A new look at the proof of K-theoretic amenability for groups acting on trees

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Abstract

We generalize the construction by Pytlik and Szwarc of uniformly bounded representations for free groups to groups acting on trees. We deduce a new version of the proof (by Alain Valette and the author, 1983) of the fact that locally compact groups acting on trees with amenable stabilizers are amenable in K-theory.

To any locally compact group G are naturally associated two C^*-algebras: the full C^*-algebra C^*G which contains the information on all unitary representations of G, and the reduced C^*algebra C^*_rG which only takes into account the unitary representations weakly contained in the regular representation in $L^2(G)$. There is a surjective morphism $\lambda : C^*G \to C^*_rG$ which is an isomorphism if and only if G is amenable.

The K-theory functor, a covariant functor from C^* algebras to abelian groups, gives rise to a morphism λ_* of abelian groups . J. Cuntz [C] has proved that λ_* is an isomorphism for some non amenable discrete groups such as free groups or $SL(2, \mathbf{Z})$. Such groups are said to be K-amenable. Strictly speaking, one requires a slightly stronger property: the isomorphism in K theory must hold not only for the group C^*-algebras of G but for the crossed products associated to the action of G by automorphisms on an auxiliary C^*-algebra. In our 1983 paper [JV1], A. Valette and myself gave a generalization of J. Cuntz's result, proving K-amenability of any locally compact group acting on a tree with amenable stabilizers. The

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most striking example was the (totally disconnected) group $SL(2)$ over the field of p-adic numbers. Details are given in [JV2].

The technical tool, both in Cuntz's paper and in our work, is G. Kasparov's equivariant KK-bifunctor [K]. An important special case is the ring $KK_G(\mathbf{C}, \mathbf{C})$ associated to a locally compact group G. If G is compact, this is nothing else as the representation ring $R(G)$. In general, it is defined as the set of homotopy classes of G-Fredholm modules. The existence of a product in a non trivial result in [K].

The only thing we shall need here is the definition of a G-Fredholm module: it is given by two unitary representations π_1 and π_2 of G respectively on Hilbert spaces H_1 and H_1, together with a bounded Fredholm operator $T : H_1 \to H_2$ such that $T\pi_1(g) - \pi_2(g)T$ is compact for any $g \in G$ and depends on g in a norm continuous manner.

The proof of [JV2] combined two ingredients:

1) The construction of a very simple Fredholm module associated to the group action on a tree. It defines a class γ in Kasparov's ring $KK_G(\mathbf{C}, \mathbf{C})$.

2) The construction of a homotopy proving that $\gamma = 1$ using the fact that the distance kernel on the set of vertices of the tree is of conditionally negative type. Or equivalently the existence of an affine action of G on the ℓ^2 space of edges of the tree.

Note that the two ingredients are of a rather different nature. The first can be generalized to other situations such as Bruhat-Tits buildings [JV3][KS1] or (hyper)bolic spaces [KS2]. The second is very specific to trees or generalizations (e.g. CAT(0) cubic complexes cf [BGH]).

I present here a perhaps more natural proof of the same result. It is inspired by the nice construction by Pytlik and Szwarc [PS] of a family of uniformly bounded representations of a free group, generalized by Valette [V1] and Szwarc [S1] to groups acting on trees. I have had this new version for quite a long time in my private notes. I thank Amaury Freslon and Jacek Brodzki for convincing me that making these notes available could be useful to others. I thank Ryszard Szwarc for pointing me reference [S1] and Jean-Pierre Schreiber for some comments and corrections.

I dedicate this short paper to the memory of Tadeusz Pytlik, who died in 2006 [S2].

1 Notations

Let $X = (X^0, X^1)$ be a tree. There is no orientation on X so that the set X^1 of edges is just a subset of $X^0 \times X^0$ stable by the map $(x, y) \mapsto (y, x)$. By hypothesis, for any x and y in X^0 there is a unique path joining x to y. We denote by $[x, y]$ the set of vertices lying between x and y and $d(x, y)$ the number of edges between x and y.

Let G be a group acting on X. In other words G acts on X^0 in such a way that the subset $X^1 \subset X^0 \times X^0$ is stable. For simplicity we consider a discrete group G but the arguments can be easily generalized to the case of a locally compact group G, such as $SL_2(\mathbf{Q}_p)$, cf. [JV2]. We consider the Hilbert space $\ell^2(X^0)$, $(\delta_x\,, x \in X^0)$ its canonical Hilbert basis and π_0 the unitary representation arising from the action of G on the set X^0 defined by $\pi_0(g)\delta_x = \delta_{gx}$. Let $\ell^2(X^1)^-$ be the quotient of $\ell^2(X^1)$ by the subspace generated by the vectors $\delta_{(x,y)} + \delta_{(y,x)}$ for $(x,y) \in X^1$ and π_1 the unitary representation of G on $\ell^2(X^1)^-$ defined by $\pi_1(g)\delta_{(x,y)} = \delta_{(gx,gy)}$

For $x \in X^0$ let $V(x)$ be the set of neighbours of x. Let the cardinality of $V(x)$ be denoted $q_x + 1$. We assume for simplicity that q_x is bounded. We define the following bounded operators S and Q on $\ell^2(X^0)$ by the following formulae:

$$S\delta_x = \sum_{y \in V(x)} \delta_y$$

$$Q\delta_x = q_x \delta_x.$$

Note that the operators Q and S only depend on the tree structure on X and therefore commute with the unitaries $\pi_0(g)$.

2 The Pytlik-Szwarc operator

We now choose an origin $x_0 \in X^0$. Let p_0 the orthogonal projection onto the vector δ_{x_0}. Let us define the operator P on $\ell^2(X^0)$ by

$$P\delta_{x_0} = 0$$

$$P\delta_x = \delta_{x'}$$

where for $x \neq x_0$ we define x' to be the unique neighbour of x lying between x_0 and x.

Proposition 1. *The operator P such defined is bounded on $\ell^2(X^0)$ and satisfies:*

$$PP^* = Q + p_0$$

$$P + P^* = S$$

Proof: For $x \neq x_0$, one has $P^*\delta_x = \sum_{y \in V(x) \setminus \{x'\}} \delta_y$ so that clearly $(P + P^*)\delta_x = S\delta_x$. Similarly $(P + P^*)\delta_{x_0} = P^*\delta_{x_0} = \sum_{y \in V(x_0)} \delta_y = S\delta_{x_0}$.

On the other hand, for $x \neq x_0$ and $y \in V(x) \setminus \{x'\}$ one has $P\delta_y = \delta_x$ so that $PP^*\delta_x = q_x\delta_x$, whereas for any $y \in V(x_0)$, $P\delta_y = \delta_{x_0}$ so that $PP^*\delta_{x_0} = (q_{x_0} + 1)\delta_{x_0}$

Corollary 1. *Let $T_t = 1 - tP + ((1 - t^2)^{1/2} - 1)p_0$ for any $t \in [0, 1]$ Then the operator T_t satisfies the following formula: $T_tT_t^* = 1 - tS + t^2Q$. In particular, the operator $T_tT_t^*$ commutes with the unitaries $\pi_0(g)$, $g \in G$.*

Indeed we have $T_t = 1 - tP + \alpha p_0$ where α satisfies $\alpha^2 + 2\alpha + t^2 = 0$. A straightforward calculation (using the obvious fact that $Pp_0 = 0$) yields $T_tT_t^* = 1 - t(P + P^*) + t^2PP^* + (2\alpha + \alpha^2)p_0$ and the result follows from the proposition.

3 Construction of new representations

Let us consider the space $D(X^0)$ of finitely supported functions on X^0 as a dense subspace of $\ell^2(X^0)$. Clearly $D(X^0)$ is stable by the operators P, P^* and $\pi_0(g)$ for $g \in G$.

Lemma 1. *For any complex number z the operator*

$$(1 - zP)^{-1} = \sum_{k=0}^{\infty} z^k P^k$$

is defined on $D(X^0)$. One has

$$(1 - zP)^{-1}\delta_x = \sum_{y \in [x_0,x]} z^{d(y,x)}\delta_y$$

Proof. Let the elements of $[x_0, x]$ be denoted x_0, x_1,..., $x_n = x$, where $n = d(x_0, x)$. Then $P^k\delta_x = \delta_{x_{n-k}}$ for $k \leq n$ and 0 for $k > n$. Note that $d(x_{n-k}, x) = k$. This makes the statement straightforward.

Theorem 1. *(Pytlik-Szwarc) Let z be a complex number such that $|z| < 1$. For any $g \in G$ the operator $\rho_z(g) = (1 - zP)^{-1}\pi_0(g)(1 - zP)$ extends to a bounded operator on $\ell^2(X^0)$, thus defining a representation ρ_z of G in $\ell^2(X^0)$. The operator $\rho_z(g) - \pi_0(g)$ is a finite rank operator and the representation ρ_z is uniformly bounded, i.e.*

$$\sup_{g \in G} \|\rho_z(g)\| < \infty.$$

In order to construct unitary representations, we shall need to modify the operators $1 - tP$ by the operators T_t as in Corollary 1. Note that the inverse operator T_t^{-1} is well defined on the space $D(X^0)$.

Theorem 2. *For any real number such that $0 < t < 1$ the operator $\tilde{\rho}_t(g) = T_t^{-1}\pi_0(g)T_t$ on $D(X^0)$ extends to a unitary operator on $\ell^2(X^0)$, thus defining a unitary representation $\tilde{\rho}_t$ of G in $\ell^2(X^0)$. The operator $\tilde{\rho}_t(g) - \pi_0(g)$ is a finite rank operator. The uniformly bounded representation ρ_t is equivalent to the unitary representation $\tilde{\rho}_t$.*

Proof of theorems 1 and 2: Let us prove that $\rho_z(g) - \pi_0(g)$ has finite rank and a norm bounded independently from g. It is enough to consider the operator

$$\rho_z(g)\pi_0(g)^{-1} - 1 = z(1 - zP)^{-1}(P - P')$$

where P' is defined just as P but replacing x_0 by gx_0. It is clear that the above operator is supported on the finite dimensional subspace generated by the δ_x's for $x \in [x_0, gx_0]$. That subspace is indeed stable by P and P' which have restrictions of norm 1. Therefore the norm of $(1 - zP)^{-1}(P - P')$ is at most $2\sum |z|^k = 2(1 - |z|)^{-1}$. This proves theorem 1.

To deduce theorem 2 note that $\tilde{\rho}_t(g) = u_t^{-1}\rho_t(g)u_t$ where $u_t = (1-p_0) + (1-t^2)^{1/2}p_0$ is an invertible operator which differs from the identity by a compact operator. It remains to show that $\tilde{\rho}_t(g)$ is unitary. Since it is invertible it is enough to compute $\tilde{\rho}_t(g)^*\tilde{\rho}_t(g) = T_t^*\pi_0(g)^{-1}(T_tT_t^*)^{-1}\pi_0(g)T_t$ which is equal to 1 since $T_tT_t^*$ commutes to $\pi_0(g)$ by the corollary to the proposition above.

Remark. The link with the original approach of [JV1][JV2] is given by an easy computation: the kernel defining the (densely defined) operator $(T_tT_t^*)^{-1}$ is

$$< T_t^{-1}\delta_x, T_t^{-1}\delta_y >= t^{d(x,y)} = e^{-\lambda d(x,y)}$$

if $t = e^{-\lambda}$.

4 The limit when t tends to 1

Let us now calculate the limit of $\tilde{\rho}_t(g)$ when $t \to 1$. Let us recall the definition of the Julg-Valette map $F : \ell^2(X^0) \to \ell^2(X^1)^-$:

$$F\delta_{x_0} = 0$$

$$F\delta_x = \delta_{(x,x')}$$

We have $Fp_0 = 0$, $F^*F = 1 - p_0$ and $FF^* = 1$.

Lemma 2. *Let $b : \ell^2(X^1)^- \to \ell^2(X^0)$ be the coboundary operator defined by $b\delta_{(x,y)} = \delta_y - \delta_x$. Then the operators F and P are related by the formulae:*

$$1 - P = bF + p_0$$

$$(1-P)F^* = b$$

Indeed, $(1-P)\delta_x = \delta_x - \delta_{x'} = bF\delta_x$ if $x \neq x_0$ and $(1-P)\delta_{x_0} = \delta_{x_0}$. The second formula follows from the first since $(1-P)F^* = bFF^* = b$.

Remark. Let $c(x,y) = \sum \delta_{(x_i,x_{i+1})}$ if the elements of $[x,y]$ are denoted $x_0 = x$, x_1,..., $x_n = y$. This is the cocycle realizing explicitly the Haagerup property for groups acting properly on trees: $\|c(x,y)\|^2 = d(x,y)$. On has $bc(x,y) = \delta_y - \delta_x$, hence by the second formula above, $c(x,y) = F(1-P)^{-1}(\delta_y - \delta_x)$.

Corollary 2. *The operator $(1-P)^{-1}b$ extends to a bounded operator and one has:*

$$(1-P)^{-1}b = F^*$$

It follows indeed from the lemma that $(1-P)F^* = b$.

Proposition 2. *For any $g \in G$ the unitary operator $\tilde{\rho}_t(g)$ $(0 < t < 1)$ converges strongly to*

$$\tilde{\rho}_1(g) = F^*\pi_1(g)F + p_0$$

when $t \to 1$.

Proof: Let us first prove that $F\tilde{\rho}_t(g)F^*$ strongly converges to $\pi_1(g)$. One has (as $p_0F^* = 0$):

$$F\tilde{\rho}_t(g)F^* = FT_t^{-1}\pi_0(g)T_tF^* = F(1-tP)^{-1}\pi_0(g)(1-tP)F^*$$

which evaluated on functions with finite support on X^1 converges to $F(1-P)^{-1}\pi_0(g)(1-P)F^*$, but this is equal to

$$F(1-P)^{-1}\pi_0(g)bFF^* = F(1-P)^{-1}b\pi_1(g) = FF^*\pi_1(g) = \pi_1(g)$$

by two applications of Corollary 2.

We deduce that $(1-p_0)\tilde{\rho}_t(g)(1-p_0)$ converges strongly to $F^*\pi_1(g)F$.

On the other hand we check that $\tilde{\rho}_t(g)\delta_{x_0} \to \delta_{x_0}$ when $t \to 1$. We have indeed $T_t\delta_{x_0} = (1-t^2)^{1/2}\delta_{x_0}$ so that $(1-tP)^{-1}\pi_0(g)T_t\delta_{x_0} = (1-t^2)^{1/2}(1-tP)^{-1}\delta_{gx_0} = (1-t^2)^{1/2}\sum t^{d(y,gx_0)}\delta_y$ where the sum is extended to the vertices y of the segment $[x_0, gx_0]$. Finally $T_t^{-1}\pi_0(g)T_t\delta_{x_0} = t^{d(x_0,gx_0)}\delta_{x_0} + (1-t^2)^{1/2}\sum t^{d(y,gx_0)}\delta_y$ where x_0 is now excluded from the sum. Hence the result.

As a consequence $\tilde{\rho}_t(g)p_0$ converges normally to p_0, and since $\tilde{\rho}_t$ is unitary, replacing g by g^{-1} we also have that $p_0\tilde{\rho}_t(g)$ converges normally to p_0.

The proposition clearly follows.

5 Classes in KK_G-theory

Let us consider the Hilbert space $\ell^2(X^0)$ equipped with the two representations $\tilde{\rho}_t$ and π_0, which differ by compact operators. The triple $(\tilde{\rho}_t, \pi_0, Id)$ defines an element of the Kasparov group $KK_G(\mathbf{C}, \mathbf{C})$. It is independent of the value on $t \in [0,1]$ since the operators $\tilde{\rho}_t(g)$ are strongly continuous in t. Now when $t = 0$ we have $\tilde{\rho}_0 = \pi_0$ so that the element is equal to 0. On the other hand when $t = 1$ we have $\tilde{\rho}_1(g) = F^*\pi_1(g)F + p_0$ so that the element is equal to $1-\gamma$ where γ is defined as in [JV1][JV2] by the triple (π_0, π_1, F).

Corollary 3. *We have the equality $\gamma = 1$ in the Kasparov group $KK_G(\mathbf{C}, \mathbf{C})$.*

Recall [JV2] that the above corollary implies the K-theoretic amenability of G under the hypothesis that the vertices of the tree have amenable stabilisers.

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