On the Gauss map of complete spacelike hypersurfaces with bounded mean curvature in the Minkowski space

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Dedicated to my daughter Larissa

Abstract

In this paper, under an appropriated restriction on the Gauss map, we obtain an extension of the Xin-Aiyama theorem concerning to complete spacelike hypersurfaces immersed with bounded mean curvature in the Minkowski space.

1 Introduction

In the last years, the study of spacelike hypersurfaces in the Minkowski space $\mathbb{L}^{n+1}$ has been of substantial interest from both the physical and mathematical aspects. From a physical point of view, that interest is motivated by their role in the study of different problems in general relativity. From a mathematical point of view, that interest is also motivated by the fact that these hypersurfaces exhibit nice Bernstein-type properties. For example, Y.L. Xin in [6] and R. Aiyama in [1] simultaneous and independently characterized the spacelike hyperplanes as the only complete constant mean curvature spacelike hypersurfaces in $\mathbb{L}^{n+1}$ having the image of its Gauss map contained in a geodesic ball of the hyperbolic space (see also [5] for a weaker first version of this result given by B. Palmer).

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In this paper we are concerning to complete spacelike hypersurfaces immersed with bounded mean curvature in $\mathbb{L}^{n+1}$ and whose Gauss map N is future-directed on $\mathbb{L}^{n+1}$ (cf. Section 2). In this setting, under an appropriate restriction on the hyperbolic image $N(\Sigma)$ and as a suitable application of the well know generalized Maximum Principle of Omori-Yau, we obtain the following extension of the Xin-Aiyama theorem:

Theorem 1.1. *Let $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ be a complete spacelike hypersurface bounded away from the past infinite of $\mathbb{L}^{n+1}$ and with bounded mean curvature $H \geq 0$. If the hyperbolic image of Σ^n is contained in the closure of a geodesic ball of center $e_{n+1} \in \mathbb{H}^n$ and whose radius ϱ satisfies* $\cosh \varrho \leq 1 + \inf_\Sigma H$*, then Σ^n is a spacelike hyperplane.*

Finally, we want to point out that our restriction on the Gauss map of the spacelike hypersurface is motivated by the fact that the hyperbolic caps of $\mathbb{L}^{n+1}$ satisfy such condition (see Remark 3.1).

2 Complete Spacelike Hypersurfaces in $\mathbb{L}^{n+1}$

Let $\mathbb{L}^{n+1}$ denote the $(n+1)$-dimensional Minkowski space, that is, the real vector space $\mathbb{R}^{n+1}$, endowed with the Lorentz metric

$$\langle v, w \rangle = \sum_{i=1}^{n} v_i w_i - v_{n+1} w_{n+1},$$

for all $v, w \in \mathbb{R}^{n+1}$.

A smooth immersion $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ of an n-dimensional connected manifold Σ^n is said to be a *spacelike hypersurface* if the induced metric via ψ is a Riemannian metric on Σ^n, which, as usual, is also denoted by $\langle , \rangle$.

Observe that $e_{n+1} = (0, \ldots, 0, 1)$ is a unit timelike vector field globally defined on $\mathbb{L}^{n+1}$, which determines a time-orientation on $\mathbb{L}^{n+1}$. Thus we can choose a unique timelike unit normal vector field N on Σ^n which is *future-directed* on $\mathbb{L}^{n+1}$ (i.e., $\langle N, e_{n+1} \rangle \leq -1$), and hence we may assume that Σ^n is oriented by N. We also note that such timelike unit normal vector field $N \in \mathfrak{X}(\Sigma)$ can be regarded as the Gauss map $N : \Sigma^n \to \mathbb{H}^n$ of Σ^n, where $\mathbb{H}^n$ denotes the n-dimensional hyperbolic space, that is,

$$\mathbb{H}^n = \{x \in \mathbb{L}^{n+1}; \langle x, x \rangle = -1,\ x_{n+1} \geq 1\}.$$

In this setting, the image $N(\Sigma)$ is called the *hyperbolic image* of Σ^n. Furthermore, given a geodesic ball $B(a, \varrho)$ in $\mathbb{H}^n$ of radius $\varrho > 0$ and centered at a point $a \in \mathbb{H}^n$, we recall that $B(a, \varrho)$ is characterized as the following

$$B(a, \varrho) = \{p \in \mathbb{H}^n; -\cosh \varrho \leq \langle p, a \rangle \leq -1\}.$$

So, if the hyperbolic image of Σ^n is contained into some $B(a, \varrho)$, then

$$1 \leq |\langle N, a \rangle| \leq \cosh \varrho.$$

Now, let A be the shape operator of Σ^n in $\mathbb{L}^{n+1}$ associated to the Gauss map N of Σ^n future-directed on $\mathbb{L}^{n+1}$. In order to set up the notation to be used later, let us denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections of $\mathbb{L}^{n+1}$ and Σ^n, respectively. Then, the Gauss and Weingarten formulas for Σ^n in $\mathbb{L}^{n+1}$ are written, respectively, as

$$\overline{\nabla}_X Y = \nabla_X Y - \langle AX, Y\rangle N,$$

and

$$A(X) = -\overline{\nabla}_X N,$$

for all tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$.

On the other hand, as in [4], the curvature tensor R of the spacelike hypersurface Σ^n is given by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where [ ] denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(\Sigma)$.

A fact well known is that the curvature tensor R of the spacelike hypersurface Σ^n can be described in terms of the shape operator A by the so-called Gauss equation given by

$$R(X,Y)Z = -\langle AX, Z\rangle AY + \langle AY, Z\rangle AX,$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$. Consequently, the *Ricci curvature tensor* Ric_Σ of Σ^n is given by

$$\begin{aligned}\mathrm{Ric}_\Sigma(X,X) &= nH\langle AX, X\rangle + \langle AX, AX\rangle \\ &= \left|AX + \frac{nH}{2}X\right|^2 - \frac{n^2H^2}{4}|X|^2,\end{aligned}$$

where $H = -\frac{1}{n}\mathrm{tr}(A)$ is the mean curvature of Σ^n.

In what follows, $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ denotes a complete spacelike hypersurface, with Gauss map N future-directed on $\mathbb{L}^{n+1}$. In this context, according to the terminology established in [2], we say that Σ^n is *bounded away from the past infinity* of $\mathbb{L}^{n+1}$ if the height function $h : \Sigma^n \to \mathbb{R}$, defined by $h(p) = \langle \psi(p), e_{n+1}\rangle$, is bounded from below on Σ^n.

In order to prove our result, we will need the generalized *Maximum Principle* due to H. Omori and S.T. Yau [3, 7].

Lemma 2.1. *Let Σ^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and $u : \Sigma^n \to \mathbb{R}$ be a smooth function which is bounded from below on Σ^n. Then there is a sequence of points $\{p_k\}$ in Σ^n such that*

$$\lim_{k\to\infty} u(p_k) = \inf_\Sigma u, \quad \lim_{k\to\infty} |\nabla u(p_k)| = 0 \quad \text{and} \quad \lim_{k\to\infty} \Delta u(p_k) \geq 0.$$

3 Proof of Theorem 1.1

Let us consider the height function $h = \langle \psi, e_{n+1} \rangle$ defined on Σ^n. We have that

$$\nabla h = e_{n+1}^{\top} = e_{n+1} + \langle N, e_{n+1} \rangle N,$$

where $e_{n+1}^{\top} \in \mathfrak{X}(\Sigma)$ denotes the tangential component of e_{n+1} in Σ^n. Consequently, from the Gauss and Weingarten formulas, the Hessian of h is given by

$$\nabla^2 h(X, Y) = \langle \nabla_X e_{n+1}^{\top}, Y \rangle = -\langle N, e_{n+1} \rangle \langle AX, Y \rangle,$$

for all $X, Y \in \mathfrak{X}(\Sigma)$. Thus, the Laplacian of h is

$$\Delta h = -\langle N, e_{n+1} \rangle \mathrm{tr}(A) = nH \langle N, e_{n+1} \rangle.$$

On the other hand, from the previous section, the Ricci curvature of Σ^n is such that

$$\mathrm{Ric}_{\Sigma} \geq -\frac{n^2 H^2}{4}.$$

Consequently, since we are supposing that $0 \leq H \leq \alpha$ for some constant α, we get

$$\mathrm{Ric}_{\Sigma} \geq -\frac{n^2 \alpha^2}{4},$$

that is, Ric_{Σ} is bounded from below on Σ^n. Thus, since Σ^n is supposed to be bounded away from the past infinite of $\mathbb{L}^{n+1}$, we are in position to apply Lemma 2.1 to the height function h, obtaining a sequence $\{p_k\}$ in Σ^n such that

$$\lim_{k \to \infty} \Delta h(p_k) \geq 0.$$

Consequently, from the boundedness on Σ^n of the functions H and $\langle N, e_{n+1} \rangle$, we get a subsequence $\{p_{k_j}\}$ of $\{p_k\}$ such that

$$0 \leq \lim_{j \to \infty} \Delta h(p_{k_j}) \leq -n \lim_{j \to \infty} H(p_{k_j}) \leq 0.$$

Then, $\lim_{j \to \infty} H(p_{k_j}) = 0$, and $\inf_{\Sigma} H = 0$. Thus, since we are supposing that $N(\Sigma) \subset B(e_{n+1}, \varrho)$ with $\cosh \varrho \leq 1 + \inf_{\Sigma} H$, we conclude that

$$1 \leq |\langle N, e_{n+1} \rangle| \leq \cosh \varrho \leq 1,$$

that is, $\langle N, e_{n+1} \rangle = -1$ on Σ^n. Therefore, Σ^n is a spacelike hyperplane. ∎

Remark 3.1. Fixed a positive constant λ, we easily verify that the hyperbolic cap

$$\Sigma_{\lambda}^n = \left\{ x \in \mathbb{L}^{n+1}; \langle x, x \rangle = -\lambda^2, \, \lambda \leq x_{n+1} \leq \sqrt{1 + \lambda^2} \right\}$$

is an example of spacelike hypersurface of the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ which has constant mean curvature

$$H_{\lambda} = \frac{1}{\lambda} > 0,$$

if we choose the Gauss map N future-direct on $\mathbb{L}^{n+1}$. Moreover, we also easily verify that the hyperbolic image $N(\Sigma_\lambda)$ is contained in the closure of the geodesic ball of center $e_{n+1} \in \mathbb{H}^n$ and with radius ϱ satisfying

$$\cosh \varrho = \sqrt{1 + \frac{1}{\lambda^2}} \leq 1 + H_\lambda.$$

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