of basic open sets of S_2. Since $\mathcal{U}$ is an open cover of S_2, there exists $U_a \in \mathcal{U}$ such that $a \in U_a$. By assumption, there exists a finite subset F of A such that

$$U_a = U_a(F) = \{a\} \cup \cup \{\langle a_\alpha, b_n \rangle : a_\alpha \in A \setminus F, n \in \omega\}$$

by the definition of the topology of S_2, thus we have

$$U_a \subseteq St(a, \mathcal{U}).$$

For each $a_\alpha \in F$, let

$$B_{a_\alpha} = \{a_\alpha\} \cup \{\langle a_\alpha, b_n \rangle : n \in \omega\}.$$

Then B_{a_α} is a compact subset of S_2 by the definition of the topology of S_2. On the other hand, for each $a_\alpha \in A \setminus F$, there exists $U_{a_\alpha} \in \mathcal{U}$ such that $a_\alpha \in U_{a_\alpha}$. Thus there exists $\alpha_n \in \omega$ such that $\langle a_\alpha, \alpha_n \rangle \in U_{a_\alpha}$. Let $C = \{\langle a_\alpha, \alpha_n \rangle : a_\alpha \in A \setminus F\}$. Then $C \cup \{a\}$ is a compact subset of S_2 by the definition of S_2 and

$$A \setminus F \subseteq St(C \cup \{a\}, \mathcal{U}).$$

Set

$$K = \{a\} \cup C \cup \{B_{a_\alpha} : a_\alpha \in F\}.$$

Then K is a compact subset of S_2 such that $S_2 = St(K, \mathcal{U})$, which shows that S_2 is K-starcompact.

We assume $S_1 \cap S_2 = \varnothing$. Let $\pi : D \times \{\mathfrak{c}\} \to A$ be a bijection and let X be the quotient image of the disjoint sum $S_1 \oplus S_2$ by identifying $\langle d_\alpha, \mathfrak{c} \rangle$ of S_1 with $\pi(\langle d_\alpha, \mathfrak{c} \rangle)$ of S_2 for every $\alpha < \mathfrak{c}$. Let $\varphi : S_1 \oplus S_2 \to X$ be the quotient map. Clearly, $\varphi(S_1)$ is a regular-close subspace of X. For each $n \in \omega$, let

$$U_n = \pi(S_1 \cup \{\langle a_\alpha, b_m \rangle : m > n, \alpha < \mathfrak{c}\}).$$

Then U_n is open in X and $\varphi(S_1) = \bigcap_{n \in \omega} U_n$. Thus $\varphi(S_1)$ is a regular-closed G_δ-subspace of X. However $\varphi(S_1)$ is not star-K-Menger, since it is homeomorphic to S_1.

Similarly to the proof that the space X in Example 2.6 is star-K-Menger, we can prove that X is star-K-Menger. Thus we complete the proof. ∎

We give a positive result on star-K-Menger spaces:

Theorem 2.8. *An open and closed subset of a star-K-Menger space is star-K-Menger.*

Proof. Let X be a star-K-Menger space and let Y be an open and closed subset of X. To show that Y is star-K-Menger. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of Y, we have to find a sequence $\{F_n : n \in \mathbb{N}\}$ of compact subsets of Y such that

$$Y = \bigcup_{n \in \mathbb{N}} St(F_n, \mathcal{U}_n).$$

For each $n \in \mathbb{N}$, let

$$\mathcal{V}_n = \mathcal{U}_n \cup \{X \setminus Y\}.$$

Then $\{\mathcal{V}_n : n \in \mathbb{N}\}$ is a sequence of open covers of X, there exists a sequence $\{F'_n : n \in \mathbb{N}\}$ of compact subsets of X such that $X = \bigcup_{n\in\mathbb{N}} St(F'_n, \mathcal{V}_n)$, since X is star-K-Menger. For each $n \in \mathbb{N}$, let $F_n = F'_n \cap Y$. Thus $\{F_n : n \in \mathbb{N}\}$ is a sequence of compact subsets of Y, since Y is a closed subset of X. For each $y \in Y$, if $y \in St(F'_n, \mathcal{V}_n)$, then $y \in St(F_n, \mathcal{U}_n)$ by the construction of $\mathcal{V}_n$. Hence the sequence $\{F_n : n \in \mathbb{N}\}$ of compact subsets of Y witnesses for $\{\mathcal{U}_n : n \in \mathbb{N}\}$ that Y is star-K-Menger. Therefore we complete the proof. ∎

For a space X, recall that the Alexandroff duplicate $A(X)$ of a space X, denoted by $A(X)$, is constructed in the following way: The underlying set $A(X)$ is $X \times \{0,1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of $\langle x, 0\rangle \in X \times \{0\}$ is a set of the form $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x in X. It is well known that a space X is compact if and only if so is $A(X)$. In the following, we give two examples to show that the result can not be generalized to star-K-Menger. For a Tychonoff space X, let βX denote the Čech-Stone compactification of X. We give a lemma from [9]. Here we give the proof for the sake of completeness.

Lemma 2.9. *Let κ be infinite cardinal and $D = \{d_\alpha : \alpha < \kappa\}$ be a discrete space of cardinality κ. Then the subspace $X = (\beta D \times \kappa^+) \cup (D \times \{\kappa^+\})$ of the product space $\beta D \times (\kappa^+ + 1)$ is K-starcompact (hence star-K-Menger).*

Proof. We only show that X is K-starcompact. To this end, let $\mathcal{U}$ be an open cover of X. For each $\alpha < \kappa$, there exist $\beta_\alpha < \kappa^+$ and $U_\alpha \in \mathcal{U}$ such that $\{d_\alpha\} \times (\beta_\alpha, \kappa^+] \subseteq U_\alpha$. Let $\beta = \sup\{\beta_\alpha : \alpha < \kappa\}$. Then $\beta < \kappa^+$. Let $K_1 = \beta D \times \{\beta + 1\}$. Then K_1 is a compact subset of X and $D \times \{\kappa^+\} \subseteq St(K_1, \mathcal{U})$. On the other hand, since $\beta D \times [0, \kappa^+)$ is countably compact, there exists a finite subset K_2 of $\beta D \times [0, \kappa^+)$ such that $\beta D \times [0, \kappa^+) \subseteq St(K_2, \mathcal{U})$. If we put $K = K_1 \cup K_2$, then K is a compact subset of X such that $X = St(K, \mathcal{U})$, which shows that X is K-starcompact. ∎

Example 2.10. *There exists a Tychonoff star-K-Menger space X such that $A(X)$ is not star-K-Menger.*

Proof. Let $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ be a discrete space of the cardinality $\mathfrak{c}$, and let

$$X = (\beta D \times \mathfrak{c}^+) \cup (D \times \{\mathfrak{c}^+\})$$

be the subspace of the product space $\beta D \times (\mathfrak{c}^+ + 1)$. Then X is star-K-Menger by Lemma 2.9.

However $A(X)$ is not star-K-Menger. In fact, since $D \times \{\mathfrak{c}^+\}$ is a discrete closed subset of X with $|D \times \{\mathfrak{c}^+\}| = \mathfrak{c}$, then the set $(D \times \{\mathfrak{c}^+\}) \times \{1\}$ is an open and closed subset of $A(X)$ with $|(D \times \{\mathfrak{c}^+\}) \times \{1\}| = \mathfrak{c}$, and each point $\langle\langle d_\alpha, \mathfrak{c}^+\rangle, 1\rangle$ is isolated for each $\alpha < \mathfrak{c}$. Hence $A(X)$ is not star-K-Menger by Theorem 2.8, since $(D \times \{\mathfrak{c}^+\}) \times \{1\}$ is not star-K-Menger. ∎

Now we give a positive result. For showing the result, we need a lemma from [2]. For a space X, the *extent* $e(X)$ of X is the supremum of cardinalities of closed discrete subsets of X.

Lemma 2.11. *For T_1-space X, $e(X) = e(A(X))$.*

Theorem 2.12. *If X is a star-K-Menger space with $e(X) < \omega_1$, then $A(X)$ is star-K-Menger.*

Proof. We show that $A(X)$ is star-K-Menger. To this end, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of $A(X)$. For each $n \in \mathbb{N}$ and each $x \in X$, choose an open neighborhood $W_{n_x} = (V_{n_x} \times \{0,1\}) \setminus \{\langle x, 1\rangle\}$ of $\langle x, 0\rangle$ satisfying that there exists some $U \in \mathcal{U}_n$ such that $W_{n_x} \subseteq U$, where V_{n_x} is an open subset of X containing x. For each $n \in \mathbb{N}$, let $\mathcal{V}_n = \{V_{n_x} : x \in X\}$, then $\mathcal{V}_n$ is an open cover of X, thus $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of open covers of X, there exists a sequence $(K_n : n \in \mathbb{N})$ of compact subsets of X such that $\bigcup_{n\in\mathbb{N}}\{St(K_n, \mathcal{V}_n) : V \in \mathcal{V}'_n\}$ is an open cover of X, since X is star-K-Menger. For each $n \in \mathbb{N}$, let $K'_n = A(K_n)$. Then K'_n is a compact subset of $A(X)$. Thus we get a sequence $(K'_n : n \in \mathbb{N})$ of compact subsets of $A(X)$ and

$$X \times \{0\} \subseteq \bigcup_{n\in\mathbb{N}} St(K'_n, \mathcal{U}_n).$$

Let $A = A(X) \setminus \bigcup_{n\in\mathbb{N}} St(K'_n, \mathcal{U}_n)$. Then A is a discrete closed subset of $A(X)$. By Lemma 2.11, the set A is countable, we can enumerate A as $\{a_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let $K''_n = K'_n \cup \{a_n\}$. Then K''_n is a compact subset of $A(X)$ and

$$A(X) = \bigcup_{n\in\mathbb{N}} St(K''_n, \mathcal{U}_n),$$

which shows that $A(X)$ is star-K-Menger. ∎

From the proof of Example 2.10, it is not difficult to show the following result.

Theorem 2.13. *If X is a T_1-space and $A(X)$ is a star-K-Menger space. Then $e(X) < \omega_1$.*

Proof. Suppose that $e(X) \geq \omega_1$. Then there exists a discrete closed subset B of X such that $|B| \geq \omega_1$. Hence $B \times \{1\}$ is an open and closed subset of $A(X)$ and every point of $B \times \{1\}$ is an isolated point. Thus $A(X)$ is not star-K-Menger by Theorem 2.8, since $B \times \{1\}$ is not star-K-Menger. ∎

We have the following corollary from Theorems 2.12 and 2.13.

Corollary 2.14. *If X is a star-K-Menger T_1-space, then $A(X)$ is star-K-Menger if and only if $e(X) < \omega_1$.*

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Institute of Mathematics, School of Mathematical Science,
Nanjing Normal University,
Nanjing 210023, P.R. China
email:songyankui@njnu.edu.cn