On a Certain Subclass of Multivalent Analytic Functions Defined by the Liu-Owa Operator

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Abstract

By making use of the principle of subordination between analytic functions, we introduce a certain subclass of multivalent analytic functions defined by the Liu-Owa operator. Results such as subordination and superordination properties, convolution properties, distortion theorems and inequality properties, are proved.

1 Introduction

Let $H[a,k]$ be the class of analytic functions of the form:

$$f(z) = a + a_k z^k + a_{k+1} z^{k+1} + \ldots \quad (z \in U)$$

and let $\mathcal{A}(p,k)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=k}^{\infty} a_{p+n} z^{p+n} \quad (p,k \in \mathbb{N} = \{1,2,3,\ldots\}) \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. For simplicity, we write $\mathcal{A}(p,1) = \mathcal{A}(p)$ and $\mathcal{A}(1,1) = \mathcal{A}$.

If $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$, or, equivalently, $g(z)$ is superordinate to $f(z)$, notation $f \prec g$ in U or $f(z) \prec$

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$g(z)$ $(z \in U)$, if there exists a Schwarz function $\omega(z)$, which (by definition) is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in U)$, such that $f(z) = g(\omega(z))$ $(z \in U)$. Indeed, it is known that

$$f(z) \prec g(z) \ \ (z \in U) \Longrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Furthermore, if the function $g(z)$ is univalent in U, then the following equivalence holds (see [8] and [9]) :

$$f(z) \prec g(z) \quad (z \in U) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

For functions $f, g \in \mathcal{A}(p,k)$, where f is given by (1.1) and g is defined by

$$g(z) = z^p + \sum_{n=k}^{\infty} b_{p+n} z^{p+n} \ ,$$

the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f*g)(z) = z^p + \sum_{n=k}^{\infty} a_{p+n} b_{p+n} z^{p+n} = (g*f)(z).$$

Motivated in essence by Jung et al. [4] , Liu and Owa [6] introduced the operator $Q^{\alpha}_{\beta,p} : \mathcal{A}(p,k) \to \mathcal{A}(p,k)$ as follows:

$$Q^{\alpha}_{\beta,p} f(z) = \binom{p+\alpha+\beta-1}{p+\beta-1} \frac{\alpha}{z^{\beta}} \int_0^z \left(1-\frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)\, dt$$
$$(\alpha > 0;\ \beta > -1;\ p \in \mathbb{N}), \quad (1.2)$$

and

$$Q^{0}_{\beta,p} f(z) = f(z) \quad (\alpha = 0;\ \beta > -1;\ p \in \mathbb{N}). \quad (1.3)$$

For $f(z) \in \mathcal{A}(p,k)$ given by (1.1) , it follows from (1.2) and (1.3) , that

$$Q^{\alpha}_{\beta,p} f(z) = z^p + \frac{\Gamma(\alpha+\beta+p)}{\Gamma(\beta+p)} \sum_{n=k}^{\infty} \frac{\Gamma(\beta+p+n)}{\Gamma(\alpha+\beta+p+n)} a_{p+n} z^{p+n}$$
$$(\alpha \geq 0; \beta > -1; p \in \mathbb{N}). \quad (1.4)$$

Using (1.4), it is easy to verify that

$$z\left(Q^{\alpha}_{\beta,p} f(z)\right)' = (\alpha+\beta+p-1)\, Q^{\alpha-1}_{\beta,p} f(z) - (\alpha+\beta-1)\, Q^{\alpha}_{\beta,p} f(z). \quad (1.5)$$

We note that $Q^{1}_{c,p} f(z) = J_{c,p}(f)(z) = \frac{c+p}{z^c} \int t^{c-1} f(z)\, dt$ $(c > -p)$, where the operator $J_{c,p}$ is the generalized Bernardi–Libera–Livingston integral operator (see [2]). Also, we note that the one-parameter family of integral operators $Q^{\alpha}_{\beta,1} = Q^{\alpha}_{\beta}$ was defined by Jung et al. [4] and studied by Aouf [1] and Gao et al. [3] .

By making use of the linear operator $Q^{\alpha}_{\beta,p}$ and the above-mentioned principle of subordination between analytic functions, we now introduce the following subclass of the class $\mathcal{A}(p,k)$ of $p-$valent analytic functions.

Definition 1. A function $f \in \mathcal{A}(p,k)$ is said to be in the class $\mathcal{N}_{p,k}^{\lambda,\mu}(\alpha,\beta;A,B)$ if it satisfies the following subordination condition:

$$(1+\lambda)\left(\frac{z^p}{Q_{\beta,p}^{\alpha}f(z)}\right)^{\mu}-\lambda\left(\frac{Q_{\beta,p}^{\alpha-1}f(z)}{Q_{\beta,p}^{\alpha}f(z)}\right)\left(\frac{z^p}{Q_{\beta,p}^{\alpha}f(z)}\right)^{\mu}\prec\frac{1+Az}{1+Bz}\quad(z\in U), \tag{1.6}$$

where (here and throughout this paper unless otherwise stated) the parameters $\alpha,\beta,p,\lambda,\mu,A$ and B satisfy the constraints:

$$\alpha\geq 1;\ \beta>-1;\lambda\in\mathbb{C};\ 0<\mu<1;\ -1\leq B\leq 1,\ A\neq B,\ A\in\mathbb{R}\text{ and }\ p,k\in\mathbb{N},$$

and all powers are understood as being principle values.

In this paper we aim at proving such results as subordination and superordination properties, convolution properties, distortion theorems and inequality properties for the class $\mathcal{N}_{p,k}^{\lambda,\mu}(\alpha,\beta;A,B)$.

2 Preliminary results

In order to establish our main results, we need the following definition and lemmas.

Definition 2 [9] . *Denote by $\mathcal{L}$ the set of all functions f that are analytic and injective on $\bar{U}\backslash E(q)$, where*

$$E(f)=\left\{\zeta\in\partial U:\lim_{z\to\zeta}f(z)=\infty\right\},$$

and such that $f^{'}(\zeta)\neq 0$ for $\zeta\in\bar{U}\backslash E(q)$.

Lemma 1 [8] . *Let the function $h(z)$ be analytic and convex (univalent) in U with $h(0)=1$. Suppose also that the function $g(z)$ given by*

$$g(z)=1+c_kz^k+c_{k+1}z^{k+1}+... \tag{2.1}$$

is analytic in U. If

$$g(z)+\frac{zg^{'}(z)}{\gamma}\prec h(z)\quad(\Re(\gamma)>0;\gamma\neq 0;\ z\in U), \tag{2.2}$$

then

$$g(z)\prec q(z)=\frac{\gamma}{k}z^{-\frac{\gamma}{k}}\int h(t)\,t^{\frac{\gamma}{k}-1}\,dt\prec h(z),$$

and $q(z)$ is the best dominant of (2.2).

Lemma 2 [11]. *Let $q(z)$ be a convex univalent function in U and let $\sigma \in \mathbb{C}$, $\eta \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}$ with*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\Re\left(\frac{\sigma}{\eta}\right)\right\}.$$

If the function $g(z)$ is analytic in U and

$$\sigma g(z) + \eta z g'(z) \prec \sigma q(z) + \eta z q'(z),$$

then $g(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 3 [9]. *Let $q(z)$ be convex univalent in U and $\kappa \in \mathbb{C}$. Further assume that $\Re(\bar{\kappa}) > 0$. If $g(z) \in H[q(0),1] \cap \mathcal{L}$, and $g(z) + \kappa z g'(z)$ is univalent in U, then*

$$q(z) + \kappa z q'(z) \prec g(z) + \kappa z g'(z),$$

implies $q(z) \prec g(z)$ and $q(z)$ is the best subordinant.

Lemma 4 [5]. *Let $\mathcal{F}$ be analytic and convex in U. If $f, g \in \mathcal{A}$* and *$f, g \prec \mathcal{F}$ then*

$$\lambda f + (1-\lambda) g \prec \mathcal{F} \quad (0 \le \lambda \le 1).$$

Lemma 5 [10]. *Let $f(z) = 1 + \sum\limits_{k=1}^{\infty} a_k z^k$ be analytic in U and $g(z) = 1 + \sum\limits_{k=1}^{\infty} b_k z^k$ be analytic and convex in U. If $f(z) \prec g(z)$, then*

$$|a_k| < |b_1| \quad (k \in \mathbb{N}).$$

3 Main results

We begin by presenting our first subordination property given by Theorem 1 below.

Theorem 1. *Let $f(z) \in \mathcal{N}_{p,k}^{\lambda,\mu}(\alpha, \beta; A, B)$ with $\Re(\lambda) > 0$. Then*

$$\begin{aligned} \left(\frac{z^p}{Q_{\beta,p}^{\alpha} f(z)}\right)^{\mu} &\prec q(z) = \frac{(\alpha+\beta+p-1)\mu}{\lambda k} \int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1} du \\ &\prec \frac{1+Az}{1+Bz} \quad (z \in U), \end{aligned} \tag{3.1}$$

and $q(z)$ is the best dominant.

Proof. Define the function $g(z)$ by

$$g(z) = \left(\frac{z^p}{Q_{\beta,p}^{\alpha} f(z)}\right)^{\mu} \quad (z \in U). \tag{3.2}$$

Then the function $g(z)$ is of the form (2.1) and is analytic in U . Differentiating (3.2) with respect to z and using the identity (1.5), we get

$$(1+\lambda)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}-\lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}=g(z)+\frac{\lambda zg'(z)}{(\alpha+\beta+p-1)\mu}. \tag{3.3}$$

As $f(z)\in\mathcal{N}^{\lambda,\mu}_{p,k}(\alpha,\beta;A,B)$, we have

$$g(z)+\frac{\lambda zg'(z)}{(\alpha+\beta+p-1)\mu}\prec\frac{1+Az}{1+Bz}\quad(z\in U).$$

Applying Lemma 1 with $\gamma=\frac{(\alpha+\beta+p-1)\mu}{\lambda}$, we get

$$\begin{aligned}\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}\prec\ q(z)&=\frac{(\alpha+\beta+p-1)\mu}{\lambda k}z^{-\frac{(\alpha+\beta+p-1)\mu}{\lambda k}}\int_0^z\frac{1+At}{1+Bt}t^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1}dt\\&=\frac{(\alpha+\beta+p-1)\mu}{\lambda k}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1}du\prec\frac{1+Az}{1+Bz},\end{aligned} \tag{3.4}$$

and $q(z)$ is the best dominant, which ends the proof. ∎

Theorem 2. *Let $q(z)$ be univalent in U, $\lambda\in\mathbb{C}^*$. Suppose also that $q(z)$ satisfies the following inequality:*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right)>\max\left\{0,-\Re\left(\frac{(\alpha+\beta+p-1)\mu}{\lambda}\right)\right\}. \tag{3.5}$$

If $f\in\mathcal{A}(p)$ satisfies the following subordination condition:

$$(1+\lambda)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}-\lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}\prec q(z)+\frac{\lambda zq'(z)}{(\alpha+\beta+p-1)\mu}, \tag{3.6}$$

then

$$\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}\prec q(z),$$

and $q(z)$ is the best dominant.

Proof. Let the function $g(z)$ be defined by (3.2). We know that (3.3) holds. Combining (3.3) and (3.6), we find that

$$g(z)+\frac{\lambda}{(\alpha+\beta+p-1)\mu}zg'(z)\prec q(z)+\frac{\lambda}{(\alpha+\beta+p-1)\mu}zq'(z). \tag{3.7}$$

By using Lemma 2 and (3.7), we easily get the assertion of Theorem 2. ∎

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 2, we get the following result.

Corollary 1. *Let $\lambda \in \mathbb{C}^*$ and $-1 \leq B < A \leq 1$. Suppose also that*

$$\Re\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -\Re\left(\frac{(\alpha+\beta+p-1)\mu}{\lambda}\right)\right\}.$$

If $f(z) \in \mathcal{A}(p)$ satisfies the following subordination:

$$(1+\lambda)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu} - \lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu} \prec \frac{1+Az}{1+Bz} + \frac{\lambda}{(\alpha+\beta+p-1)\mu}\frac{(A-B)z}{(1+Bz)^2},$$

then

$$\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu} \prec \frac{1+Az}{1+Bz},$$

and the function $\frac{1+Az}{1+Bz}$ is the best dominant.

By making use of Lemma 3, we now derive the following superordination result.

Theorem 3 . *Let $q(z)$ be convex univalent in U, $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$. Also let*

$$\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu} \in H[q(0), 1] \cap \mathcal{L}$$

and

$$(1+\lambda)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu} - \lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}$$

be univalent in U. If

$$q(z) + \frac{\lambda}{(\alpha+\beta+p-1)\mu}zq'(z) \prec (1+\lambda)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu} - \lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu},$$

then

$$q(z) \prec \left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu},$$

and the function $q(z)$ is the best subdominant.

Proof. Let the function $g(z)$ be defined by (3.2). Then

$$\begin{aligned} & q(z)+\frac{\lambda z q'(z)}{(\alpha+\beta+p-1)\mu} \\ \prec\ & (1+\lambda)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}-\lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu} \\ =\ & g(z)+\frac{\lambda z g'(z)}{(\alpha+\beta+p-1)\mu}. \end{aligned}$$

Applying Lemma 3 yields the assertion of Theorem 3. ∎

Taking $q(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ in Theorem 3, we get the following corollary.

Corollary 2. *Let $q(z)$ be convex univalent in U and $-1\le B<A\le 1$, $\lambda\in\mathbb{C}$ with $\Re(\lambda)>0$. Also let*

$$\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}\in H[q(0),1]\cap\mathcal{L},$$

and

$$(1+\lambda)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}-\lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}$$

be univalent in U. If

$$\begin{aligned} & \frac{1+Az}{1+Bz}+\frac{\lambda}{(\alpha+\beta+p-1)\mu}\frac{(A-B)z}{(1+Bz)^2} \\ & \qquad \prec (1+\lambda)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}-\lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}, \end{aligned}$$

then

$$\frac{1+Az}{1+Bz}\prec\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu},$$

and the function $\frac{1+Az}{1+Bz}$ is the best subdominant.

Combining Theorem 2 and Theorem 3, we easily get the following "Sandwich-type result".

Theorem 4. *Let $q_1(z)$ be convex univalent and let $q_2(z)$ be univalent in U, $\lambda\in\mathbb{C}$ with $\Re(\lambda)>0$. Let $q_2(z)$ satisfies (3.5). If*

$$\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}\in H[q_1(0),1]\cap\mathcal{L},$$

and

$$(1+\lambda)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}-\lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}$$

is univalent in U, and if also

$$\begin{aligned} q_1(z) &+ \frac{\lambda z q_1'(z)}{(\alpha+\beta+p-1)\mu} \\ &\prec (1+\lambda)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu} - \lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu} \\ &\prec q_2(z) + \frac{\lambda z q_2'(z)}{(\alpha+\beta+p-1)\mu}, \end{aligned}$$

then

$$q_1(z) \prec \left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu} \prec q_2(z),$$

and $q_1(z)$ and $q_2(z)$ are the best subordinant and dominant respectively.

Theorem 5. *If $\lambda, \mu > 0$ and $f(z) \in \mathcal{N}^{0,\mu}_{p,k}(\alpha,\beta;1-2\rho,-1)$ $(0 \le \rho < 1)$, then $f(z) \in \mathcal{N}^{\lambda,\mu}_{p,k}(\alpha,\beta;1-2\rho,-1)$ for $|z| < R$, where*

$$R = \left(\sqrt{\left(\frac{\lambda k}{(\alpha+\beta+p-1)\mu}\right)^2 + 1} - \frac{\lambda k}{(\alpha+\beta+p-1)\mu}\right)^{\frac{1}{k}}. \tag{3.8}$$

The bound R is the best possible.

Proof. We begin by writing

$$\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu} = \rho + (1-\rho)g(z) \quad (z \in U; 0 \le \rho < 1). \tag{3.9}$$

Then, clearly, the function $g(z)$ is of the form (2.1), it is analytic and it has a positive real part in U. Differentiating (3.9) with respect to z and using the identity (1.5), we obtain

$$\begin{aligned} \frac{1}{1-\rho}&\left\{(1+\lambda)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu} - \lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu} - \rho\right\} \\ &= g(z) + \frac{\lambda z g'(z)}{(\alpha+\beta+p-1)\mu}. \end{aligned} \tag{3.10}$$

By making use of the following well-known estimate (*see* [7]):

$$\frac{\left|zg'(z)\right|}{\Re\{g(z)\}} \le \frac{2kr^k}{1-r^{2k}} \quad (|z| = r < 1)$$

(3.10) leads to

$$\Re\left(\frac{1}{1-\rho}\left\{(1+\lambda)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}-\lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}-\rho\right\}\right)$$

$$\geq \Re\{g(z)\}\left(1-\frac{2\lambda k r^k}{(\alpha+\beta+p-1)\mu\left(1-r^{2k}\right)}\right). \tag{3.11}$$

It is seen that the right-hand side of (3.11) is positive, provided that $r < R$, where R is given by (3.8).

In order to show that the bound R is the best possible, we consider the function $f(z) \in \mathcal{A}(p,k)$ defined by

$$\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}=\rho+(1-\rho)\frac{1+z^k}{1-z^k}\quad (z\in U; 0\leq\rho<1).$$

Noting that

$$\frac{1}{1-\rho}\left\{(1+\lambda)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}-\lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}-\rho\right\}$$

$$=\frac{1+z^k}{1-z^k}+\frac{2\lambda k z^k}{(\alpha+\beta+p-1)\mu\left(1-z^k\right)^2}=0, \tag{3.12}$$

for $|z| = R$, we conclude that the bound is the best possible, which ends the proof. ∎

Theorem 6. *Let* $f(z) \in \mathcal{N}^{\lambda,\mu}_{p,k}(\alpha,\beta;A,B)$ *with* $\Re(\lambda) > 0$. *Then*

$$f(z)=\left(z^p\left(\frac{1+B\omega(z)}{1+A\omega(z)}\right)^{\frac{1}{\mu}}\right)*\left(z^p+\frac{\Gamma(\beta+p)}{\Gamma(\alpha+\beta+p)}\sum_{n=k}^{\infty}\frac{\Gamma(\alpha+\beta+p+n)}{\Gamma(\beta+p+n)}z^{p+n}\right), \tag{3.13}$$

where $\omega(z)$ *is an analytic function with* $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in U)$.

Proof. Suppose that $f(z) \in \mathcal{N}^{\lambda,\mu}_{p,k}(\alpha,\beta;A,B)$ with $\Re(\lambda) > 0$. It follows from (3.1) that

$$\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}=\frac{1+A\omega(z)}{1+B\omega(z)}, \tag{3.14}$$

where $\omega(z)$ is an analytic function with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in U)$. By virtue of (3.14), we easily find that

$$Q^{\alpha}_{\beta,p}f(z)=z^p\left(\frac{1+B\omega(z)}{1+A\omega(z)}\right)^{\frac{1}{\mu}}. \tag{3.15}$$

Combining (1.4) and (3.15), we have

$$\left(z^p+\frac{\Gamma(\alpha+\beta+p)}{\Gamma(\beta+p)}\sum_{n=k}^{\infty}\frac{\Gamma(\beta+p+n)}{\Gamma(\alpha+\beta+p+n)}z^{p+n}\right)*f(z)=z^p\left(\frac{1+B\omega(z)}{1+A\omega(z)}\right)^{\frac{1}{\mu}}. \tag{3.16}$$

The assertion (3.13) of Theorem 6 can now easily be derived from (3.16). ∎

Theorem 7. *Let $f\in\mathcal{N}_{p,k}^{\lambda,\mu}(\alpha,\beta;A,B)$ with $\Re(\lambda)>0$. Then*

$$\frac{1}{z^p}\left[\left(1+Ae^{i\theta}\right)^{\frac{1}{\mu}}\left(z^p+\frac{\Gamma(\alpha+\beta+p)}{\Gamma(\beta+p)}\sum_{n=k}^{\infty}\frac{\Gamma(\beta+p+n)}{\Gamma(\alpha+\beta+p+n)}z^{p+n}\right)\right.$$
$$\left.*f(z)-z^p\left(1+Be^{i\theta}\right)^{\frac{1}{\mu}}\right]\neq 0\qquad(z\in U;\ 0<\theta<2\pi). \tag{3.17}$$

Proof. Suppose that $f(z)\in\mathcal{N}_{p,k}^{\lambda,\mu}(\alpha,\beta;A,B)$ with $\Re(\lambda)>0$. We know that (3.1) holds, implying that

$$\left(\frac{z^p}{Q_{\beta,p}^{\alpha}f(z)}\right)^{\mu}\neq\frac{1+Ae^{i\theta}}{1+Be^{i\theta}}\quad(z\in U;\ 0<\theta<2\pi). \tag{3.18}$$

It is easy to see that the condition (3.18) can be written as follows:

$$\frac{1}{z^p}\left[Q_{\beta,p}^{\alpha}f(z)\left(1+Ae^{i\theta}\right)^{\frac{1}{\mu}}-z^p\left(1+Be^{i\theta}\right)^{\frac{1}{\mu}}\right]\neq 0\quad(z\in U;\ 0<\theta<2\pi). \tag{3.19}$$

Combining (1.4) and (3.19), we easily get the convolution property (3.17). ∎

Theorem 8. *Let $\lambda_2\geq\lambda_1\geq 0$ and $-1\leq B_1\leq B_2<A_2\leq A_1\leq 1$. Then*

$$\mathcal{N}_{p,k}^{\lambda_2,\mu}(\alpha,\beta;A_2,B_2)\subset\mathcal{N}_{p,k}^{\lambda_1,\mu}(\alpha,\beta;A_1,B_1). \tag{3.20}$$

Proof. Suppose that $f\in\mathcal{N}_{p,k}^{\lambda_2,\mu}(\alpha,\beta;A_2,B_2)$. We know that

$$(1+\lambda_2)\left(\frac{z^p}{Q_{\beta,p}^{\alpha}f(z)}\right)^{\mu}-\lambda_2\left(\frac{Q_{\beta,p}^{\alpha-1}f(z)}{Q_{\beta,p}^{\alpha}f(z)}\right)\left(\frac{z^p}{Q_{\beta,p}^{\alpha}f(z)}\right)^{\mu}\prec\frac{1+A_2z}{1+B_2z}.$$

As $-1\leq B_1\leq B_2<A_2\leq A_1\leq 1$, we easily find that

$$(1+\lambda_2)\left(\frac{z^p}{Q_{\beta,p}^{\alpha}f(z)}\right)^{\mu}-\lambda_2\left(\frac{Q_{\beta,p}^{\alpha-1}f(z)}{Q_{\beta,p}^{\alpha}f(z)}\right)\left(\frac{z^p}{Q_{\beta,p}^{\alpha}f(z)}\right)^{\mu}\prec\frac{1+A_2z}{1+B_2z}\prec\frac{1+A_1z}{1+B_1z}, \tag{3.21}$$

which means that $f\in\mathcal{N}_{p,k}^{\lambda_2,\mu}(\alpha,\beta;A_1,B_1)$. Thus the assertion (3.20) holds for $\lambda_2=\lambda_1\geq 0$. If $\lambda_2>\lambda_1\geq 0$, by Theorem 1 and (3.21), we know that $f\in\mathcal{N}_{p,k}^{0,\mu}(\alpha,\beta;A_1,B_1)$, that is,

$$\left(\frac{z^p}{Q_{\beta,p}^{\alpha}f(z)}\right)^{\mu}\prec\frac{1+A_1z}{1+B_1z}, \tag{3.22}$$

At the same time, we have

$$(1+\lambda_1)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}-\lambda_1\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}=$$
$$\left(1-\frac{\lambda_1}{\lambda_2}\right)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}+\frac{\lambda_1}{\lambda_2}\left[(1+\lambda_2)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}\right.$$
$$\left.-\lambda_2\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}\right]. \quad (3.23)$$

Moreover

$$0\le\frac{\lambda_1}{\lambda_2}<1,$$

and the function $\frac{1+A_1z}{1+B_1z}\,(-1\le B_1<A_1\le 1; z\in U)$ is analytic and convex in U. Combining $(3.21)-(3.23)$ and Lemma 4, we find that

$$(1+\lambda_1)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}-\lambda_1\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}\prec\frac{1+A_1z}{1+B_1z},$$

which means that $f\in\mathcal{N}^{\lambda_1,\mu}_{p,k}(\alpha,\beta;A_1,B_1)$, which implies that the assertion (3.20) of Theorem 8 holds. ■

Theorem 9. *Let $f\in\mathcal{N}^{\lambda,\mu}_{p,k}(\alpha,\beta;A,B)$ with $\lambda>0$ and $-1\le B<A\le 1$. Then*

$$\frac{(\alpha+\beta+p-1)\mu}{\lambda k}\int_0^1\frac{1-Au}{1-Bu}\,u^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1}du$$
$$<\ \Re\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}<\frac{(\alpha+\beta+p-1)\mu}{\lambda k}\int_0^1\frac{1+Au}{1+Bu}\,u^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1}du \quad (3.24)$$

The extremal function of (3.24) *is given by*

$$Q^{\alpha}_{\beta,p}F(z)=z^p\left(\frac{(\alpha+\beta+p-1)\mu}{\lambda k}\int_0^1\frac{1+Az^nu}{1+Bz^nu}\,u^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1}du\right)^{-\frac{1}{\mu}}. \quad (3.25)$$

Proof. Let $f(z)\in\mathcal{N}^{\lambda,\mu}_{p,k}(\alpha,\beta;A,B)$ with $\lambda>0$. From Theorem 1, we know that (3.1) holds, which implies that

$$\Re\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu}<\sup_{z\in U}\Re\left\{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}\int_0^1\frac{1+Azu}{1+Bzu}\,u^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1}du\right\}$$
$$\le\frac{(\alpha+\beta+p-1)\mu}{\lambda k}\int_0^1\sup_{z\in U}\Re\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1}du$$
$$<\frac{(\alpha+\beta+p-1)\mu}{\lambda k}\int_0^1\frac{1+Au}{1+Bu}\,u^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1}du, \quad (3.26)$$

and

$$\begin{aligned}\Re\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu} &> \inf_{z\in U}\Re\left\{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}\int_0^1 \frac{1+Azu}{1+Bzu}\,u^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1}du\right\}\\ &\geq \frac{(\alpha+\beta+p-1)\mu}{\lambda k}\int_0^1 \inf_{z\in U}\Re\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1}du\\ &> \frac{(\alpha+\beta+p-1)\mu}{\lambda k}\int_0^1 \frac{1-Au}{1-Bu}\,u^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1}du. \qquad (3.27)\end{aligned}$$

Combining (3.26) and (3.27), we get (3.24). By noting that the function $Q^{\alpha}_{\beta,p}F(z)$, defined by (3.25), belongs to the class $\mathcal{N}^{\lambda,\mu}_{p,k}(\alpha,\beta;A,B)$, we obtain that equality (3.24) is sharp. This completes the proof. ∎

In a similar way, applying the method used in the proof of Theorem 9, we easily get the following result.

Corollary 3. *Let $f\in\mathcal{N}^{\lambda,\mu}_{p,k}(\alpha,\beta;A,B)$ with $\lambda>0$ and $-1\leq A<B\leq 1$. Then*

$$\begin{aligned}&\frac{(\alpha+\beta+p-1)\mu}{\lambda k}\int_0^1 \frac{1+Au}{1+Bu}\,u^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1}du\\ &< \Re\left(\frac{z^p}{Q^{\alpha}_{\beta,p}f(z)}\right)^{\mu} < \frac{(\alpha+\beta+p-1)\mu}{\lambda k}\int_0^1 \frac{1-Au}{1-Bu}\,u^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1}du. \qquad (3.28)\end{aligned}$$

The extremal function of (3.28) *is given by* (3.25).

In view of Theorem 9 and Corollary 3, we easily derive the following distortion theorems for the class $\mathcal{N}^{\lambda,\mu}_{p,k}(\alpha,\beta;A,B)$.

Corollary 4. *Let $f(z)\in\mathcal{N}^{\lambda,\mu}_{p,k}(\alpha,\beta;A,B)$ with $\lambda>0$ and $-1\leq B<A\leq 1$. Then for $|z|=r<1$, we have*

$$\begin{aligned}&r^p\left(\frac{(\alpha+\beta+p-1)\mu}{\lambda k}\int_0^1 \frac{1-Aur}{1-Bur}\,u^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1}du\right)^{\frac{1}{\mu}}\\ &< \left|Q^{\alpha}_{\beta,p}f(z)\right| < r^p\left(\frac{(\alpha+\beta+p-1)\mu}{\lambda k}\int_0^1 \frac{1+Aur}{1+Bur}\,u^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1}du\right)^{\frac{1}{\mu}}. \qquad (3.29)\end{aligned}$$

The extremal function of (3.29) *is defined by* (3.25).

Corollary 5. *Let $f(z)\in\mathcal{N}^{\lambda,\mu}_{p,k}(\alpha,\beta;A,B)$ with $\lambda>0$ and $-1\leq A<B\leq 1$. Then for $|z|=r<1$, we have*

$$\begin{aligned}&r^p\left(\frac{(\alpha+\beta+p-1)\mu}{\lambda k}\int_0^1 \frac{1+Aur}{1+Bur}\,u^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1}du\right)^{\frac{1}{\mu}}\\ &< \left|Q^{\alpha}_{\beta,p}f(z)\right| < r^p\left(\frac{(\alpha+\beta+p-1)\mu}{\lambda k}\int_0^1 \frac{1-Aur}{1-Bur}\,u^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1}du\right)^{\frac{1}{\mu}}. \qquad (3.30)\end{aligned}$$

The extremal function of (3.30) *is given by* (3.25).

By noting that

$$\left(\Re\left(v\right)\right)^{\frac{1}{2}} \leq \Re\left(v^{\frac{1}{2}}\right) \leq |v|^{\frac{1}{2}} \quad \left(v \in \mathbb{C}; \Re\left(v\right) \geq 0\right).$$

we easily derive from Theorem 9 and Corollary 3 the following results.

Corollary 6. *Let* $f \in \mathcal{N}_{p,k}^{\lambda,\mu}\left(\alpha,\beta;A,B\right)$ *with* $\lambda > 0$ *and* $-1 \leq B < A \leq 1$. *Then*

$$\begin{aligned} &\left(\frac{\left(\alpha+\beta+p-1\right)\mu}{\lambda k}\int_0^1 \frac{1-Au}{1-Bu}\,u^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1}du\right)^{\frac{1}{2}} \\ <\ & \Re\left(\frac{z^p}{Q_{\beta,p}^{\alpha}f\left(z\right)}\right)^{\frac{\mu}{2}} < \left(\frac{\left(\alpha+\beta+p-1\right)\mu}{\lambda k}\int_0^1 \frac{1+Au}{1+Bu}\,u^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1}du\right)^{\frac{1}{2}}. \end{aligned}$$

The extremal function is given by (3.25).

Corollary 7. *Let* $f \in \mathcal{N}_{p,k}^{\lambda,\mu}\left(\alpha,\beta;A,B\right)$ *with* $\lambda > 0$ *and* $-1 \leq A < B \leq 1$. *Then*

$$\begin{aligned} &\left(\frac{\left(\alpha+\beta+p-1\right)\mu}{\lambda k}\int_0^1 \frac{1+Au}{1+Bu}\,u^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1}du\right)^{\frac{1}{2}} \\ <\ & \Re\left(\frac{z^p}{Q_{\beta,p}^{\alpha}f\left(z\right)}\right)^{\frac{\mu}{2}} < \left(\frac{\left(\alpha+\beta+p-1\right)\mu}{\lambda k}\int_0^1 \frac{1-Au}{1-Bu}\,u^{\frac{(\alpha+\beta+p-1)\mu}{\lambda k}-1}du\right)^{\frac{1}{2}}. \end{aligned}$$

The extremal function is given by (3.25).

Theorem 10. *Let*

$$f\left(z\right) = z^p + \sum_{n=k}^{\infty} a_{p+n}z^{p+n} \in \mathcal{N}_{p,k}^{\lambda,\mu}\left(\alpha,\beta;A,B\right). \tag{3.31}$$

Then

$$\left|a_{p+k}\right| \leq \frac{\Gamma\left(\beta+p\right)\Gamma\left(\alpha+\beta+p+k\right)}{\Gamma\left(\alpha+\beta+p-1\right)\Gamma\left(\beta+p+k\right)}\left|\frac{A-B}{\left[\lambda k+\mu\left(\alpha+\beta+p-1\right)\right]}\right|. \tag{3.32}$$

The estimate (3.32) *is sharp, with the extremal function defined by* (3.25).

Proof. Combining (1.6) and (3.31), we obtain

$$\begin{aligned} &\left(1-\lambda\right)\left(\frac{z^p}{Q_{\beta,p}^{\alpha}f\left(z\right)}\right)^{\mu} + \lambda\left(\frac{Q_{\beta,p}^{\alpha-1}f\left(z\right)}{Q_{\beta,p}^{\alpha}f\left(z\right)}\right)\left(\frac{z^p}{Q_{\beta,p}^{\alpha}f\left(z\right)}\right)^{\mu} \\ =\ & 1+\left[-\mu\left(\alpha+\beta+p-1\right)-k\lambda\right]\frac{\Gamma\left(\alpha+\beta+p-1\right)\Gamma\left(\beta+p+k\right)}{\Gamma\left(\beta+p\right)\Gamma\left(\alpha+\beta+p+k\right)}a_{p+k}z^k+... \\ \prec\ & \frac{1+Az}{1+Bz} = 1+\left(A-B\right)z+.... \end{aligned} \tag{3.33}$$

Applying Lemma 5 to (3.33) yields

$$\left|\left[-\mu\left(\alpha+\beta+p-1\right)-k\lambda\right]\frac{\Gamma\left(\alpha+\beta+p-1\right)\Gamma\left(\beta+p+k\right)}{\Gamma\left(\beta+p\right)\Gamma\left(\alpha+\beta+p+k\right)}a_{p+k}\right|\leq\left|A-B\right|. \tag{3.34}$$

From (3.34), we easily derive (3.32), which completes the proof. ■

Remark. Putting $\alpha = 1$ and $\beta = c$ $(c > -p)$ in the above results, we obtain the corresponding results for the generalized Bernardi–Libera–Livingston integral operator $J_{c,p}(f)$.

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