On certain results of C. Bereanu and J. Mawhin

Stanisław Sędziwy

Abstract

It is shown that the assumption of the singularity of φ–Laplacian permits to get for the scalar differential equations the existence results of the Dirichlet, Dirichlet–Neumann, Neuman–Steklov or periodic problems using a simple elementary argument.

1 Introduction

In [1] and [2] C. Bereanu and J. Mawhin considered the boundary value problems for the scalar differential equation

$$(\varphi(u'))' = f(t, u, u'), \tag{1.1}$$

with a singular φ–Laplacian, i.e. assuming that φ is an increasing homeomorphism such that $\varphi : (-a, a) \to \mathbb{R}$ ($a \in (0, \infty)$, $\varphi(0) = 0$).

Among others, using the Leray–Schauder theory, they proved the existence of solutions to various boundary value problems under, as they claim, rather general conditions (only the continuity of f is required). The paper [3] presents generalization of works [1], [2] to the vector differential equations as well as new results.

Rather general conditions on f and boundary functions could be assumed since the condition $\varphi : (-a, a) \to \mathbb{R}$ is in fact very strong: it permits to define the compact set K_0 containing all possible solutions of BVPs in question. Once the set K_0 is known it is possible, in the scalar case, to prove results of [1], [2], [3] by elementary methods.

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2 Results

Theorem 1. *Let $\varphi : (-a, a) \to \mathbb{R}$, $\varphi(0) = 0$, be an increasing homeomorphism and let $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Then:*

(A) (cf [1, Corollary 1],[2, Corollary 1]) If $|B - A| < aT$, then there exists at least one solution of (1.1) subject to Dirichlet boundary conditions

$$u(0) = A, \quad u(T) = B. \tag{2.1}$$

(B) (cf [3, Corollary 2, 4]) For each A and C, if $|C| < a$ then the boundary value problems (1.1),

$$u(0) = A \quad u'(T) = C, \quad \textit{Dirichlet–Neumann} \tag{2.2}$$

$$u'(0) = C, \quad u(T) = A \quad \textit{Neumann–Dirichlet} \tag{2.3}$$

have at least one solution.

Theorem 2. *Let $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$, $g_0, g_1 : \mathbb{R} \to \mathbb{R}$ be continuous.*

Suppose there exists $R > 0$ such that f satisfies one of the following conditions:

$$\begin{aligned} &\int_0^T f(t, u(t), u'(t))\, dt - (g_T(u(T)) - g_0(u(0)) > 0 \quad if \min_{t\in[0,T]} u(t) \geq R, \\ &\int_0^T f(t, u(t), u'(t))\, dt - (g_T(u(T)) - g_0(u(0)) < 0 \quad if \max_{t\in[0,T]} u(t) \leq -R, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\int_0^T f(t, u(t), u'(t))\, dt > 0 \quad if \min_{t\in[0,T]} u(t) \geq R, \\ &\int_0^T f(t, u(t), u'(t))\, dt < 0 \quad if \max_{t\in[0,T]} u(t) \leq -R, \end{aligned} \tag{2.5}$$

then:

(C) (cf [2, Thm 2],[3, Cor.2]) equation (1.1) with Neumann–Steklov boundary conditions

$$\varphi(u'(0)) = g_0(u(0)), \quad \varphi(u'(T)) = g_T(u(T)) \tag{2.6}$$

has at least one solution, provided (2.5) holds,

(D) (cf [1, Thm 2]) BVP (1.1),

$$u(0) = u(T), \quad u'(0) = u'(T) \tag{2.7}$$

has a solution, provided (2.5) holds.

Remark 1. *Without the loss of generality weak inequalities (2.4) appearing in [2] may be replaced by the strong ones.*

An immediate consequence of the differential inequalities theory (see e.g. [4, Ch. III]) is the following remark.

Remark 2. *If the initial problem for (1.1) has the unique solution and $f(t, u, w)$ is increasing with respect to u, then solutions of (1.1), subject to boundary conditions (2.1), (2.2) or (2.3) are unique for arbitrary values of parameters A, B or A, C.*

3 Proofs

Observe that if $u(t)$ is the solution of (1.1), then $|u'(t)| < a$ and any of conditions (2.1), (2.2) or (2.3) implies that $|u(t) - A| < aT$ for $t \in [0, T]$.

Similarly, the sign conditions imply that solutions to BVP (1.1),(2.6) satisfy $|u(t)| < R$ for $t \in [0, T]$.

In both cases solutions to (1.1) with mentioned boundary conditions are in the compact sets $K_0 = [A - aT, A + aT] \times [-a, a]$ or $K_0 = [-R, R] \times [-a, a]$.Let $K = [0, T] \times K_0$.

Denote $M = \max_K |f(t, u, w)|$.

Proof of Theorem 1. Replace (1.1) by the equivalent first order system

$$u' = \varphi^{-1}(v), \quad v' = f(t, u, \varphi^{-1}(v)). \tag{3.1}$$

Assume additionally that for any S the initial value problem (IVP) (3.1), $(u(0), v(0)) = (A, S)$ have the unique solution $(u(t, S), v(t, S))$.

Proof of (A). Choose $a_1, a_2 \in (-a, a)$ as follows: if $B < A$, then $-a < a_1 < (B - A)/T$ and for $B > A$ let $(B - A)/T < a_2 < a$. Set $v_i = \varphi(a_i)$.

The formula $|v(t, S) - S| \leq \int_0^T |f(t, u(t, S), \varphi^{-1}(v(t, S)))| \, dt \leq MT$ implies that $\lim_{S\to\pm\infty} v(t, S) = \pm\infty$ uniformly in $t \in [0, T]$. Since $\lim_{z\to\pm\infty} \varphi^{-1}(z) = \pm a$, there exist constants S_i, $i = 1, \ldots, 4$ such that $\varphi^{-1}(v(t, S))$ satisfy for $t \in [0, T]$ the inequalities

$$\varphi^{-1}(v(t,S)) \begin{cases} < \varphi^{-1}(v_1) = a_1, & for\ S = S_1, \\ > \varphi^{-1}(v_2) = a_2, & for\ S = S_2, \\ > \varphi^{-1}(v_2/2) > 0, & for\ S = S_3, \\ < \varphi^{-1}(v_1/2) < 0, & for\ S = S_4, \end{cases}$$

from which, by the formulae $u(T, S_i) = A + \int_0^T \varphi^{-1}(v(t, S_i)) \, dt$ and $a_i T = B - A$, it follows that:

$$\begin{array}{llll} \text{for} & B < A & u(T, S_1) < B, & u(T, S_4) > A > B, \\ \text{for} & B > A & u(T, S_2) > B, & u(T, S_3) < A < B \quad \text{and} \\ \text{for} & A = B & u(T, S_3) < A, & u(T, S_4) > A. \end{array}$$

The continuity of $u(T, \cdot)$ and the inequalities above imply in each case the existence of a number D such that $u(T, D) = B$, completing the proof of (A).

Proof of (B). BVPs (1.1), (2.2), (1.1), (2.3) are equivalent to BVPs for (3.1) subject to one of the boundary conditions

$$u(0) = A, \quad \varphi^{-1}(v(T)) = C, \tag{3.2}$$

$$\varphi^{-1}(v(0)) = C, \quad u(T) = A. \tag{3.3}$$

To show the existence of solution to BVP (3.1), (3.2) note that since $S - MT \leq v(t, S) \leq S + MT$, $t \in [0, T]$, the conclusion follows from the continuity of $v(T, \cdot)$, inequality $|\varphi^{-1}(v(T, S)| = |C| < a$ and the observation that $\lim_{S\to\pm\infty} \varphi^{-1}(v(T, S)) = \pm a$. The remaining case is proven similarly.

The case of lack of uniqueness to IVPs is reduced to the previous one by a standard procedure (cf [4, Ch.1, Thm 2.4]). It consists in approximation (3.1) by equations with the uniqueness property:

$$u' = g_n(v) \quad v' = h_n(t,u,v), \tag{3.4}$$

with smooth with respect to arguments u, v right hand sides, such that

$$\lim_{n\to\infty}(g_n(v), h_n(t,u,v)) = (\varphi^{-1}(v), f(t,u,\varphi^{-1}(v))$$

uniformly in a compact set K_1 containing K_0 in its interior (cf [4, Ch. 1]).

By the Ascoli theorem, the sequence $\{(u_n(t,S_n), v(t,S_n))\}$ of solutions to BVPs for (3.4), contains the subsequence converging to the solution of the corresponding BVP. ■

Proof of Theorem 2. Proof of (C)

Boundary conditions of (3.1) are equivalent to

$$v(0) = g_0(u(0)), \quad v(T) = g_T(u(T)). \tag{3.5}$$

At first assume additionally that IVP (3.1), $(u(0), v(0)) = (A,B))$ is uniquely solvable. Since $(u(t,A,B), v(t,A,B))$ satisfies conditions

$$u(t) = \int_0^t \varphi^{-1}(v(s))\,ds + A, \quad v(t) = \int_0^t f(s,u(s),\varphi^{-1}(v(s)))\,ds + B, \tag{3.6}$$

(to simplify notations arguments A, B in u, v are dropped) from (3.5) it follows that $(u(t,A,B), v(t,A,B))$ is the solution of BVP (3.1),(3.5) iff

$$\int_0^T f(t,u(t),u'(t))\,dt - (g_T(u(T)) - g_0(u(0))) = 0.$$

By (3.6), $A - aT \le u(t,A,B) \le A + aT$, for $t \in [0,T]$, so for sufficiently large $P > 0$ for any B and all $t \in [0,T]$ we have

$$u(t,-P,B) < -R, \quad u(t,P,B) > R \tag{3.7}$$

which, by (2.4) and the intermediate value theorem completes the proof of (C).

Proof of (D)

By (3.6), $B - MT \le v(t,A,B) \le B + MT$, hence there exists $Q > 0$ such that for any A and $t \in [0,T]$

$$v(t,A,-Q) < 0, \quad v(t,A,Q) > 0. \tag{3.8}$$

Let $K = (-R,R) \times (-Q,Q)$ and define the map $\Phi : \overline{K} \to \mathbb{R}^2$ by $\Phi(A,B) = (u(T,A,B) - u(0,A,B), v(T,A,B) - v(0,A,B))$.

Observe that by (3.7), (3.8)

$$\Phi(-R,[-Q,Q]) < 0, \quad \Phi(R,[-Q,Q]) > 0,$$
$$\Phi([-R,R],-Q) < 0, \quad \Phi([-R,R],-Q) < 0$$

hence for all $\alpha \in [1/2, 1]$ and every $(A,B) \in \partial K$,

$$\alpha\Phi(A,B) \neq (1-\alpha)\Phi(-A,-B)$$

which implies that Φ vanishes in a certain point of K (cf [5, 3.31. Corollary]), i.e. conditions (2.7) hold. This completes the proof in the uniqueness case.

The case of non uniqueness is treated as in Theorem 1. ■

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Theoretical Computer Science,
Faculty of Mathematics and Computer Science,
Jagiellonian University
ul. Łojasiewicza 6, 31-348 Kraków, Poland
email:sedziwy@tcs.uj.edu.pl