# Lichnerowicz inequality on foliated manifold with a parallel 2-form

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**Abstract**

In this paper, we prove that if $(M, g)$ is a closed orientable Riemannian manifold with a transversely oriented harmonic $g$-Riemannian foliation of codimension $q$ on $M$ and if there exists a parallel basic 2-form on $M$ and a positive constant $k$ such that the transversal Ricci curvature satisfies $Ric_{\nabla}(Z, Z) \geq k(q-1)|Z|^2$ for every transverse vector field $Z$, then the smallest nonzero eigenvalue $\lambda_B$ of the basic Laplacian $\Delta_B$ satisfies $\lambda_B \geq 2k(q-1)$.

## 1 Introduction

In [1] the authors gave a foliated version of Lichnerowicz and Obata theorems. They proved that if $M$ is a closed Riemannian manifold with a Riemannian foliation of codimension $q$, and if the normal Ricci curvature satisfies $Ric^{\perp}(X, X) \geq a(q-1)|X|^2$ for every $X$ in the normal bundle for some fixed $a > 0$, then the smallest eigenvalue $\lambda_B$ of the basic Laplacian satisfies $\lambda_B \geq aq$. In this paper, we assume that the manifold is endowed with a nontrivial parallel basic 2-form, and we give a new estimation of the first non zero eigenvalue of the basic Laplacian.

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## 2 Preliminaries

Let $(M, g)$ be a Riemannian manifold of dimension $n$ with a foliation $\mathcal{F}$ of codimension $q$. The foliation is given by an integrable sub-bundle $L$ of the tangent bundle $TM$ over $M$. Let $L^\perp$ indicates the orthogonal complement bundle of $L$ and $\pi : TM \longrightarrow L^\perp$ the projection of $TM$ on $L^\perp$ parallel to $L$. In what follows, for any sub-bundle $E$ of $TM$ we denote by $\Gamma E$ the space of sections of $E$.

Let $\nabla^M$ be the Riemannian connection on $(M, g)$. We can define an adapted connection $\nabla$ on $L^\perp$ by the following:

$$\nabla_X Z = \begin{cases} \pi([X, Z]) & \text{if } X \in \Gamma(L) \\ \pi(\nabla^M_X Z) & \text{if } X \in \Gamma(L^\perp) \end{cases}$$

for any $Z \in \Gamma(L^\perp)$.

Let $X, Y \in \Gamma TM$, the *torsion* $T_\nabla$ of $\nabla$ is given by

$$T_\nabla(X, Y) = \nabla_X \pi(Y) - \nabla_Y \pi(X) - \pi([X, Y]),$$

and the *curvature* $R_\nabla$ of $\nabla$ is defined by

$$R_\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

We know that $T_\nabla = 0$ and $R_\nabla(X, Y) = 0$ for $X, Y \in \Gamma L$ (see [7]).

A local orthonormal frame $(E_i)_{0 \le i \le n}$ of $TM$ is adapted with respect to the foliation $\mathcal{F}$ if $E_i \in \Gamma(L)$ for $0 \le i \le p$, and $E_i \in \Gamma(L^\perp)$ for $p + 1 \le i \le n$ where $p = n - q$ is the dimension of $\mathcal{F}$.

The tension field $\tau$ of the foliation $\mathcal{F}$ is given by $\tau = \pi(\sum_{i=1}^{p} \nabla^M_{E_i} E_i)$.

Let $Y, Z \in \Gamma(E^\perp)$, the transversal Ricci operator and the transversal Ricci curvature are defined respectively by

$$\rho_\nabla(Z) = \sum_{i=p+1}^{n} R_\nabla(Z, E_i)E_i, \qquad Ric_\nabla(Z, Y) = g(\rho_\nabla(Z), Y).$$

The transversal divergence operator is given by

$$div_\nabla Z = \sum_{i=p+1}^{n} g(\nabla_{E_i} Z, E_i) = \sum_{i=p+1}^{n} g(\nabla^M_{E_i} Z, E_i).$$

If $div_M$ is the standard divergence operator, then we have

$$div_M Z = div_\nabla Z - g(\tau, Z).$$

The foliation $\mathcal{F}$ is harmonic if all the leaves of $\mathcal{F}$ are minimal submanifolds of $M$, that is $\tau = 0$ [7]. The foliation $\mathcal{F}$ is $g$-Riemannian if it is bundle like with respect to the metric $g$, that is $(\nabla_X g)(Y, Z) = 0$ for $X \in \Gamma(L)$ and $Y, Z \in \Gamma(L^\perp)$. The set $\mathcal{V}^\perp(\mathcal{F}) = \{Z \in \Gamma L^\perp / \nabla_X Z = \pi[X, Z] = 0 \text{ for all } X \in \Gamma L\}$ is called the space of transverse fields. The set of basic forms is defined by

$$\Omega^*_B(\mathcal{F}) = \{\omega \in \Omega^*(M) / i_X \omega = 0, L_X \omega = 0 \text{ for all } X \in \Gamma L\}.$$

It's an easy task to show that $Z \in \mathcal{V}^{\perp}(\mathcal{F})$ if and only if $\omega = i_Z g \in \Omega^1_B(\mathcal{F})$. So $\mathcal{V}^{\perp}(\mathcal{F})$ is isomorph to $\Omega^1_B(\mathcal{F})$.

A form $\omega \in \Omega^*_B(\mathcal{F})$ is parallel if $\nabla_X \omega = 0$ for all $X \in \Gamma L^{\perp}$. The exterior differential $d$ restricts to $d_B : \Omega^*_B(\mathcal{F}) \longrightarrow \Omega^{*+1}_B(\mathcal{F})$. The adjoint of $d_B$, with respect to the induced scalar product $\langle .,. \rangle_B$ on $\Omega^*_B(\mathcal{F})$ is denoted by $\delta_B : \Omega^*_B(\mathcal{F}) \longrightarrow \Omega^{*-1}_B(\mathcal{F})$ and we have the basic Laplacian

$$\Delta_B = \delta_B d_B + d_B \delta_B.$$

For a transversely oriented harmonic $g$-Riemannian foliation $\mathcal{F}$, it follows from [2] and [3] that $\delta_B$ is given on $\omega \in \Omega^1_B(\mathcal{F})$ by the formula

$$\delta_B \omega = -\sum_{i=p+1}^{n} (\nabla_{E_i} \omega) E_i$$

which gives the Bochner-Weitzenbock formula

$$\Delta_B \omega = -tr_B \nabla^2 \omega + \rho_{\nabla}(\omega), \tag{2.1}$$

where $tr_B \nabla^2 = \sum_{i=p+1}^{n} \nabla^2_{E_i,E_i}$

## 3 Some computational results.

In the following, for $Z \in \Gamma L^{\perp}$, we denote by $A_Z$ the endomorphism of $\Gamma L^{\perp}$ defined by $A_Z(Y) = \nabla_Y Z$.

**Proposition 1.** *Let $Z \in \mathcal{V}^{\perp}(\mathcal{F})$ and $\alpha \in \Omega^2_B(\mathcal{F})$. Let $\Theta$ be the endomorphism of $\Gamma L^{\perp}$ associated to $\alpha$ with respect to the metric $g$. If $\alpha$ is parallel, then we have*

$$\sum_{i=p+1}^{n} \nabla_{E_i}(A_Z \circ \Theta)(E_i) = \frac{1}{2} \sum_{i=p+1}^{n} R(E_i, \Theta(E_i))Z. \tag{3.2}$$

*Proof.* Since $\Theta$ is antisymmetric with respect to $g$, we have

$$\begin{aligned}
\sum_{i=p+1}^{n} \nabla^2_{E_i,\Theta(E_i)} Z &= \sum_{i,j=p+1}^{n} g(\Theta(E_i), E_j) \nabla^2_{E_i,E_j} Z \\
&- \sum_{i,j=p+1}^{n} g(\Theta(E_j), E_i) \nabla^2_{E_i,E_j} Z \\
&= -\sum_{j=p+1}^{n} \nabla^2_{\Theta(E_j),E_j} Z.
\end{aligned}$$

Furthermore $\Theta$ is parallel with respect to the connection $\nabla$, hence

$$\begin{aligned}
\sum_{i=p+1}^{n} \nabla_{E_i}(A_Z \circ \Theta)(E_i) &= \sum_{i=p+1}^{n} \nabla_{E_i}(A_Z)\Theta(E_i) = \sum_{i=p+1}^{n} \nabla^2_{E_i,\Theta(E_i)} Z. \\
&= \tfrac{1}{2}\Big( \sum_{i=p+1}^{n} \nabla^2_{E_i,\Theta(E_i)} Z - \sum_{i=p+1}^{n} \nabla^2_{\Theta(E_i),E_i} Z \Big) \\
&= \tfrac{1}{2} \sum_{i=p+1}^{n} R(E_i, \Theta(E_i))Z,
\end{aligned}$$

and we get the desired equality. ∎

In the next, we assume that $\mathcal{F}$ is $g$-Riemannian and we use the following special (orthonormal) moving frames on $M$. For $x \in M$, let $\{e_i\}_{i=1}^n \subset T_xM$ be an (oriented) orthonormal basis with $\{e_i\}_{i=1}^p \subset L_x$ and $\{e_i\}_{i=p+1}^n \subset L_x^\perp$. Let $U$ be a distinguished (flat) neighborhood of $x$ for $\mathcal{F}$ with local (Riemannian) submersion $f : U \longrightarrow N$ ($N$ is a Riemannian manifold of dimension $q$). For $i = p+1, ..., n$, let $E_i \in \Gamma(U, L)$ be the pull back of the extension of $f_* e_i$ to a vector field on $N$ by parallel transport along geodesic segments emanating from$f(x)$ (use [6] Prop 4.2). Then, we complete $\{E_i\}_{i=p+1}^n$ by the Gram-Schmidt process to a moving frame $\{E_i\}_{i=1}^n$ by adding $E_i \in \Gamma(U, L)$ with $(E_i)_x = e_i$, $i = 1, ..., p$. We have then for $i, j = p+1, ..., n$:

$$\nabla_{e_i} E_j = (\nabla_{E_i} E_j)_x = 0. \tag{3.3}$$

**Proposition 2.** *Under the assumptions of proposition 1, we have*

$$|\Theta \circ A_Z|^2 = g(tr_B \nabla^2 Z, \Theta^2(Z)) - div_\nabla A_Z^* \circ \Theta^2(Z) \tag{3.4}$$

*Proof.* Let $x \in M$, we use the under orthonormal frame. Since $Z \in \mathcal{V}^\perp(\mathcal{F})$, we have $A_Z(E_k) = 0$ for $k = 1, ..., p$, hence

$$\begin{aligned}
|\Theta \circ A_Z|^2 &= \sum_{k=p+1}^{n} g(\Theta \circ A_Z(E_k), \Theta \circ A_Z(E_k)) \\
&= \sum_{i,k=p+1}^{n} g(A_Z(E_k), E_i) g(\Theta(E_i), \Theta \circ A_Z(E_k)) \\
&= S + T,
\end{aligned}$$

where

$$S = \sum_{i,k=p+1}^{n} E_k.(g(Z, E_i) g(\Theta(E_i), \Theta \circ A_Z(E_k)))$$

and

$$T = -\sum_{i,j,k=p+1}^{n} (g(Z, E_i) E_k.(g(\nabla_{E_k} Z, E_j) g(\Theta(E_i), \Theta(E_j))).$$

Observe that at the point $x$, we have

$$\begin{aligned}
S &= \sum_{k=p+1}^{n} E_k.g(\Theta(Z), \Theta \circ A_Z(E_k)) \\
&= -\sum_{k=p+1}^{n} E_k.g(A_Z^* \circ \Theta^2(Z), E_k) \\
&= -div_\nabla (A_Z^* \circ \Theta^2(Z)),
\end{aligned}$$

and since $\Theta$ is parallel, we get

$$\begin{aligned}
T &= -\sum_{i,j,k=p+1}^{n} g(Z, E_i)(g(\nabla_{E_k} \nabla_{E_k} Z, E_j) g\big(\Theta(E_i), \Theta(E_j)\big) \\
&= g\big(tr_B \nabla^2 Z, \Theta^2(Z)\big).
\end{aligned}$$

Hence the statement of the proposition follows. ∎

**Proposition 3.** *Under the assumptions of proposition 1, we have*

$$\begin{aligned} tr(\Theta \circ A_Z^* \circ \Theta \circ A_Z) &= g\big(\rho_\nabla(Z), \Theta^2(Z)\big) \\ &+ div_\nabla \Theta \circ A_Z^* \circ \Theta(Z). \end{aligned} \tag{3.5}$$

*Proof.* First we have

$$\begin{aligned} tr(\Theta \circ A_Z^* \circ \Theta \circ A_Z) &= \sum_{k=p+1}^{n} g(\Theta \circ A_Z^* \circ \Theta \circ A_Z(E_k), E_k) \\ &= \sum_{i,k=p+1}^{n} g(A_Z(E_k), E_i) g(\Theta \circ A_Z^* \circ \Theta(E_i), E_k) \\ &= P + Q, \end{aligned}$$

with

$$\begin{aligned} P &= \sum_{i,k=p+1}^{n} E_k.(g(Z, E_i) g(\Theta \circ A_Z^* \circ \Theta(E_i), E_k)) \\ &= \sum_{k=p+1}^{n} E_k.g(\Theta \circ A_Z^* \circ \Theta(Z), E_k)) \\ &= div_\nabla \Theta \circ A_Z^* \circ \Theta(Z), \end{aligned}$$

and

$$\begin{aligned} Q &= -\sum_{i,k=p+1}^{n} g(Z, E_i) E_k.g(\Theta \circ A_Z^* \circ \Theta(E_i), E_k)) \\ &= -\sum_{i,j,k=p+1}^{n} g(Z, E_i) E_k.\big(g(\Theta(E_i), E_j) g(\Theta \circ A_Z^*(E_j), E_k)\big) \\ &= \sum_{i,j,k=p+1}^{n} g(Z, E_i) E_k.\big(g(\Theta(E_i), E_j) g(A_Z^*(E_j), \Theta(E_k))\big) \\ &= \sum_{i,j,k=p+1}^{n} g(Z, E_i) E_k.\big(g(\Theta(E_i), E_j) g(E_j, A_Z \circ \Theta(E_k))\big) \\ &= \sum_{i,j,k,l=p+1}^{n} g(Z, E_i) E_k.\big(g(\Theta(E_i), E_j) g(A_Z(E_l), E_j) g(\Theta(E_k), E_l)\big) \\ &= \sum_{i,j,k=p+1}^{n} g(Z, E_i) E_k.\big(g(A_Z \circ \Theta(E_k), E_j) g(\Theta(E_i), E_j)\big) \\ &= \sum_{i,j,k=p+1}^{n} g(Z, E_i) g(\nabla_{E_k}(A_Z \circ \Theta)(E_k), E_j) g(\Theta(E_i), E_j) \\ &= \frac{1}{2} \sum_{k=p+1}^{n} g(R(E_k, \Theta(E_k))Z, \Theta(Z)), \end{aligned}$$

where we use the formula (3.2) in the last equality.

Now, under the first Bianchi identity and by the vertu of both antisymmetry and parallelism of $\Theta$ we have

$$\begin{aligned}\sum_{k=p+1}^{n} R(E_k,\Theta(E_k))Z &= 2\sum_{k=p+1}^{n} R(E_k,Z)\Theta(E_k)\\ &= 2\Theta(\sum_{k=p+1}^{n} R(E_k,Z)E_k) = -2\Theta(\rho_\nabla(Z)).\end{aligned}$$

This completes the proof. ■

**Proposition 4.** *Under the assumptions of proposition 1, we have*

$$\begin{aligned} g(tr_B\nabla^2 Z+\rho_\nabla(Z),\Theta^2(Z)) &\geq div_\nabla\Theta\circ A_Z^*\circ\Theta(Z)\\ &- div_\nabla A_Z^*\circ\Theta^2(Z)\end{aligned} \tag{3.6}$$

$$g\big(\rho_\nabla(Z),\Theta^2(Z)\big) = -Ric_\nabla(\Theta(Z),\Theta(Z)) \tag{3.7}$$

*Proof.* i) By straightforward calculation, we have

$$\begin{aligned}\sum_{i,j=p+1}^{n}\big(g(\Theta\circ A_Z(E_i),E_j)+g(\Theta\circ A_Z(E_j),E_i)\big)^2 &= |\Theta\circ A_Z|^2\\ &+ tr(\Theta\circ A_Z^*\circ\Theta\circ A_Z).\end{aligned}$$

So the relation (3.6) follows from equations (3.4) and (3.5).

ii) Since $\Theta$ is parallel antisymmetric,

$$\begin{aligned} g\big(\rho_\nabla(Z),\Theta^2(Z)\big) &= -\sum_{k=p+1}^{n} g\big(\Theta(R(Z,E_k)E_k),\Theta(Z)\big)\\ &= -\sum_{k=p+1}^{n} g\big(R(Z,E_k)\Theta(E_k),\Theta(Z)\big)\\ &= -\sum_{k=p+1}^{n} g\big(R(\Theta(E_k),\Theta(Z))Z,E_k\big)\\ &= \sum_{k=p+1}^{n} g\big(R(E_k,\Theta(Z))Z,\Theta(E_k)\big)\\ &= -\sum_{k=p+1}^{n} g\big(\Theta(R(E_k,\Theta(Z))Z,E_k\big)\\ &= -Ric_\nabla(\Theta(Z),\Theta(Z)).\end{aligned}$$

and the proof is complete ■

**Remark 5.** *We know that $L_Z=\nabla_Z-A_Z$. If $\nabla\alpha=0$ and $L_Z\alpha=0$, then $\Theta\circ A_Z$ is symmetric. If furthermore $\omega=i_Zg$ is closed, then $\Theta\circ A_Z=-A_Z\circ\Theta$.*

## 4 The main theorem.

Let $\Omega^0(M)$ be the space of smooth functions on $M$. The set of smooth basic function is given by $\Omega^0_B(\mathcal{F}) = \{f \in \Omega^0(M)/X.f = 0 \text{ for all } X \in \Gamma L\}$. The basic Laplacian $\Delta_B$ acting on $f \in \Omega^0_B(\mathcal{F})$ is given by $\Delta_B f = \delta_B d_B f$. For a harmonic $g$-Riemannian foliation we have $\Delta_B f \in \Omega^0_B(\mathcal{F})$.

**Theorem 6.** *Let (M,g) be a closed orientable Riemannian manifold and let $\mathcal{F}$ be a transversely oriented harmonic g-Riemannian foliation of codimension q on M. Suppose that there exists a nontrivial parallel 2-form $\alpha \in \Omega^2_B(\mathcal{F})$ and a positive constant k such that the transversal Ricci curvature satisfies $Ric_\nabla(Z,Z) \geq k(q-1)|Z|^2$ for every $Z \in \mathcal{V}^\perp(\mathcal{F})$. Then the smallest nonzero eigenvalue $\lambda_B$ of the basic Laplacian $\Delta_B$ satisfies*

$$\lambda_B \geq 2k(q-1).$$

*Proof.* Let $Z \in \mathcal{V}^\perp(\mathcal{F})$ and $\omega = i_Z g$. Let $\Theta$ be the endomorphism of $\Gamma L^\perp$ associated to $\alpha$. By formula (2.1) and (3.7) we have

$$\begin{aligned} g(tr\nabla^2_B Z + \rho_\nabla Z, \Theta^2(Z)) &= tr_B \nabla^2\omega(\Theta^2(Z)) - Ric_\nabla(\Theta(Z), B(Z)) \\ &\leq -\Delta_B\omega(\Theta^2(Z)) - 2k(q-1)|\Theta(Z)|^2. \end{aligned} \tag{4.8}$$

Since $\mathcal{F}$ is harmonic, by integrating the inequality (3.6) over $M$ and by taking into account the inequality (4.8) we get

$$\int_M (-\Delta_B\omega(\Theta^2(Z)) - 2k(q-1)|\Theta(Z)|^2) d_M \geq 0. \tag{4.9}$$

Let $f$ be an eigenfunction of $\Delta_B$ with eigenvalue $\lambda_B > 0$ and let $Z = \nabla f$ be the gradient of $f$. Since $\Delta_B d_B f = d_B \Delta_B f = \lambda_B d_B f$, hence from the inequality (4.9) we obtain

$$\big(\lambda_B - 2k(q-1)\big) \int_M |\Theta(\nabla f)|^2 d_M \geq 0.$$

and the theorem follows. ∎

Let $\xi \in \mathcal{V}^\perp(\mathcal{F})$. We recall that $\xi$ is a transverse Killing field if $L_\xi g(Y,Z) = 0$ for all $Y, Z \in \Gamma L^\perp$. Since $L_\xi g(Y,Z) = g(A_\xi(Y), Z) + g((Y, A_\xi(Z))$, we have

**Corollary 7.** *Let (M,g) be a closed Riemannian manifold and let $\mathcal{F}$ be a harmonic g-Riemannian foliation of codimension q on M. Suppose that there exists a nontrivial transverse Killing field $\xi \in \mathcal{V}^\perp(\mathcal{F})$ such that $\nabla^2\xi = 0$ and a positive constant k such that the transversal Ricci curvature satisfies $Ric_\nabla(Z,Z) \geq k(q-1)|Z|^2$ for every $Z \in \mathcal{V}^\perp(\mathcal{F})$. Then the smallest nonzero eigenvalue $\lambda_B$ of the basic Laplacian $\Delta_B$ satisfies*

$$\lambda_B \geq 2k(q-1).$$

**Corollary 8.** *Let (M,g) be a closed Riemannian manifold and let $\mathcal{F}$ be a harmonic g-Riemannian foliation of codimension $q \geq 3$ on M. Suppose that there exists a positive constant k such that the transversal Ricci curvature satisfies $Ric_\nabla(Z,Z) \geq k(q-1)|Z|^2$ for every $Z \in \mathcal{V}^\perp(\mathcal{F})$. If the smallest nonzero eigenvalue $\lambda_B$ of the basic Laplacian $\Delta_B$ satisfies*

$$kq \leq \lambda_B < 2k(q-1),$$

*then any parallel 2-form $\alpha \in \Omega^2_B(\mathcal{F})$ is trivial. In particular any transverse Killing field $\xi$ on M which satisfies $\nabla^2\xi = 0$ is trivial.*

**Remark 9.** *If $q = 2$, then $kq = 2k(q-1)$, ie our estimation coincides with that in [1].*

**Example 10.** *Let $(M_1, g_1, \mathcal{F}_1)$ be a $g_1$-Riemannian harmonic foliation of codimension two on closed Riemannian manifold. There exists a basic function $\lambda$ such that $Ric_{\nabla^1} = \lambda g_1$ on $M$. Suppose that the transversal Ricci curvature $Ric_{\nabla^1} > 0$. So the first non zero eigenvalue of the basic Laplacian satisfies $\lambda_B^1 \geq 2k$ where $k > 0$ is the minimum of the function $\lambda$. Let $\nu_1 = E^*_{p+1} \wedge E^*_{p+2}$ be the transversal volume form of $\mathcal{F}_1$. The form $\nu_1$ is basic parallel. In fact let $X \in \Gamma(L_1)$, so*

$$\begin{aligned}(L_X\nu_1)(E_{p+1}, E_{p+2}) &= X.\nu_1(E_{p+1}, E_{p+2}) - \nu_1([X, E_{p+1}], E_{p+2}) \\ &- \nu_1(E_{p+1}, [X, E_{p+2}]) \\ &= -g_1([X, E_{p+1}], E_{p+1}) - g_1, [X, E_{p+2}], (E_{p+2}) \\ &= -\tfrac{1}{2}((\nabla^1_X g_1)(E_{p+1}, E_{p+1}) + (\nabla^1_X g_1)(E_{p+2}, E_{p+2})) = 0,\end{aligned}$$

*because $\mathcal{F}$ is $g_1$-Riemannian. Now let $Z \in \Gamma(L_1^\perp)$ so*

$$\begin{aligned}(\nabla_Z\nu_1)(E_{p+1}, E_{p+2}) &= -\nu(\nabla^{M_1}_Z E_{p+1}, E_{p+2}) - \nu_1(E_{p+1}, \nabla^{M_1}_Z E_{p+2}) \\ &= -g_1(\nabla^{M_1}_Z E_{p+1}, E_{p+1}) - g_1(\nabla^{M_1}_Z E_{p+2}, E_{p+2}) = 0.\end{aligned}$$

*Now let $(M_2, g_2, \mathcal{F}_2)$ be an other $g_2$-Riemannian harmonic foliation of codimension $q-2 \geq 1$ with a transversal Ricci curvature $Ric_{\nabla^2} \geq kg_2$ (for example, take $\mathcal{F}_2$ a transversely elliptic foliation). The product foliation $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ on $M = M_1 \times M_2$ is also $g_1 \times g_2$-Riemannian harmonic of codimension $q$ with transversal Ricci curvature $Ric_\nabla \geq k(g_1 \times g_2)$. By the new estimation the smallest non zero eigenvalue of the basic Laplacian satisfies $\lambda_B \geq 2k$. Whereas the estimation given in [1] is $\lambda_B \geq \frac{q}{q-1}k$.*

**Example 11.** *Assume that $\mathcal{F}$ is a Kähler foliation (see [4]). That is*
*i) $\mathcal{F}$ is $g$-Riemannian, i.e. $\nabla g = 0$, ii) there is a holonomy invariant almost complex structure $J : L^\perp \longrightarrow L^\perp$, where dim $L^\perp = q = 2m$ (real dimension), with respect to which the metric $g$ is transversely Hermitian, i.e. $g(JX, JY) = g(X, Y)$ for $X, Y \in \Gamma(L^\perp)$, iii) $\nabla$ is almost complex structure, i.e. $\nabla J = 0$. Note that $\alpha(X, Y) = g(X, JY)$ for $X, Y \in \Gamma(L^\perp)$ and $i_\xi\alpha = 0$ for $\xi \in \Gamma L$ defines a basic 2-form $\alpha$, which is closed as a consequence of $\nabla g = 0$ and $\nabla J = 0$.*

*Clearly $\alpha^m \neq 0$ at all points of $M$. Let $f \in \Omega^0_B(\mathcal{F})$; in the one hand $L_{\nabla f}\alpha^m = \Delta_B f.\alpha^m$ and in the other hand $L_{\nabla f}\alpha^m = mL_{\nabla f}\alpha \wedge \alpha^{m-1}$. Consequently $L_{\nabla f}\alpha = \frac{1}{m}\Delta_B f.\alpha$. We deduce that*

*i) $\nabla f$ is a transversal Kähler field ($L_{\nabla f}\alpha = 0$) if and only if the function $f$ is harmonic. In this case $J \circ A_{\nabla f} = -A_{\nabla f} \circ J$. Let $x \in M$; since $(A_{\nabla f})_x$ is diagonalisable and $J_x$ is antisymmetric anti-commuting with $(A_{\nabla f})_x$, hence $(A_{\nabla f})_x$ is of even rang.*

*ii) $\nabla f$ is a transversal Liouville field ($L_{\nabla f}\alpha = \alpha$) if and only if $\Delta_B f = m$.*

*Let $\chi_\mathcal{F}$ be the characteristic form of $\mathcal{F}$ and let $\bar{*} : \Omega^r_B(\mathcal{F}) \longrightarrow \Omega^{q-r}_B(\mathcal{F})$ be the transversal star operator, we have*

$$\bar{*}L_{\nabla f} = \frac{1}{m}\nabla_B f. \wedge \alpha^{m-1}, \quad \|L_{\nabla f}\|^2_{\Omega^*_B(\mathcal{F})} = \int_M L_{\nabla f} \wedge \bar{*}L_{\nabla f} \wedge \chi_\mathcal{F} = \frac{\|\Delta_B f\|_2^2}{m^2}.$$

*We deduce that if $\mathcal{F}$ is a harmonic Kähler foliation of codimension $q$ on a closed manifold such that $Ric_\nabla \geq k(q-1)$ and if $f$ is an eigenfunction of $\Delta_B$, then $\|L_{\nabla f}\|_{\Omega^*_B(\mathcal{F})} \geq \frac{4k(q-1)}{q}\|f\|_2$.*

*Now we give an example of a harmonic Kähler foliation with constant sectional curvature see* [5] *page* 273. *Let* $P^m C$ *be the complex projective space. This is the quotient of the Euclidian sphere* $S^{2m+1}$ *under the canonical* $S^1$*-action. We obtain the Hopf fibration*

$$S^1 \to S^{2m+1} \to P^m C = SU(m+1)/S(U(1) \times U(m)),$$

*which gives rice naturally to a harmonic (and totally geodesic) Kähler (and symmetric) foliation on* $S^{2m+1}$. *The transversal holomorphic sectional curvature is* 4*; therefore* $Ric_\nabla = 4q$ *and* $k = \frac{4q}{q-1}$. *By the new estimation the smallest non zero eigenvalue of the basic Laplacian satisfies* $\lambda_B \geq 8q$. *Whereas the estimation given in* [1] *is* $\lambda_B \geq \frac{4q^2}{q-1}$.

We end the paper by the following question. If in Theorem 6 the equality occurs, is the leaf space isometric to the space of orbit of a discrete subgroup of $O(q-2) \times O(2)$ acting on the standard product $(q-2)$-sphere with 2-sphere of constant curvature $k$?

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