Classification of Lorentz surfaces with parallel mean curvature vector in non-flat pseudo-Riemannian space forms $S^4_2(1)$ and $H^4_2(-1)$

Dan Yang Yu Fu

Abstract

Lorentz surfaces with parallel mean curvature vector in $\mathbb{E}^4_2$ have been classified in [14]. In this paper, we continue to classify Lorentz surfaces with parallel mean curvature vector in pseudo-Riemannian space forms $S^4_2(1)$ and $H^4_2(-1)$. Consequently, we achieve the complete classification of Lorentz surfaces with parallel mean curvature vector in 4-dimensional neutral indefinite space form with index 2.

1 Introduction

Let $\mathbb{E}^m_t$ denote the pseudo-Euclidean m-space equipped with pseudo-Euclidean metric of index t given by

$$g_0 = -\sum_{i=1}^{t} dx_i^2 + \sum_{j=t+1}^{n} dx_j^2,$$

Received by the editors in September 2012 - In revised form in November 2012.
Communicated by F. Bourgeois.
2010 *Mathematics Subject Classification* : 53B25, 53C40, 53C42.
Key words and phrases : Lorentz surface; Parallel mean curvature vector; Marginally trapped surface; Pseudo-Riemannian space form.

where $(x_1, ..., x_n)$ is the rectangular coordinate system of $\mathbb{E}^m_t$. Put

$$S^k_s(c_0, r^2) = \left\{ x \in \mathbb{E}^{k+1}_s | \langle x - c_0, x - c_0 \rangle = 1/r^2 > 0 \right\}, \tag{1.1}$$

$$H^k_s(c_0, -r^2) = \left\{ x \in \mathbb{E}^{k+1}_{s+1} | \langle x - c_0, x - c_0 \rangle = -1/r^2 < 0 \right\}, \tag{1.2}$$

where $\langle , \rangle$ is the associated inner product and c_0 is a fixed point. Then $S^k_s(c_0, r^2)$ and $H^k_s(c_0, -r^2)$ are complete semi-Riemannian manifolds with index s of constant curvature r^2 and $-r^2$, respectively. We denote $S^k_s(c_0, r^2)$ and $H^k_s(c_0, -r^2)$ by $S^k_s(r^2)$ and $H^k_s(-r^2)$ when c_0 is the origin. In general relativity, the Lorentz manifolds $\mathbb{E}^k_1$, $S^k_1(r^2)$ and $H^k_1(-r^2)$ are known as the Minkowski, de Sitter and anti-de Sitter space, respectively.

It is well known that submanifolds with parallel mean curvature vector play important roles in differential geometry, theory of harmonic maps as well as in physics. Surfaces with parallel mean curvature vector in Euclidean space were classified in the early 1970s by Chen and Yau[10]. Further, spacelike surfaces with parallel mean curvature vector in arbitrary indefinite space forms were completely classified (see [8, 11, 12, 13]). However, the study of the classification of Lorentz surfaces is less relative to the spacelike surfaces. In [14], we firstly classified Lorentz surfaces with parallel mean curvature vector in $\mathbb{E}^4_2$. Soon after, Lorentz surfaces with parallel mean curvature vector in pseudo-Euclidean spaces with arbitrary codimension and index were classified in [15] and [16], independently. (For an up-to-date survey on submanifolds with parallel mean curvature vector, see [17]). Hence it is an interesting problem to classify all Lorentz surfaces with parallel mean curvature vector in non-flat pseudo-Riemannian space forms.

In this paper, we achieve the classification of Lorentz surfaces with parallel mean curvature vector in 4-dimensional pseudo-Riemannian space forms $S^4_2(1)$ and $H^4_2(-1)$. Our results state that there exist 19 families of Lorentz surfaces in $S^4_2(1)$ and $H^4_2(-1)$, respectively.

2 Preliminaries

2.1 Basic formulas

Let $R^4_2(c)$ denote an 4-dimensional pseudo-Riemannian space form with index 2 of constant curvature c. Then the curvature tensor $\widetilde{R}$ of $R^4_2(c)$ is given by

$$\widetilde{R}(X, Y)Z = c\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\}.$$

Let M be a Lorentz surface in $R^4_2(c)$. Denote by ∇ and $\tilde{\nabla}$ the Levi Civita connections of M and $R^4_2(c)$, respectively. For vector fields X and Y tangent to M and vector field ξ normal to M, the formulas of Gauss and Weingarten are given by (cf. [10, 18, 19])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X Y, \tag{2.2}$$

where h, A and D are the second fundamental form, the shape operator and the normal connection, respectively. It is well known that h and A are related by

$$\langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle. \tag{2.3}$$

We define the mean curvature $H = \frac{1}{2}\text{trace } h$. The equation of Gauss and Codazzi are given respectively by

$$\begin{aligned} \langle R(X,Y)Z, W \rangle &= c\{\langle Y,Z \rangle \langle X,W \rangle - \langle X,Z \rangle \langle Y,W \rangle\} \\ &+ \langle h(Y,Z), h(X,W) \rangle - \langle h(X,Z), h(Y,W) \rangle, \\ (\bar{\nabla}_X h)(Y,Z) &= (\bar{\nabla}_Y h)(X,Z), \end{aligned}$$

where R is the curvature tensor of M and $\bar{\nabla} h$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.4}$$

We denote R^D the curvature tensor associated with the normal connection D, i.e.,

$$R^D(X,Y) = D_X D_Y - D_Y D_X - D_{[X,Y]},$$

then for vector fields X and Y tangent to M and vector field ξ, η normal to M, the Ricci equation is given by

$$\langle R^D(X,Y)\xi, \eta \rangle = \langle [A_\xi, A_\eta] X, Y \rangle.$$

2.2 Basic definitions

A surface in a pseudo-Riemannian 3-manifold (or a light cone) is called *CMC* if its mean curvature vector H satisfies $\langle H, H \rangle = \text{constant} \neq 0$.

A vector v is called *spacelike* (*timelike*) if $\langle v, v \rangle > 0$ ($\langle v, v \rangle < 0$). A nonzero vector v is called *lightlike* if $\langle v, v \rangle = 0$. A curve $z : I \to \mathbb{E}^m_t$ defined on an open interval $I \subset \mathbf{R}$ is called *null* if its velocity vector $z'(x)$ is lightlike for each $x \in I$.

A surface in a semi-Riemannian manifold is called *marginally trapped* if its mean curvature vector is lightlike. Recently, marginally trapped surfaces have been studied from a mathematical viewpoint, such as in [1, 2, 3, 4, 5, 6, 7, 8, 9].

2.3 Light cones

The light cone $\mathcal{LC}^{n-1}_s(c_0)$ with vertex c_0 in $\mathbb{E}^n_s$ is defined by

$$\mathcal{LC}^{n-1}_s(c_0) = \{x \in \mathbb{E}^n_s : \langle x - c_0, x - c_0 \rangle = 0\}.$$

We simply denote the light cone $\mathcal{LC}^{n-1}_s(0)$ by $\mathcal{LC}$ if there is no confusion possible.

The light cone $\mathcal{LC}^{n-1}_s$ can be naturally embedded in $S^n_s(1)$ via

$$\iota : \mathcal{LC}^{n-1}_s \subset \mathbb{E}^n_s \to S^n_s(1) \subset \mathbb{E}^{n+1}_s : y \mapsto (y, 1) \in \mathbb{E}^{n+1}_s.$$

The light cone $\mathcal{LC}^{n-1}_s$ can be naturally embedded in $H^n_s(-1)$ via

$$\iota : \mathcal{LC}^{n-1}_s \subset \mathbb{E}^n_s \to H^n_s(-1) \subset \mathbb{E}^{n+1}_{s+1} : y \mapsto (1, y) \in \mathbb{E}^{n+1}_{s+1}.$$

2.4 Moving frames.

We assume that M is a Lorentz surface in $R^4_2(c)$, $\{e_1, e_2\}$ is a local tangent frame and $\{e_3, e_4\}$ is a local normal frame, which satisfy

$$\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0, \quad \langle e_1, e_2\rangle = -1, \tag{2.5}$$

$$\langle e_3, e_3\rangle = \langle e_4, e_4\rangle = 0, \quad \langle e_3, e_4\rangle = -1. \tag{2.6}$$

If we let

$$\nabla_X e_1 = \omega_1^1(X)e_1 + \omega_1^2(X)e_2, \quad \nabla_X e_2 = \omega_2^1(X)e_1 + \omega_2^2(X)e_2,$$

then from (2.5) we obtain that $\omega_1^2 = \omega_2^1 = 0$ and $\omega_1^1 = -\omega_2^2$. If we put $\omega = \omega_1^1$, then

$$\nabla_X e_1 = \omega(X)e_1, \quad \nabla_X e_2 = -\omega(X)e_2. \tag{2.7}$$

We put $\omega(e_1) = \omega_1$ and $\omega(e_2) = \omega_2$. Similarly, for some one-form ϕ, we have

$$D_X e_3 = \phi(X)e_3, \quad D_X e_4 = -\phi(X)e_4. \tag{2.8}$$

2.5 Some lemmas

We introduce some results for later use.

Lemma 2.1. *[20]. There exist local coordinates (x, y) on M_1^2 such that the metric of the surface is given by $g = -m^2(x, y)(dx \otimes dy + dy \otimes dx)$ for some positive function $m(x, y)$. The Levi-Civita connection of the surface is then given by*

$$\nabla_{\partial_x}\partial_x = \frac{2m_x}{m}\partial_x, \quad \nabla_{\partial_x}\partial_y = 0, \quad \nabla_{\partial_y}\partial_y = \frac{2m_y}{m}\partial_y, \tag{2.9}$$

and its Gaussian curvature is $K = 2(mm_{xy} - m_x m_y)/m^4$.

Similar to the Lemma 2.2 of [14], we establish the following lemma.

Lemma 2.2. *Let M be a Lorentz surface in $R^4_2(c)$ with parallel mean curvature vector, then (1) $\phi = 0$, which implies $R^D = 0$; (2) $\langle H, H\rangle$ is constant.*

3 Lorentz surfaces in $S^4_2(1)$

Let $\mathcal{K}_a = \{(x_1, x_2, \cdots, x_5) \in \mathbb{E}^5_2 : x_5 = x_1 + a\}$. For any two vectors $a = (a_1, \ldots, a_5)$, $b = (b_1, \ldots, b_5)$ in $\mathbb{E}^5_2$, we put $a * b = (a_1 b_1, \ldots, a_5 b_5)$. The following theorem classifies Lorentz surfaces with parallel mean curvature vector in $S^4_2(1)$.

Theorem 3.1. *Let M be a Lorentz surface with parallel mean curvature vector in de Sitter space-time $S_2^4(1) \subset \mathbb{E}_2^5$, then L is congruent to a surface of the following 19 families.*

1. *A minimal Lorentz surface of $S_2^4(1)$;*

2. *A Lorentz surface of curvature one with constant lightlike mean curvature vector, lying in $\mathcal{K}_0 \cap S_2^4(1)$, which is defined by*
$$L(x,y) = \left(f(x,y), \quad \frac{xy-1}{x+y}, \quad \frac{xy+1}{x+y}, \quad \frac{y-x}{x+y}, \quad f(x,y) \right),$$
for some function $f(x,y)$.

3. *A Lorentz surface of curvature one defined by $L = -\frac{p(x)}{x+y} + q(x)$, where $p(x)$ is a curve lying in the light cone $\mathcal{LC}$ and $q(x)$ is a null curve satisfying $\langle p', q' \rangle = 0, \langle p, q' \rangle = -2, \langle p', p' \rangle = 4$;*

4. *A Lorentz marginally trapped surface of curvature one in $S_2^4(1)$ and lies in $\mathcal{LC}_2^3 = \{(y,1) \in \mathbb{E}_2^5 : \langle y,y \rangle = 0, y \in \mathbb{E}_2^4\} \subset S_2^4(1)$, which is defined by*
$$L(x,y) = \frac{1}{x+y}(u(x) * z(y) + v(x) * w(y)) + c_0,$$
where $c_0 = (0,0,0,0,1)$, $u''(x) + c_1(x)u(x) = v''(x) + c_1(x)v(x) = z''(y) + c_2(y)z(y) = w''(y) + c_2(y)w(y) = 0$ and $\langle u'(x) * z(y) + v'(x) * w(y), u(x) * z'(y) + v(x) * w'(y) \rangle = -2$ for some functions $c_1(x)$ and $c_2(y)$.

5. *a non-flat Lorentz surface which lies in $S_2^4(c_0, r^2) \cap S_2^4(1)$ such that the mean curvature vector H' of M in $S_2^4(c_0, r^2) \cap S_2^4(1)$ satisfies $\langle H', H' \rangle = 1 - r^2$.*

6. *a non-flat Lorentz surface which lies in $H_1^4(c_0, -r^2) \cap S_2^4(1)$ such that the mean curvature vector H' of M in $H_1^4(c_0, -r^2) \cap S_2^4(1)$ satisfies $\langle H', H' \rangle = 1 + r^2$.*

7. *A flat marginally trapped surface defined by*
$$L = \left(u, u^2 + \frac{1}{2}, u^2 + 1, \frac{1}{2}\sin 2v, \cos 2v \right).$$

8. *A non-flat CMC surface lying in $S_2^4(c_0, r^2) \cap S_2^4(1)$ such that the mean curvature vector H' of M in $S_2^4(c_0, r^2) \cap S_2^4(1)$ satisfies $\langle H', H' \rangle = 1 - r^2 - 2a$ for a nonzero real number a.*

9. *a non-flat CMC surface lying in $H_1^4(c_0, -r^2) \cap S_2^4(1)$ such that the mean curvature vector H' of M in $H_1^4(c_0, -r^2) \cap S_2^4(1)$ satisfies $\langle H', H' \rangle = 1 + r^2 - 2a$ for a nonzero real number a.*

10. *A flat surface defined by*
$$L = \left(\frac{\cos\sqrt{m}u}{\sqrt{2m}}, \frac{\sin\sqrt{m}u}{\sqrt{2m}}, \frac{\cos\sqrt{n}v}{\sqrt{2n}}, \frac{\sin\sqrt{n}v}{\sqrt{2n}}, \pm\sqrt{1 + \frac{1}{2m} - \frac{1}{2n}} \right),$$
where $m = a(2a+3)$ and $n = 2a^2 - 5a + 4$ for $a \in (-\infty, -\frac{3}{2}) \cup (0, \frac{1}{2}) \cup (\frac{1}{2}, +\infty)$.

11. *A flat surface defined by*

$$L=\left(\frac{u}{\sqrt{2}},\ u^2+\frac{13}{8},\ u^2+\frac{15}{8},\ \frac{\cos 2v}{2\sqrt{2}},\ \frac{\sin 2v}{2\sqrt{2}}\right).$$

12. *A flat surface defined by*

$$L=\left(\frac{u}{\sqrt{2}},\ u^2+\frac{29}{16},\ u^2+\frac{33}{16},\ \frac{\cos 4v}{4\sqrt{2}},\ \frac{\sin 4v}{4\sqrt{2}}\right).$$

13. *A flat surface defined by*

$$L=\left(\pm\sqrt{\frac{1}{2n}-\frac{1}{2m}-1},\ \frac{\sinh\sqrt{-m}u}{\sqrt{-2m}},\ \frac{\cosh\sqrt{-m}u}{\sqrt{-2m}},\ \frac{\cos\sqrt{n}v}{\sqrt{2n}},\ \frac{\sin\sqrt{n}v}{\sqrt{2n}}\right),$$

where $m=a(2a+3)$ *and* $n=2a^2-5a+4$ *for* $a\in(-\frac{3}{2},0)$.

14. *A non-flat CMC surface lying in* $S^4_2\cap\pi$, *where* π *is hyperplane of index 2 in* $\mathbb{E}^5_2$.

15. *A non-flat CMC surface lying in* $S^4_2\cap\pi$, *where* π *is hyperplane of index 1 in* $\mathbb{E}^5_2$.

16. *A non-flat surface defined by*

$$L(x,y)=-\frac{1}{x+y}u(y)+v(y),$$

where $u(y)$ *is a curve lying in the light cone* $\mathcal{LC}$ *and* $v(y)$ *is a null curve satisfying* $\langle u',v'\rangle=0$, $\langle u',u'\rangle=\frac{4}{1-2a}$ *and* $\langle u,v'\rangle=\frac{2}{2a-1}$ *for a real number* $a<1/2$.

17. *A non-flat CMC surface lying in* $\mathcal{LC}^3_2=\{(y,1)\in\mathbb{E}^5_2:\langle y,y\rangle=0, y\in\mathbb{E}^4_2\}\subset S^4_2(1)$ *defined by*

$$L(x,y)=\frac{1}{x+y}(u(x)*z(y)+v(x)*w(y))+c_0,$$

where u,v,z,w *are curves in* $\mathbb{E}^5_2$ *satisfying* $u''(x)+c_1(x)u(x)=v''(x)+c_1(x)v(x)=z''(y)+c_2(y)z(y)=w''(y)+c_2(y)w(y)=0$, *and* $\langle u'(x)*z(y)+v'(x)*w(y),\ u(x)*z'(y)+v(x)*w'(y)\rangle=\frac{2}{2a-1}$ *for functions* $c_1(x)$, $c_2(y)$ *and for a real number* $a<1/2$.

18. *A non-flat surface defined by*

$$L(x,y)=\frac{1}{x-y}u(y)+v(y),$$

where $u(y)$ *is a curve lying in the light cone* $\mathcal{LC}$ *and* $v(y)$ *is a null curve satisfying* $\langle u',v'\rangle=0$, $\langle u',u'\rangle=\frac{4}{1-2a}$ *and* $\langle u,v'\rangle=\frac{2}{2a-1}$ *for a real number* $a>1/2$.

19. *A non-flat CMC surface lying in* $\mathcal{LC}_2^3 = \{(y,1) \in \mathbb{E}_2^5 : \langle y,y\rangle = 0, y \in \mathbb{E}_2^4\} \subset S_2^4(1)$ *defined by*

$$L(x,y) = \frac{1}{x-y}(u(x) * z(y) + v(x) * w(y)) + c_0,$$

where $c_0 = (0,0,0,0,1)$, u, v, z, w *are curves in* $\mathbb{E}_2^4$ *satisfying* $u''(x) + c_1(x)u(x) = v''(x) + c_1(x)v(x) = z''(y) + c_2(y)z(y) = w''(y) + c_2(y)w(y) = 0$, *and* $\langle u'(x) * z(y) + v'(x) * w(y),\ u(x) * z'(y) + v(x) * w'(y)\rangle = -\frac{2}{(2a-1)}$ *for functions* $c_1(x)$, $c_2(y)$ *and for a real number* $a > 1/2$.

Remark 3.2. Case (2) $-$ (7) are marginally trapped Lorentz surfaces with parallel mean curvature vector in $S_2^4(1) \subset \mathbb{E}_2^5$.

Proof. Since M is a Lorentz surface in $S_2^4(1)$ with parallel mean curvature vector, then $\langle H, H\rangle$ is constant and $H = 0$, or H is lightlike, or $\langle H, H\rangle$ is a nonzero constant.

If $H = 0$, we get case (1).

If H is lightlike, then M is a marginally trapped Lorentz surface in $S_2^4(1)$ with parallel mean curvature vector. There exists a pseudo-orthonormal frame $\{e_3, e_4\}$ satisfying (2.6) such that $-H = h(e_1, e_2) = e_3$. Let us regard $S_2^4(1)$ as a hypersurface of $\mathbb{E}_2^5$ via (1.1). Denote by ∇^S and $\widetilde{\nabla}$ be the Levi-Civita connections of $S_2^4(1)$ and $\mathbb{E}_2^5$, respectively. Let $\widetilde{D}$ and $\widetilde{A}$ be the normal connection and the shape operator of M in $\mathbb{E}_2^5$ respectively; Let D and A the corresponding quantities for M in $S_2^4(1)$. Then we have

$$\widetilde{D}\xi = D\xi, \quad A_\xi = \widetilde{A}_\xi, \quad \widetilde{\nabla}_X\xi = \nabla^S_X\xi \tag{3.1}$$

for any normal vector field ξ of M in $S_2^4(1)$ and any $X \in TM$. Since M has parallel mean curvature vector H, from Lemma 2.2 we have

$$\widetilde{D}e_3 = \widetilde{D}e_4 = De_3 = De_4 = 0. \tag{3.2}$$

Let

$$h(e_1,e_1) = \alpha e_3 + \beta e_4, \quad h(e_1,e_2) = e_3, \quad h(e_2,e_2) = \gamma e_3 + \delta e_4, \tag{3.3}$$

for some functions α, β, γ, δ. By (2.3), (2.5) and (2.6), we have

$$A_{e_3} = \begin{pmatrix} 0 & \delta \\ \beta & 0 \end{pmatrix}, \qquad A_{e_4} = \begin{pmatrix} 1 & \gamma \\ \alpha & 1 \end{pmatrix}. \tag{3.4}$$

From Lemma 2.2 and the Ricci equation we have $[A_{e_3}, A_{e_4}] = 0$, which implies that $\alpha\delta = \beta\gamma$. It follows from (3.3) and the Gauss equation that the Gauss curvature K of M is given by

$$K = 1 + 2\alpha\delta = 1 + 2\beta\gamma. \tag{3.5}$$

Case (A): $K \neq 1$. It follows from (3.5) that $\alpha, \beta, \delta, \gamma \neq 0$. In this case, (2.7), (3.2) and (3.3) show that Codazzi equation (2.4) reduces to

$$e_2(\alpha) = 2w_2\alpha, \quad e_2(\beta) = 2w_2\beta, \quad e_1(\gamma) = -2w_1\gamma, \quad e_1(\delta) = -2w_1\delta, \tag{3.6}$$

which together with $\alpha\delta = \beta\gamma$ forces that $e_i(\ln \mid \alpha/\beta \mid) = e_i(\ln \mid \gamma/\delta \mid) = 0$ for $i = 1, 2$. Then there exists a nonzero real number c such that

$$\alpha = c\beta, \qquad \gamma = c\delta. \tag{3.7}$$

By (2.7) and (3.6), we have $[\beta^{-\frac{1}{2}}e_1, \delta^{-\frac{1}{2}}e_2] = 0$. Then there exists a coordinate system $\{x, y\}$ such that

$$\frac{\partial}{\partial x} = \beta^{-\frac{1}{2}}e_1, \quad \frac{\partial}{\partial y} = \delta^{-\frac{1}{2}}e_2, \quad g = -(\beta\delta)^{-\frac{1}{2}}dxdy. \tag{3.8}$$

We denote $\rho = (\beta\delta)^{-\frac{1}{2}}$, a direct computation shows that the Levi-Civita connection of g satisfies

$$\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = (\ln\rho)_x\frac{\partial}{\partial x}, \quad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = 0, \quad \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = (\ln\rho)_y\frac{\partial}{\partial y}. \tag{3.9}$$

Moreover, from (3.3), (3.4), (3.7) and (3.8), we have

$$h(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = ce_3 + e_4, \quad h(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = \rho e_3, \quad h(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = ce_3 + e_4, \tag{3.10}$$

$$A_{e_3} = \begin{pmatrix} 0 & \rho^{-1} \\ \rho^{-1} & 0 \end{pmatrix}, \qquad A_{e_4} = \begin{pmatrix} 1 & c\rho^{-1} \\ c\rho^{-1} & 1 \end{pmatrix}. \tag{3.11}$$

By (3.9), (3.10), and (2.1), we have

$$L_{xx} = (\ln\rho)_x L_x + ce_3 + e_4, \quad L_{xy} = \rho(e_3 + L), \quad L_{yy} = (\ln\rho)_y L_y + ce_3 + e_4. \tag{3.12}$$

The compatibility condition of this system is given by Poisson equation:

$$(\ln\rho)_{xy} = 2c\rho^{-1} + \rho. \tag{3.13}$$

Moreover, if we let $\xi = -ce_3 + e_4, \eta = ce_3 + e_4$, then

$$D\xi = D\eta = 0, \quad \langle\xi, \xi\rangle = 2c, \quad \langle\eta, \eta\rangle = -2c, \quad \langle\xi, \eta\rangle = 0, \quad A_\xi = I.$$

Consider the map $\psi : M \to \mathbb{E}^5_2$ defined by $\psi(p) = L(p) + \xi(p)$. Then we have $\tilde{\nabla}_X\psi = 0$ for $X \in TM$. So $\psi = L + \xi$ is a constant vector, say $c_0 \in \mathbb{E}^5_2$. Thus, $L - c_0 = -\xi$ and hence

$$\langle L - c_0, L - c_0\rangle = 2c = \text{constant}. \tag{3.14}$$

Case (A.a): $c > 0$. In this case, (3.14) implies that M lies in $S^4_2(c_0, r^2)$ with $r^2 = 1/2c$. The mean curvature vector H' of M in $S^4_2(c_0, r^2)$ and the mean curvature vector H in $\mathbb{E}^5_2$ are related by $H = H' - r^2(L - c_0)$. Since M is marginally trapped in $S^4_2(1)$, we have $1 = \langle H', H'\rangle + r^2$. This gives $\langle H', H'\rangle = 1 - r^2$. We can conclude that M is non-flat. In fact, if M is flat,We choose $\rho = 1$ and hence from (3.13) we have that $c = -1/2$. This is a contradiction. Hence we get case (5).

Case (A.b): $c < 0$. In this case (3.14) implies that M lies in $S^4_2(1) \cap H^4_1(c_0, -r^2)$ with $-r^2 = 1/2c$. Since the mean curvature vector H' of M in $H^4_1(c_0, -r^2)$ and H in $\mathbb{E}^5_2$ are related by $H = H' + r^2(L - c_0)$, and since M is marginally trapped in $S^4_2(1)$, we have $1 = \langle H', H' \rangle - r^2$. This gives $\langle H', H' \rangle = 1 + r^2$. If M is non-flat, we get case (6).

If M is flat, we choose $\rho = 1$ and hence $c = -1/2$ from (3.13). In this case, the PDE system (3.12) becomes

$$L_{xx} = -\frac{1}{2}e_3 + e_4, \quad L_{xy} = e_3 + L, \quad L_{yy} = -\frac{1}{2}e_3 + e_4.$$

We put $x = (u+v)/\sqrt{2}$, $y = (u-v)/\sqrt{2}$, then

$$L_{uv} = 0, \qquad L_{uuu} = 0, \qquad L_{vvv} = -4L_v.$$

Solving these system of differential equation, we obtain

$$L = c_1 u + c_2 u^2 + c_3 \sin 2v + c_4 \cos 2v + c_5,$$

for some vectors $c_i \in \mathbb{E}^5_2, i = 1, \cdots, 5$. After choosing suitable initial conditions, we obtain case (7).

Case (B): $K = 1$. It follows from (3.5) that $\alpha\delta = \beta\gamma = 0$ and M is a Lorentz surface of curvature one. From Lemma 2.1, we may choose coordinates $\{x, y\}$ on M so that the metric tensor of M is given by

$$g = -\frac{2}{(x+y)^2} dxdy. \tag{3.15}$$

The Levi-Civita connection of the surface M is then given by

$$\nabla_{\partial_x}\partial_x = -\frac{2}{x+y}\partial_x, \qquad \nabla_{\partial_x}\partial_y = 0, \qquad \nabla_{\partial_y}\partial_y = -\frac{2}{x+y}\partial_y. \tag{3.16}$$

And (3.3) becomes

$$h(\partial_x, \partial_x) = 2\frac{\alpha e_3 + \beta e_4}{(x+y)^2}, \quad h(\partial_x, \partial_y) = 2\frac{e_3}{(x+y)^2}, \quad h(\partial_y, \partial_y) = 2\frac{\gamma e_3 + \delta e_4}{(x+y)^2}. \tag{3.17}$$

It follows from (3.16) and (3.17) that $L : M \to S^4_2(1) \subset \mathbb{E}^5_2$ satisfies

$$L_{xx} = -\frac{2}{x+y}L_x + \frac{2}{(x+y)^2}(\alpha e_3 + \beta e_4), \tag{3.18}$$

$$L_{xy} = \frac{2}{(x+y)^2}(e_3 + L), \tag{3.19}$$

$$L_{yy} = -\frac{2}{x+y}L_y + \frac{2}{(x+y)^2}(\gamma e_3 + \delta e_4). \tag{3.20}$$

Case (B.a): $\delta = \beta = 0$. In this case, $A_{e_3} = 0$, which together with (3.1) and (3.2) shows that e_3 is a constant lightlike vector in $\mathbb{E}^5_2$. So, without loss of generality, we may put $e_3 = (1,0,0,0,1) \in \mathbb{E}^5_2$. It follows that M lies in $\mathcal{K}_0 \cap S^4_2(1)$ and the mean curvature vector of M in $S^4_2(1)$ is a constant lightlike vector in $\mathbb{E}^5_2$.

On the other hand, the compatibility conditions of the system (3.18)−(3.20) are given by

$$\alpha_y = \frac{2}{x+y}\alpha, \qquad \gamma_x = \frac{2}{x+y}\gamma. \tag{3.21}$$

Hence there exist functions $p(x)$ and $q(y)$ such that $\alpha = p(x)(x+y)^2$ and $\gamma = q(y)(x+y)^2$. Then (3.18) and (3.20) become

$$L_{xx} = -\frac{2}{x+y}L_x + 2p(x)e_3, \qquad L_{yy} = -\frac{2}{x+y}L_y + 2q(y)e_3. \tag{3.22}$$

Solving equation (3.22) gives

$$L = f(x,y)e_3 + \frac{c_1xy + c_2x + c_3y + c_4}{x+y},$$

where

$$f(x,y) = 2\Big(\iint p(x)dx^2 + \iint q(y)dy^2\Big) - \frac{4}{x+y}\Big(\iiint p(x)dx^3 + \iiint q(y)dy^3\Big).$$

From (3.19), we have $c_2 + c_3 + 2e_3 = 0$. After choosing suitable initial conditions, we obtain case (2).

Case (B. b): $\delta = \gamma = 0$. In this case, equation (3.20) becomes

$$L_{yy} = -\frac{2}{x+y}L_y. \tag{3.23}$$

Solving (3.23), we have

$$L = -\frac{1}{x+y}p(x) + q(x)$$

for some $\mathbb{E}^5_2$-valued functions $p(x)$ and $q(x)$. Thus we have

$$L_x = \frac{1}{(x+y)^2}p(x) - \frac{1}{x+y}p'(x) + q'(x), \quad L_y = -\frac{1}{(x+y)^2}p(x). \tag{3.24}$$

By using (3.24) and $g = -\frac{2}{(x+y)^2}dxdy$, we obtain

$$\langle p,p\rangle = \langle p',q'\rangle = \langle q',q'\rangle = 0, \quad \langle p,q'\rangle = -2, \quad \langle p',p'\rangle = 4.$$

This gives case (3).

Case (B. c): $\alpha = \beta = 0$. After interchanging x and y, we get case (3) as well.

Case (B. d): $\alpha = \gamma = 0$. In this case, $A_{e_4} = I$. So $\widetilde{\nabla}_X e_4 = -X$ for $X \in TM$ and hence $L + e_4$ is a constant vector in $\mathbb{E}^5_2$, say c_0. Since e_4 is tangent to $S^4_2(1)$, we find $\langle L, e_4\rangle = 0$. Combining this with $\langle L, L\rangle = 1$ gives $\langle c_0, c_0\rangle = 1$. Hence c_0 is a unit spacelike vector. Without loss of generality, we may put $c_0 = (0,0,0,0,1)$. On the other hand, from $\langle L-c_0, L-c_0\rangle = \langle e_4, e_4\rangle = 0$ we get $\langle L, c_0\rangle = 1$. It follows from $\langle L, L\rangle = \langle L, c_0\rangle = 1$ that $x_1^2+x_2^2 = x_3^2+x_4^2$, $x_5 = 1$, where $x_1, \cdots, x_5$ are coordinates of L in $\mathbb{E}_2^5$. So, M lies in $\mathcal{LC}_2^3 = \{(y,1) \in \mathbb{E}_2^5 : \langle y, y\rangle = 0, y \in \mathbb{E}_2^4\} \subset S_2^4(1)$.

On the other hand, the compatibility condition of the system (3.18)$-$(3.20) are given by

$$\beta_y = \frac{2}{x+y}\beta, \qquad \delta_x = \frac{2}{x+y}\delta.$$

Hence there exist function $c_1(x)$ and $c_2(y)$ such that $\beta = c_1(x)(x+y)^2/2$, $\delta = c_2(y)(x+y)^2/2$. Then (3.18) and (3.20) become

$$L_{xx} = -\frac{2}{x+y}L_x + c_1(x)(c_0 - L), \quad L_{yy} = -\frac{2}{x+y}L_y + c_2(y)(c_0 - L). \tag{3.25}$$

Solving (3.25) gives

$$L(x,y) = \frac{1}{x+y}f(x,y) + c_0, \tag{3.26}$$

where $f(x,y)$ is a vector-valued function lying in the light cone $\mathcal{LC}(c_0)$ and satisfying

$$f_{xx} = -c_1(x)f, \qquad f_{yy} = -c_2(y)f. \tag{3.27}$$

Equation (3.27) implies that

$$f(x,y) = u(x) * z(y) + v(x) * w(y), \tag{3.28}$$

where u, v, z, w are curves in $\mathbb{E}_2^5$ satisfying

$$u''(x) + c_1(x)u(x) = v''(x) + c_1(x)v(x) = z''(y) + c_2(y)z(y) = w''(y) + c_2(y)w(y) = 0.$$

Hence (3.26) and (3.28) imply that

$$L(x,y) = \frac{1}{x+y}(u(x) * z(y) + v(x) * w(y)) + c_0. \tag{3.29}$$

By applying (3.29) and $g = -\frac{2}{(x+y)^2}dxdy$, we have

$$\langle u'(x) * z(y) + v'(x) * w(y),\ u(x) * z'(y) + v(x) * w'(y)\rangle = -2.$$

This gives case (4).

If $\langle H, H\rangle$ is a nonzero constant. Then there exists a pseudo-orthonormal frame $\{e_3, e_4\}$ satisfying (2.6) such that $-H = h(e_1, e_2) = e_3 + ae_4 (a \neq 0)$. Let

$$h(e_1,e_1) = \alpha e_3 + \beta e_4, \quad h(e_1,e_2) = e_3 + ae_4, \quad h(e_2,e_2) = \gamma e_3 + \delta e_4, \tag{3.30}$$

for some functions α, β, γ, δ. Then

$$A_{e_3} = \begin{pmatrix} a & \delta \\ \beta & a \end{pmatrix}, \qquad A_{e_4} = \begin{pmatrix} 1 & \gamma \\ \alpha & 1 \end{pmatrix}. \tag{3.31}$$

It follows from Lemma 2.2 and the Ricci equation that $[A_{e_3}, A_{e_4}] = 0$, which implies that $\alpha\delta = \beta\gamma$. By (3.30) and Gauss equation, we have

$$K = 1 - 2a + 2\alpha\delta = 1 - 2a + 2\beta\gamma. \tag{3.32}$$

Case (A): $K \neq 1 - 2a$. Then from (3.32) we have $\alpha, \beta, \gamma, \delta \neq 0$. It follows from (2.7), (3.2) and (3.30) that Codazzi equation (2.4) also reduces to

$$e_2(\alpha) = 2w_2\alpha, \quad e_2(\beta) = 2w_2\beta, \quad e_1(\gamma) = -2w_1\gamma, \quad e_1(\delta) = -2w_1\delta, \tag{3.33}$$

which together with $\alpha\delta = \beta\gamma$ shows that there exists a nonzero real number c such that $\alpha = c\beta, \gamma = c\delta$, and $[\beta^{-\frac{1}{2}}e_1, \delta^{-\frac{1}{2}}e_2] = 0$. Then there exists a coordinate system $\{x, y\}$ such that

$$\frac{\partial}{\partial x} = \beta^{-\frac{1}{2}}e_1, \quad \frac{\partial}{\partial y} = \delta^{-\frac{1}{2}}e_2, \quad g = -(\beta\delta)^{-\frac{1}{2}}dxdy. \tag{3.34}$$

Denote by $\rho = (\beta\delta)^{-\frac{1}{2}}$, then Gauss curvature in (3.32) becomes

$$K = 1 - 2a + 2c/\rho^2, \tag{3.35}$$

and the Levi-Civita connection of g still satisfies (3.9). Moreover, from (3.30), (3.31) and (3.34) we have

$$h(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = ce_3 + e_4, \quad h(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = \rho(e_3 + ae_4), \quad h(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = ce_3 + e_4, \tag{3.36}$$

$$A_{e_3} = \begin{pmatrix} a & \rho^{-1} \\ \rho^{-1} & a \end{pmatrix}, \qquad A_{e_4} = \begin{pmatrix} 1 & c\rho^{-1} \\ c\rho^{-1} & 1 \end{pmatrix}. \tag{3.37}$$

By applying (3.9) and (3.36), we have that $L : M \to S^4_2 \subset \mathbb{E}^5_2$ satisfies

$$L_{xx} = (\ln\rho)_x L_x + ce_3 + e_4, \; L_{xy} = \rho(e_3 + ae_4 + L), \; L_{yy} = (\ln\rho)_y L_y + ce_3 + e_4. \tag{3.38}$$

The compatibility condition is

$$(\ln\rho)_{xy} = 2c\rho^{-1} + (1 - 2a)\rho. \tag{3.39}$$

Case (A.a): $ca \neq 1$. Let $\xi = -ce_3 + e_4, \eta = ce_3 + e_4$, then

$$D\xi = D\eta = 0, \quad \langle\xi, \xi\rangle = 2c, \quad \langle\eta, \eta\rangle = -2c, \quad \langle\xi, \eta\rangle = 0, \quad A_\xi = (1 - ca)I.$$

Consider the map $\psi : M \to \mathbb{E}^5_2$ defined by $\psi(p) = L(p) + \frac{1}{1-ca}\xi(p)$. Then $\tilde{\nabla}_X\psi = 0$ for $X \in TM$ and $\psi = L + \frac{1}{1-ca}\xi$ is a constant vector, say $c_0 \in \mathbb{E}^5_2$. So $L - c_0 = \frac{1}{ca-1}\xi$ and hence

$$\langle L - c_0, L - c_0\rangle = \frac{2c}{(ca-1)^2} = \text{constant}. \tag{3.40}$$

Case (A.a.1): $c > 0$. Equation (3.40) implies that M lies in $S^4_2(c_0, r^2)$ with $r^2 = (ca-1)^2/(2c)$. Since the mean curvature vector H' of M in $S^4_2(c_0, r^2)$ and the mean curvature vector H in $\mathbb{E}^5_2$ are related by $H = H' - r^2(L - c_0)$, hence we have $\langle H, H\rangle = \langle H', H'\rangle + r^2$. This gives $\langle H', H'\rangle = 1 - r^2 - 2a$. If M is non-flat, we obtain case (8) of Theorem 3.1.

If M is flat, it follows from (3.35) that $\rho^2 = \frac{2c}{2a-1}$. We choose $\rho = 1$ and hence $c = a - 1/2 > 0$, then $a > 1/2$. In this case, (3.36) and (3.37) become

$$h(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = h(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = (a - 1/2)e_3 + e_4, \quad h(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = e_3 + ae_4, \tag{3.41}$$

$$A_{e_3} = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}, \qquad A_{e_4} = \begin{pmatrix} 1 & a-1/2 \\ a-1/2 & 1 \end{pmatrix}. \tag{3.42}$$

By applying (3.41) we have that $L : M \to S^4_2 \subset \mathbb{E}^5_2$ satisfies

$$L_{xx} = L_{yy} = (a - \frac{1}{2})e_3 + e_4, \quad L_{xy} = e_3 + ae_4 + L.$$

Put $x = (u+v)/\sqrt{2}$ and $y = (u-v)/\sqrt{2}$, then

$$L_{uv} = 0, \quad L_{uuu} = -mL_u, \quad L_{vvv} = -nL_v. \tag{3.43}$$

where $m = a(2a+3)$, $n = 2a^2 - 5a + 4$. Solving equation (3.43) we obtain

$$L = c_1 \cos\sqrt{m}u + c_2 \sin\sqrt{m}u + c_3 \cos\sqrt{n}v + c_4 \sin\sqrt{n}v + c_5.$$

After choosing suitable initial conditions, we obtain case (10) for $a > 1/2$.

Case (A.a.2): $c < 0$. Equation (3.40) implies that M lies in $H^4_1(c_0, -r^2) \cap S^4_2(1)$ with $r^2 = -(ca-1)^2/(2c)$. Since the mean curvature vector H' of M in $H^4_2(c_0, r^2)$ and the mean curvature vector H in $\mathbb{E}^5_2$ are related by $H = H' + r^2(L - c_0)$, hence we have $\langle H, H\rangle = \langle H', H'\rangle - r^2$. This gives $\langle H', H'\rangle = 1 + r^2 - 2a$. If M is non-flat, we obtain case (9) of Theorem 3.1.

If M is flat, similar to case (A.a.1) we also choose $\rho = 1$ and $c = a - \frac{1}{2} < 0$, then $a < \frac{1}{2}$. Just like case (A.a.1), we put $x = (u+v)/\sqrt{2}$ and $y = (u-v)/\sqrt{2}$, then

$$L_{uv} = 0, \quad L_{uuu} = -mL_u, \quad L_{vvv} = -nL_v, \tag{3.44}$$

where $m = a(2a+3)$, $n = 2a^2 - 5a + 4$.

(a) If $a = 0$ or $a = -\frac{3}{2}$, then $m = 0$, and $n = 4$ or $n = 16$. Solving (3.44), we have

$$L = c_1 u^2 + c_2 u + c_3 \cos\sqrt{n}v + c_4 \sin\sqrt{n}v + c_5.$$

After choosing suitable initial conditions, we obtain case (11) or case (12).

(b) If $-\frac{3}{2} < a < 0$, solving (3.44) we have

$$L = c_1 \cosh\sqrt{-m}u + c_2 \sinh\sqrt{-m}u + c_3 \cos\sqrt{n}v + c_4 \sin\sqrt{n}v + c_5.$$

After choosing suitable initial conditions, we obtain case (13).

(c) If $a < -\frac{3}{2}$ or $0 < a < \frac{1}{2}$, solving (3.44) we have

$$L = c_1 \cos \sqrt{m}u + c_2 \sin \sqrt{m}u + c_3 \cos \sqrt{n}v + c_4 \sin \sqrt{n}v + c_5.$$

After choosing suitable initial conditions, we also obtain case (10).

Case (A.b): $ca = 1$. It follows from (3.37) that $A_{-ce_3+e_4} = 0$, which implies that $-ce_3 + e_4$ is a constant vector, say $c_0 \in \mathbb{E}^5_2$ and it is easy to check that $\langle c_0, c_0 \rangle = 2c$.

Case (A.b.1): $c > 0$. In this case, M lies in $S^4_2 \cap \pi$, where π is hyperplane of index 2 in $\mathbb{E}^5_2$. If M is non-flat, we get case (14). If M is flat, from (3.35) we have $\rho^2 = \frac{2c}{2a-1}$ and hence ρ is constant. We choose $\rho = 1$, then $c = \frac{\sqrt{17}-1}{4}$ and $a = \frac{\sqrt{17}+1}{4}$. Then we obtain case (10) for $a = \frac{\sqrt{17}+1}{4}$.

Case (A.b.2): $c < 0$. In this case, M lies in $S^4_2 \cap \pi$, where π is hyperplane of index 1 in $\mathbb{E}^5_2$. If M is non-flat, we get case (15). If M is flat, similar to case (A.b.1) we get $c = \frac{-\sqrt{17}-1}{4}$ and $a = \frac{1-\sqrt{17}}{4}$. Then we obtain case (13) for $a = \frac{1-\sqrt{17}}{4}$.

Case (B): $K = 1 - 2a$. In this case, M is a Lorentz surface of constant Gauss curvature and $\alpha\delta = \beta\gamma = 0$ from (3.32). We choose the metric of the surface given by Lemma 2.1 and define $e_1 = \frac{1}{m}\partial_x, e_2 = \frac{1}{m}\partial_y$. Then from (3.30), we have

$$h(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = m^2(\alpha e_3 + \beta e_4),\ h(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = m^2(e_3 + a e_4),\ h(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = m^2(\gamma e_3 + \delta e_4). \tag{3.45}$$

From (3.31), we have

$$\begin{aligned} \tilde{\nabla}_{\frac{\partial}{\partial x}} e_3 &= -a\frac{\partial}{\partial x} - \beta\frac{\partial}{\partial y}, \qquad \tilde{\nabla}_{\frac{\partial}{\partial y}} e_3 = -\delta\frac{\partial}{\partial x} - a\frac{\partial}{\partial y}, \\ \tilde{\nabla}_{\frac{\partial}{\partial x}} e_4 &= -\frac{\partial}{\partial x} - \alpha\frac{\partial}{\partial y}, \qquad \tilde{\nabla}_{\frac{\partial}{\partial y}} e_4 = -\gamma\frac{\partial}{\partial x} - \frac{\partial}{\partial y}. \end{aligned} \tag{3.46}$$

It follows from (2.9) and (3.45) that

$$L_{xx} = \frac{2m_x}{m}L_x + m^2(\alpha e_3 + \beta e_4), \tag{3.47}$$

$$L_{xy} = m^2(e_3 + a e_4 + L), \tag{3.48}$$

$$L_{yy} = \frac{2m_y}{m}L_y + m^2(\gamma e_3 + \delta e_4). \tag{3.49}$$

The compatibility conditions of system (3.46) are given by

$$\alpha_y + \alpha\frac{2m_y}{m} = 0, \quad \beta_y + \beta\frac{2m_y}{m} = 0, \quad \delta_x + \delta\frac{2m_x}{m} = 0, \quad \gamma_x + \gamma\frac{2m_x}{m} = 0. \tag{3.50}$$

The compatibility condition of the system (3.47)-(3.49) is given by

$$(\ln m)_{xy} = (\frac{1}{2} - a)m^2.$$

Case (B.a): $K > 0$, i.e. $a < \frac{1}{2}$. One can choose local coordinates (x, y) such that

$$m(x, y) = \frac{1}{\sqrt{\frac{1}{2} - a}(x + y)}.$$

Case (B.a.1): $\alpha = \beta = 0$. Equation (3.47) becomes

$$L_{xx} = -\frac{2}{x+y} L_x.$$

Solving this equation and from $g = -\frac{1}{(\frac{1}{2}-a)(x+y)^2} dxdy$, we can get case (16).

Case (B.a.2): $\gamma = \delta = 0$. After interchanging x and y, we get case (16) as well.

Case (B.a.3): $\alpha = \gamma = 0$. In this case, $A_{e_4} = I$ and hence $L + e_4$ is a constant vector in $\mathbb{E}^5_2$, say c_0. It is easy to check that c_0 is a spacelike vector and $\langle L, L \rangle = \langle L, c_0 \rangle = 1$. Without loss of generality, we put $c_0 = (0,0,0,0,1)$. Hence we have $x_1^2 + x_2^2 = x_3^2 + x_4^2$, $x_5 = 1$, where $x_1, \cdots, x_5$ are coordinates of L in $\mathbb{E}^5_2$. So M lies in $\mathcal{LC}^3_2 = \{(y,1) \in \mathbb{E}^5_2 : \langle y, y \rangle = 0, y \in \mathbb{E}^4_2\} \subset S^4_2(1)$.

On the other hand, from (3.50) we have $\beta = c_1(x)/m^2, \delta = c_2(y)/m^2$ for some functions $c_1(x)$, $c_2(y)$. Hence (3.47) and (3.49) become

$$L_{xx} = -\frac{2}{x+y} L_x + c_1(x)(c_0 - L), \quad L_{yy} = -\frac{2}{x+y} L_y + c_2(y)(c_0 - L). \tag{3.51}$$

Solving this equation and from $g = \frac{1}{(a-1/2)(x+y)^2} dxdy$, we obtain case (17).

Case (B.a.4): $\beta = \delta = 0$. Similar to case (B.a.3), we get case (17) as well.

Case (B.b): $K < 0$, i.e. $a > \frac{1}{2}$. In this case, one can choose local coordinates (x, y) such that

$$m(x,y) = \frac{1}{\sqrt{a - \frac{1}{2}}(x-y)}. \tag{3.52}$$

Case (B.b.1): $\alpha = \beta = 0$. Equation (3.47) becomes

$$L_{xx} = -\frac{2}{x-y} L_x.$$

Solving this equation and from $g = -\frac{1}{(a-1/2)(x-y)^2} dxdy$, we have case (18).

Case (B.b.2): $\gamma = \delta = 0$. After interchanging x and y, we get case (18) as well.

Case (B.b.3): $\alpha = \gamma = 0$. similar to case (B.a.3), we conclude that M lies in $\mathcal{LC}^3_2 = \{(y,1) \in \mathbb{E}^5_2 : \langle y, y \rangle = 0, y \in \mathbb{E}^4_2\} \subset S^4_2(1)$. In this case, (3.47) and (3.49) become

$$L_{xx} = -\frac{2}{x-y} L_x + c_3(x)(c_0 - L), \quad L_{yy} = \frac{2}{x-y} L_y + c_4(y)(c_0 - L).$$

Solving this equation and from $g = -\frac{1}{(a-1/2)(x-y)^2} dxdy$, we obtain case (19).

Case (B.b.4): $\beta = \delta = 0$. In this case, similar to case (B.2.3), we get case (19) as well.

Conversely, it is easy to verify that each of the 19 types of Lorentz surfaces in $S^4_2(1)$ has parallel mean curvature vector. ∎

4 Lorentz surfaces in $H^4_2(-1)$

In this section, we will give the classification of Lorentz surfaces in $H^4_2(-1)$ with parallel mean curvature vector. The classification and the proof are similar to the ones in $S^4_2(1)$. In fact, the map $\phi : \mathbb{E}^5_2 \to \mathbb{E}^5_3 : (x_1, x_2, x_3, x_4, x_5) \mapsto (x_3, x_4, x_5, x_1, x_2)$ takes $S^4_2(1)$ into $H^4_2(-1)$ and is a conformal map with factor -1. So we omit the proof here.

Let $\mathcal{G}_b = \{(x_1, x_2, \cdots, x_5) \in \mathbb{E}^5_3 : x_5 = x_1 + b\}$. For any two vectors $a = (a_1, \ldots, a_5)$, $b = (b_1, \ldots, b_5)$ in $\mathbb{E}^5_3$, we put $a * b = (a_1 b_1, \ldots, a_5 b_5)$. The following theorem classifies Lorentz surfaces with parallel mean curvature vector in $H^4_2(-1)$.

Theorem 4.1. *Let M be a Lorentz surface with parallel mean curvature vector in de Sitter space-time $H^4_2(-1) \subset \mathbb{E}^5_3$, then L is congruent to a surface of the following 19 families.*

1. *A minimal Lorentz surface of $H^4_2(-1)$;*
2. *A Lorentz surface of curvature -1 with constant lightlike mean curvature vector, lying in $\mathcal{G}_0 \cap H^4_2(-1)$, which is defined by*

$$L(x,y) = \left(f(x,y), \quad \frac{x+y}{x-y}, \quad \frac{xy-1}{x-y}, \quad \frac{xy+1}{x-y}, \quad f(x,y) \right),$$

   for some function $f(x,y)$.

3. *A Lorentz surface of curvature -1 defined by $L = -\frac{p(y)}{x-y} + q(y)$, where $p(y)$ is a curve lying in the light cone $\mathcal{LC}$ and $q(y)$ is a null curve satisfying $\langle p', q' \rangle = 0, \langle p, q' \rangle = -2, \langle p', p' \rangle = -4$;*
4. *A Lorentz marginally trapped surface of curvature -1 in $H^4_2(-1)$ and lies in $\mathcal{LC}^3_2 = \{(1, y) \in \mathbb{E}^5_3 : \langle y, y \rangle = 0, y \in \mathbb{E}^4_2\} \subset H^4_2(-1)$, which is defined by*

$$L(x,y) = \frac{1}{x-y}(u(x) * z(y) + v(x) * w(y)) + c_0,$$

   where $c_0 = (1,0,0,0,0)$, $u''(x) + 2c_1(x)u(x) = v''(x) + 2c_1(x)v(x) = z''(y) + 2c_2(y)z(y) = w''(y) + 2c_2(y)w(y) = 0$ and $\langle u'(x) * z(y) + v'(x) * w(y), u(x) * z'(y) + v(x) * w'(y) \rangle = -2$ for some functions $c_1(x)$ and $c_2(y)$.

5. *a non-flat Lorentz surface which lies in $H^4_2(c_0, -r^2) \cap H^4_2(-1)$ such that the mean curvature vector H' of M in $H^4_2(c_0, -r^2) \cap H^4_2(-1)$ satisfies $\langle H', H' \rangle = -1 + r^2$.*
6. *a non-flat Lorentz surface which lies in $S^4_3(c_0, r^2) \cap H^4_2(-1)$ such that the mean curvature vector H' of M in $S^4_3(c_0, r^2) \cap H^4_2(-1)$ satisfies $\langle H', H' \rangle = -1 - r^2$.*
7. *A flat marginally trapped surface defined by*

$$L = \left(\frac{\sin 2u}{2\sqrt{2}}, \frac{\cos 2u}{2\sqrt{2}}, v^2 + \frac{15}{8}, v^2 + \frac{13}{8}, \frac{v}{\sqrt{2}} \right).$$

8. *A non-flat CMC surface lying in* $H_2^4(c_0, -r^2) \cap H_2^4(-1)$ *such that the mean curvature vector* H' *of* M *in* $H_2^4(c_0, -r^2) \cap H_2^4(-1)$ *satisfies* $\langle H', H' \rangle = -1 - 2a + r^2$ *for a nonzero real number* a.

9. *a non-flat CMC surface lying in* $S_3^4(c_0, r^2) \cap H_2^4(-1)$ *such that the mean curvature vector* H' *of* M *in* $S_3^4(c_0, r^2) \cap H_2^4(-1)$ *satisfies* $\langle H', H' \rangle = -1 - 2a - r^2$ *for a nonzero real number* a.

10. *A flat surface defined by*

$$L = \left(\pm\sqrt{1 - \frac{1}{2m} + \frac{1}{2n}}, \frac{\cos\sqrt{m}u}{\sqrt{2m}}, \frac{\sin\sqrt{m}u}{\sqrt{2m}}, \frac{\cos\sqrt{n}v}{\sqrt{2n}}, \frac{\sin\sqrt{n}v}{\sqrt{2n}} \right),$$

where $m = 2a^2 + 5a + 4$, $n = 2a - 3$ *for* $a \in (-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, 0) \cup (\frac{3}{2}, 0)$.

11. *A flat surface defined by*

$$L = \left(v^2 + \frac{15}{8}, \frac{\cos 2u}{2\sqrt{2}}, \frac{\sin 2u}{2\sqrt{2}}, \frac{v}{\sqrt{2}}, v^2 + \frac{13}{8} \right).$$

12. *A flat surface defined by*

$$L = \left(v^2 + \frac{33}{16}, \frac{\cos 4u}{4\sqrt{2}}, \frac{\sin 4u}{4\sqrt{2}}, \frac{v}{\sqrt{2}}, v^2 + \frac{29}{16} \right).$$

13. *A flat surface defined by*

$$L = \left(\frac{\cos\sqrt{m}u}{\sqrt{2m}}, \frac{\sin\sqrt{m}u}{\sqrt{2m}}, \frac{\sinh\sqrt{-n}v}{\sqrt{-2n}}, \frac{\cosh\sqrt{-n}v}{\sqrt{-2n}}, \pm\sqrt{\frac{1}{2m} - \frac{1}{2n} - 1} \right),$$

where $m = 2a^2 + 5a + 4$, $n = 2a - 3$ *for* $a \in (0, \frac{3}{2})$.

14. *A non-flat CMC surface lying in* $H_2^4(-1) \cap \pi$, *where* π *is hyperplane of index 3 in* $\mathbb{E}_3^5$.

15. *A non-flat CMC surface lying in* $H_2^4(-1) \cap \pi$, *where* π *is hyperplane of index 2 in* $\mathbb{E}_3^5$.

16. *A non-flat surface defined by*

$$L(x, y) = -\frac{1}{x + y} u(y) + v(y),$$

where $u(y)$ *is a curve lying in the light cone* $\mathcal{LC}$ *and* $v(y)$ *is a null curve satisfying* $\langle u', v' \rangle = 0$, $\langle u', u' \rangle = -\frac{4}{1+2a}$ *and* $\langle u, v' \rangle = \frac{2}{2a+1}$ *for a real number* $a < -1/2$.

17. *A non-flat CMC surface lying in* $\mathcal{LC}_2^3 = \{(1, y) \in \mathbb{E}_3^5 : \langle y, y \rangle = 0, y \in \mathbb{E}_2^4\} \subset H_2^4(-1)$ *defined by*

$$L(x, y) = \frac{1}{x + y}(u(x) * z(y) + v(x) * w(y)) + c_0,$$

where $c_0 = (1,0,0,0,0)$, u, v, z, w *are curves in* $\mathbb{E}^4_2$ *satisfying* $u''(x) - c_1(x)u(x) = v''(x) - c_1(x)v(x) = z''(y) - c_2(y)z(y) = w''(y) - c_2(y)w(y) = 0$, *and* $\langle u'(x) * z(y) + v'(x) * w(y),\ u(x) * z'(y) + v(x) * w'(y)\rangle = \frac{1}{a+1/2}$ *for functions* $c_1(x)$, $c_2(y)$ *and for a real number* $a < -1/2$.

18. *A non-flat surface defined by*

$$L(x,y) = -\frac{1}{x-y}u(y) + v(y),$$

where $u(y)$ *is a curve lying in the light cone* $\mathcal{LC}$ *and* $v(y)$ *is a null curve satisfying* $\langle u', v'\rangle = 0$, $\langle u', u'\rangle = -\frac{4}{2a+1}$ *and* $\langle u, v'\rangle = -\frac{2}{2a+1}$ *for a real number* $a > -1/2$.

19. *A non-flat CMC surface lying in* $\mathcal{LC}^3_2 = \{(1,y) \in \mathbb{E}^5_3 : \langle y, y\rangle = 0, y \in \mathbb{E}^4_2\} \subset H^4_2(-1)$ *defined by*

$$L(x,y) = \frac{1}{x-y}(u(x) * z(y) + v(x) * w(y)) + c_0,$$

where $c_0 = (1,0,0,0,0)$, u, v, z, w *are curves in* $\mathbb{E}^5_3$ *satisfying* $u''(x) - c_1(x)u(x) = v''(x) - c_1(x)v(x) = z''(y) - c_2(y)z(y) = w''(y) - c_2(y)w(y) = 0$, *and* $\langle u'(x) * z(y) + v'(x) * w(y),\ u(x) * z'(y) + v(x) * w'(y)\rangle = -\frac{2}{(2a+1)}$ *for functions* $c_1(x)$, $c_2(y)$ *and for a real number* $a > -1/2$.

Remark 4.2. Case (2)$-$(7) are marginally trapped Lorentz surfaces with parallel mean curvature vector in $H^4_2(-1) \subset \mathbb{E}^5_3$.

References

[1] Chen, B. Y.: Pseudo-Riemannian Geometry, δ-invariants and Applications. Word Scientific, Hackensack, NJ, 2011.

[2] Chen, B. Y.: Classification of marginally trapped Lorentzian flat surfaces in $\mathbb{E}^4_2$ and its application to biharmonic surfaces. *J. Math. Anal. Appl.* **340**,861-875(2008).

[3] Chen, B. Y.: Classification of quasi-minimal slant surfaces in Lorentzian complex space forms. *Acta. Math. Hungar.* **122**,4, 307-328(2009).

[4] Chen, B. Y., Dillen. F.: Classification of marginally trapped Lagrangian surfaces in Lorentzian complex space forms. *J. Math. Phys.* **48**,013509, 23 p(2007).

[5] Chen, B. Y., Garay, O. J.: Complete classification of quasi-minimal surfaces with parallel mean curvature vector in neutral pseudo-Euclidean 4-space $\mathbb{E}^4_2$. *Result. Math.* **55**, 23-38(2009).

[6] Chen, B. Y., Ishikawa, S.: Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces. *Kyushu J. Math.* **52**, 1-18(1998).

[7] Chen, B. Y., Van der Veken J.: Marginally trapped surfaces in Lorentzian space forms with positive relative nullity. *Class. Quantum Grav.* 2007 **24**, 551-563(2007).

[8] Chen, B. Y., Van der Veken, J.: Classification of marginally trapped surfaces with parallel mean curvature vector in Lorentzian space forms. *Houston J. Math.*. **36**,421-449(2010).

[9] Chen, B. Y., Yang, D.: Addendum to "Classification of marginally trapped Lorentzian flat surfaces in and its application to biharmonic surfaces". *J. Math. Anal. Appl.* **361**,280-282(2010).

[10] Chen, B. Y.: Geometry of Submanifolds. New York, M. Dekker 1973.

[11] Chen, B. Y., Van der Veken, J.: Complete classification of parallel surfaces in 4-dimensional Lorentzian space forms. *Tohoku Math. J.* **61**,1-40(2009).

[12] Chen, B. Y.: Classification of spatial surfaces with parallel mean curvature vector in pseudo-Euclidean spaces of arbitrary dimension. *J. Math. Phys.***50**, 043503, 14 p(2009).

[13] Chen, B. Y.: Complete classification of spatial surfaces with parallel mean curvature vector in arbitrary non-flat pseudo-Riemannian space forms. *Cent. Eur. J. Math.* **7**(3), 400-428(2009).

[14] Hou, Z. H., Yang, D.: Classification of Lorentzian surfaces with parallel mean curvature vector in E^4_2. *Acta Math. Hungar.* **128** (1-2),59-81(2010).

[15] Chen, B. Y.: Complete classification of Lorentz surfaces with parallel mean curvature vector in arbitrary pseudo-Euclidean space. *Kyushu J. Math.* **64**,2, 261-179(2010).

[16] Fu, Y., Hou, Z. H.: Classification of Lorentzian surfaces with parallel mean curvature vector in pseudo-Euclidean spaces. *J. Math. Anal. Appl.* **371**, 25-40(2010).

[17] Chen, B. Y.: Submanifolds with parallel mean curvature vector in Riemannian and indefinite space forms. *Arab J. Math. Sci.* **16**, 1-46(2010).

[18] Chen, B. Y.: Total Mean Curvature and Submanifolds of Finite Type. World Scientific, New Jersey 1984. **128**,59-81(2010).

[19] O'Neill, B.: Semi-Riemannian geometry with applications to relativity. Academic Press, New York, 1982.

[20] Chen, B. Y.: Dependence of the Gauss-Codazzi equations and the Ricci equation of Lorentz surface. *Publ. Math. Debrecen.* **74**,341-349(2009).

College of Sciences, Shenyang University,
Shenyang 110044, P. R. China
Email : danyang82@yahoo.cn

School of Mathematics and Quantitative Economics,
Dongbei University of Finance and Economics,
Dalian 116025, P. R. China
Email yu_fu@yahoo.cn