On the quasi-equivalence of orthogonal bases in non-archimedean metrizable locally convex spaces

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Abstract. We prove that any non-archimedean metrizable locally convex space E with a regular orthogonal basis has the quasi-equivalence property, i.e. any two orthogonal bases in E are quasi-equivalent. In particular, the power series spaces $A_1(a)$ and $A_\infty(a)$, the most known and important examples of non-archimedean nuclear Fréchet spaces, have the quasi-equivalence property. We also show that the Fréchet spaces: $\mathbb{K}^\mathbb{N}, c_0 \times \mathbb{K}^\mathbb{N}, c_0^\mathbb{N}$ have the quasi-equivalence property.

1 Introduction

In this paper all linear spaces are over a non-archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot| : \mathbb{K} \to [0, \infty)$. For fundamentals of locally convex Hausdorff spaces (lcs) and normed spaces we refer to [9], [10] and [11]. Orthogonal bases in locally convex spaces are studied in [4], [5], [6] and [12].

Let E be a metrizable lcs with an orthogonal basis (x_n). Clearly, for any sequence (α_n) of non-zero scalars and any permutation σ of $\mathbb{N}$ the sequence $(\alpha_n x_{\sigma(n)})$ is an orthogonal basis in E. It is interesting to know whether any orthogonal basis (y_n) in E is equivalent to one of the above bases. In other words whether any two orthogonal bases in E are quasi-equivalent. It is easy to see that it is so if E is a

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normed space with an orthogonal basis. It is not known whether any metrizable lcs with an orthogonal basis has the quasi-equivalence property.

Developing the ideas of [8] and [7] (see also [1] and [2]) we prove that any two regular orthogonal bases in a metrizable lcs are semi equivalent (Proposition 3), and any two orthogonal bases in a metrizable lcs with a regular orthogonal basis are quasi-equivalent (Theorem 5). In particular, the power series spaces $A_1(a)$ and $A_\infty(a)$ (see [3]) have the quasi-equivalence property (Corollary 6).

We also show that the Fréchet spaces: $\mathbb{K}^{\mathbb{N}}, c_0 \times \mathbb{K}^{\mathbb{N}}$ and $c_0^{\mathbb{N}}$ possess the quasi-equivalence property (Proposition 8).

2 Preliminaries

The linear span of a subset A of a linear space E is denoted by $\mathrm{lin}A$.

Let E, F be locally convex spaces. A map $T : E \to F$ is called a *linear homeomorphism* if T is linear, one-to-one, surjective and the maps T, T^{-1} are continuous. If there exists a linear homeomorphism $T : E \to F$, then we say that E is *isomorphic* to F.

Sequences (x_n) and (y_n) in a lcs E are *equivalent* if there exists a linear homeomorphism P between the linear spans of (x_n) and (y_n), such that $Px_n = y_n$ for all $n \in \mathbb{N}$. Sequences (x_n) and (y_n) in a Fréchet space E are equivalent if and only if there exists a linear homeomorphism P between the closed linear spans of (x_n) and (y_n), such that $Px_n = y_n$ for all $n \in \mathbb{N}$.

Sequences (x_n) and (y_n) in a lcs E are *semi equivalent* if for some sequence (α_n) of non-zero scalars the sequences $(\alpha_n x_n)$ and (y_n) are equivalent, and *quasi-equivalent* if for some permutation σ of $\mathbb{N}$ the sequences $(x_{\sigma(n)})$ and (y_n) are semi equivalent.

A sequence (x_n) in a lcs E is a *Schauder basis* in E if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{\infty} \alpha_n x_n$ with $(\alpha_n) \subset \mathbb{K}$ and the coefficient functionals $f_n : E \to \mathbb{K}, x \to \alpha_n \, (n \in \mathbb{N})$ are continuous.

By a *seminorm* on a linear space E we mean a function $p : E \to [0, \infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}, x \in E$ and $p(x + y) \leq \max\{p(x), p(y)\}$ for all $x, y \in E$. A seminorm p on E is a *norm* if $\ker p := \{x \in E : p(x) = 0\} = \{0\}$.

The set of all continuous seminorms on a metrizable lcs E is denoted by $\mathcal{P}(E)$. A non-decreasing sequence $(p_k) \subset \mathcal{P}(E)$ is a *base* in $\mathcal{P}(E)$ if for every $p \in \mathcal{P}(E)$ there exists $k \in \mathbb{N}$ with $p \leq p_k$. A sequence (p_k) of norms on E is a *base of norms in* $\mathcal{P}(E)$ if it is a base in $\mathcal{P}(E)$.

Any metrizable lcs E possesses a base (p_k) in $\mathcal{P}(E)$. Every metrizable lcs E with a continuous norm has a base (p_k) of norms in $\mathcal{P}(E)$.

A metrizable lcs E is *of finite type* if $\dim(E/\ker p) < \infty$ for any $p \in \mathcal{P}(E)$, and *of countable type* if E contains a linearly dense countable set.

A *Fréchet space* is a metrizable complete lcs.

A *Banach space* is a normed Fréchet space. Any infinite-dimensional Banach space of countable type is isomorphic to the Banach space c_0 of all sequences in $\mathbb{K}$ converging to zero (with the sup-norm) ([10], Theorem 3.16).

Let p be a seminorm on a linear space E. A sequence $(x_n) \subset E$ is 1-*orthogonal with respect to* p if $p(\sum_{i=1}^{n} \alpha_i x_i) = \max_{1 \leq i \leq n} p(\alpha_i x_i)$ for all $n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in \mathbb{K}$.

A sequence (x_n) in a metrizable lcs E is 1-*orthogonal with respect to* $(p_k) \subset \mathcal{P}(E)$ if (x_n) is 1-orthogonal with respect to p_k for any $k \in \mathbb{N}$.

A sequence (x_n) in a metrizable lcs E is *orthogonal* if it is 1-orthogonal with respect to some base (p_k) in $\mathcal{P}(E)$. A linearly dense orthogonal sequence of non-zero elements in a metrizable lcs E is an *orthogonal basis* in E.

Every orthogonal basis in a metrizable lcs E is a Schauder basis in E ([5], Proposition 1.4) and every Schauder basis in a Fréchet space F is an orthogonal basis in F ([5], Proposition 1.7).

A metrizable lcs E with an orthogonal basis has the *quasi-equivalence property* if any two orthogonal bases in E are quasi-equivalent.

An orthogonal basis (x_n) in a metrizable lcs E is *regular* if there exists a base of norms (p_k) in $\mathcal{P}(E)$ such that (x_n) is 1-orthogonal with respect to (p_k) and

$$\frac{p_k(x_n)}{p_{k+1}(x_n)} \geq \frac{p_k(x_{n+1})}{p_{k+1}(x_{n+1})} \quad \text{for all } k, n \in \mathbb{N};$$

in this case we will say that (x_n) is regular with respect to (p_k). If (x_n) is regular with respect to (p_k), then it is regular with respect to any subsequence (p_{k_m}) of (p_k).

Let $a = (a_n)$ be a non-decreasing sequence of positive real numbers with $a_n \to \infty$. Then the following spaces are nuclear Fréchet spaces (see [3]):

(1) $A_1(a) = \{(\alpha_n) \subset \mathbb{K} : \lim_n |\alpha_n|(\frac{k}{k+1})^{a_n} = 0 \text{ for all } k \in \mathbb{N}\}$ with the base (p_k) of norms: $p_k((\alpha_n)) = k \max_n |\alpha_n|(\frac{k}{k+1})^{a_n}, k \in \mathbb{N}$;

(2) $A_\infty(a) = \{(\alpha_n) \subset \mathbb{K} : \lim_n |\alpha_n| k^{a_n} = 0 \text{ for all } k \in \mathbb{N}\}$ with the base (q_k) of norms: $q_k((\alpha_n)) = \max_n |\alpha_n| k^{a_n}, k \in \mathbb{N}$.

$A_1(a)$ and $A_\infty(a)$ are the *power series spaces* (*of finite* and *infinite type*, respectively). The standard basis (e_n) in $A_1(a)$ and $A_\infty(a)$ is regular with respect to the bases of norms (p_k) and (q_k), respectively.

3 Results

For arbitrary subsets A, B in a linear space E and a linear subspace L of E we denote $d(A, B, L) = \inf\{|\beta| : \beta \in \mathbb{K}, A \subset (\beta B + L)\}$ (we put $\inf \emptyset = +\infty$). Let $d_n(A, B) = \inf\{d(A, B, L) : L < E, \dim L \leq (n-1)\}, n \in \mathbb{N}$.

It is easy to check the following

Remark 1. *If $A' \subset A \subset E, B \subset B' \subset E$ and $n \in \mathbb{N}$, then $d_n(A', B') \leq d_n(A, B)$. If $a, b \in (\mathbb{K} \setminus \{0\}), A, B \subset E$ and $n \in \mathbb{N}$, then $d_n(aA, bB) = |ab^{-1}| d_n(A, B)$.*

We will need the following

Lemma 2. *Let (f_n) be the sequence of coefficient functionals associated with a basis (x_n) in a lcs E. Let $(a_k), (b_k) \subset (0, \infty)$ with $a_k b_k^{-1} \geq a_{k+1} b_{k+1}^{-1}$ for all $k \in \mathbb{N}$. Put $A = \{x \in E : |f_k(x)| \leq a_k, k \in \mathbb{N}\}$ and $B = \{x \in E : |f_k(x)| \leq b_k, k \in \mathbb{N}\}$. Then for any $n \in \mathbb{N}$ and $\alpha \in \mathbb{K}$ with $|\alpha| > 1$ we have $|\alpha|^{-1} a_n b_n^{-1} \leq d_n(A, B) < |\alpha| a_n b_n^{-1}$.*

Proof. Clearly, there exists $\beta \in \mathbb{K}$ with $a_n b_n^{-1} \leq |\beta| < |\alpha| a_n b_n^{-1}$. Let $x \in A$. Since $|\beta^{-1} f_k(x)| \leq a_n^{-1} b_n a_k \leq b_k$ for any $k \geq n$, then $(\sum_{k=n}^{\infty} \beta^{-1} f_k(x) x_k) \in B$. Hence $x \in (\beta B + \operatorname{lin}\{x_i : 1 \leq i < n\})$. Thus $A \subset (\beta B + \operatorname{lin}\{x_i : 1 \leq i < n\})$. This follows that $d_n(A, B) < |\alpha| a_n b_n^{-1}$.

Now, assume that $L < E$, $\beta \in \mathbb{K}$, $|\beta| < |\alpha|^{-1}a_n b_n^{-1}$ and $A \subset (\beta B + L)$. For $1 \le i \le n$ let $\beta_i \in \mathbb{K}$ with $|\alpha|^{-1}a_i \le |\beta_i| < a_i$. Then $\beta_i x_i \in A \subset (\beta B + L), 1 \le i \le n$. Thus for any $1 \le i \le n$ there exists $u_i \in B$ such that $v_i = (\beta_i x_i - \beta u_i) \in L$. We shall prove that $v_1, \ldots, v_n$ are linearly independent in L. Let $c_1, \ldots, c_n \in \mathbb{K}$ with $\sum_{i=1}^n c_i v_i = 0$. Put $\alpha_k^i = f_k(u_i)$ for $1 \le i \le n, k \in \mathbb{N}$. Then $|\alpha_k^i| \le b_k$ for all $1 \le i \le n, k \in \mathbb{N}$, and

$$\sum_{i=1}^n c_i\beta_i x_i = \sum_{i=1}^n \beta c_i u_i = \sum_{i=1}^n \beta c_i (\sum_{k=1}^\infty \alpha_k^i x_k) = \sum_{k=1}^\infty (\sum_{i=1}^n \beta c_i \alpha_k^i) x_k.$$

Hence $(*)$ $c_k\beta_k = \sum_{i=1}^n \beta c_i \alpha_k^i, 1 \le k \le n$. For any $1 \le k \le n$ there exists $1 \le i_k \le n$ with $|\alpha_k^{i_k}| = \max_{1\le i\le n} |\alpha_k^i|$. Let $1 \le k \le n$. Put $\alpha_k = \alpha_k^{i_k}$ if $\alpha_k^{i_k} \ne 0$, and let $\alpha_k \in \mathbb{K}$ with $0 < |\alpha_k| \le b_k$ if $\alpha_k^{i_k} = 0$. By $(*)$, we get

$$(**) \quad c_k\beta_k\alpha_k^{-1} = \sum_{i=1}^n (c_i\beta_i\alpha_i^{-1})(\beta_i^{-1}\alpha_i\beta\alpha_k^i\alpha_k^{-1}).$$

For any $1 \le i \le n$ we have

$$|\beta_i^{-1}||\alpha_i||\beta||\alpha_k^i\alpha_k^{-1}| < (|\alpha|a_i^{-1})b_i(|\alpha|^{-1}a_i b_i^{-1})1 = 1.$$

Thus $c := \max_{1\le k\le n} \max_{1\le i\le n} |\beta_i^{-1}\alpha_i\beta\alpha_k^i\alpha_k^{-1}| < 1$. By $(**)$, we obtain

$$|c_k\beta_k\alpha_k^{-1}| \le c \max_{1\le i\le n} |c_i\beta_i\alpha_i^{-1}|, 1 \le k \le n.$$

Hence $c_k = 0$ for all $1 \le k \le n$.

Thus $\dim L \ge n$ and $d_n(A, B) \ge |\alpha|^{-1}a_n b_n^{-1}$. ■

Now, we can prove the following

Proposition 3. *Any two regular orthogonal bases (x_n) and (y_n) in a metrizable lcs E are semi equivalent.*

Proof. Let $\alpha \in \mathbb{K}$ with $|\alpha| > 1$. Assume that (x_n) and (y_n) are regular with respect to bases of norms (p_k) and (q_k) in $\mathcal{P}(E)$, respectively. Without loss of generality we can assume that

$$(*) \quad p_n \le |\alpha|^{-2}q_n \le |\alpha|^{-4}p_{n+1} \le |\alpha|^{-6}q_{n+1} \text{ for all } n \in \mathbb{N}.$$

Set $U_n = \{x \in E : p_n(x) \le 1\}$ and $V_n = \{y \in E : q_n(y) \le 1\}$ for $n \in \mathbb{N}$. Let (f_n) and (g_n) be the sequences of coefficient functionals associated with the bases (x_n) and (y_n), respectively. Put $a_{n,k} = p_n(x_k)$ and $b_{n,k} = q_n(y_k)$ for all $n, k \in \mathbb{N}$. Then $a_{t,k}a_{s,k}^{-1} \ge a_{t,k+1}a_{s,k+1}^{-1}$ and $b_{t,k}b_{s,k}^{-1} \ge b_{t,k+1}b_{s,k+1}^{-1}$ for all $t, s, k \in \mathbb{N}$ with $t \le s$.

Since $p_n(x) = \max_k |f_k(x)|a_{n,k}$ and $q_n(y) = \max_k |g_k(y)|b_{n,k}$ for $x, y \in E, n \in \mathbb{N}$, then $U_n = \{x \in E : |f_k(x)| \le a_{n,k}^{-1}, k \in \mathbb{N}\}$ and $V_n = \{y \in E : |g_k(y)| \le b_{n,k}^{-1}, k \in \mathbb{N}\}$ for $n \in \mathbb{N}$. Let $t, s \in \mathbb{N}$ with $t \le s$. By Lemma 2, we obtain

$(*_1)$ $|\alpha|^{-1}a_{t,n}a_{s,n}^{-1} \le d_n(U_s, U_t) \le |\alpha|a_{t,n}a_{s,n}^{-1}, n \in \mathbb{N};$

$(*_2)$ $|\alpha|^{-1}b_{t,n}b_{s,n}^{-1} \le d_n(V_s, V_t) \le |\alpha|b_{t,n}b_{s,n}^{-1}, n \in \mathbb{N}.$

Let $n \in \mathbb{N}$. Let $i, j \in \mathbb{N}$ with $i \le j$. By $(*)$, we have $U_i \supset \alpha^2 V_i \supset \alpha^2 V_j \supset \alpha^4 U_{j+1}$. Hence, by Remark 1, we get

$$|\alpha|^4 d_n(U_{j+1}, U_i) = d_n(\alpha^4 U_{j+1}, U_i) \le d_n(\alpha^2 V_j, \alpha^2 V_i) = d_n(V_j, V_i).$$

Using $(*_1)$ and $(*_2)$ we obtain that $|\alpha|^3 a_{i,n} a_{j+1,n}^{-1} \le |\alpha| b_{i,n} b_{j,n}^{-1}$.

Now, let $i, j \in \mathbb{N}$ with $j < i$. Then we have $V_j \supset \alpha^2 U_{j+1} \supset \alpha^2 U_i \supset \alpha^4 V_i$. Hence we get $|\alpha|^4 d_n(V_i, V_j) \le d_n(U_i, U_{j+1})$. Thus $|\alpha|^3 b_{j,n} b_{i,n}^{-1} \le |\alpha| a_{j+1,n} a_{i,n}^{-1}$.

We have proved that $|\alpha|^2 a_{i,n} b_{i,n}^{-1} \le a_{j+1,n} b_{j,n}^{-1}$ for all $n, i, j \in \mathbb{N}$.

Let $n \in \mathbb{N}, A_n = \sup_i a_{i,n} b_{i,n}^{-1}$ and $B_n = \inf_j a_{j+1,n} b_{j,n}^{-1}$. Then $|\alpha|^2 A_n \le B_n$. Thus there exists $c_n \in \mathbb{K}$ with $A_n \le |c_n| < B_n$. Hence for any $k \in \mathbb{N}$ we get $a_{k,n} b_{k,n}^{-1} \le |c_n| < a_{k+1,n} b_{k,n}^{-1}$; so $a_{k,n} \le |c_n| b_{k,n} < a_{k+1,n}$ for all $k, n \in \mathbb{N}$.

We have shown that $p_k(x_n) \le q_k(c_n y_n) < p_{k+1}(x_n)$ for all $k, n \in \mathbb{N}$. It follows that $p_k(\sum_{n=1}^m \alpha_n x_n) \le q_k(\sum_{n=1}^m \alpha_n c_n y_n) \le p_{k+1}(\sum_{n=1}^m \alpha_n x_n)$ for all $k, m \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{K}$. This proves that the bases (x_n) and $(c_n y_n)$ in E are equivalent. Thus the bases $(x_n), (y_n)$ are semi equivalent. ∎

Our next result is the following

Proposition 4. *Let E be a metrizable lcs with a regular orthogonal basis (x_n). Then for any orthogonal basis (y_n) in E there exists a permutation σ of $\mathbb{N}$ such that $(y_{\sigma(n)})$ is a regular orthogonal basis in E.*

Proof. Let (f_n) and (g_n) be the sequences of coefficient functionals associated with the bases (x_n) and (y_n), respectively. Since

$$1 = |g_n(y_n)| = |g_n(\sum_{k=1}^{\infty} f_k(y_n) x_k)| = |\sum_{k=1}^{\infty} f_k(y_n) g_n(x_k)| \le \max_k |f_k(y_n) g_n(x_k)|, n \in \mathbb{N},$$

then for any $n \in \mathbb{N}$ there exists $t_n \in \mathbb{N}$ with $|f_{t_n}(y_n) g_n(x_{t_n})| \ge 1$. For every $s \in \mathbb{N}$ we have $f_s(x_s) = f_s(\sum_{n=1}^{\infty} g_n(x_s) y_n) = \sum_{n=1}^{\infty} f_s(y_n) g_n(x_s)$. Hence we obtain $|f_s(y_n) g_n(x_s)| \to_n 0, s \in \mathbb{N}$. Thus for any $s \in \mathbb{N}$ the set $\{n \in \mathbb{N} : t_n = s\}$ is finite. Therefore there exists a permutation σ of $\mathbb{N}$ such that the sequence $(t_{\sigma(n)})$ is non-decreasing.

Assume that (x_n) is regular with respect to a base of norms (p_k) in $\mathcal{P}(E)$. For any $k \in \mathbb{N}$ there exist a norm q_k on E and $s_k \in \mathbb{N}$ with $p_k \le q_k \le p_{s_k}$ such that (y_n) is 1-orthogonal with respect to q_k. For all $n, k \in \mathbb{N}$ we obtain

$$p_k(f_{t_n}(y_n) x_{t_n}) \le \max_m p_k(f_m(y_n) x_m) = p_k(y_n) \le |g_n(x_{t_n})|^{-1} \max_m q_k(g_m(x_{t_n}) y_m) =$$

$$|g_n(x_{t_n})|^{-1} q_k(x_{t_n}) \le p_{s_k}(f_{t_n}(y_n) x_{t_n}).$$

Hence $(*)$ $p_k(f_{t_n}(y_n) x_{t_n}) \le p_k(y_n) \le p_{s_k}(f_{t_n}(y_n) x_{t_n})$ for all $k, n \in \mathbb{N}$.

Put $r_k(x) = \max_n |g_n(x)| p_k(f_{t_n}(y_n) x_{t_n}), k \in \mathbb{N}, x \in E$.

By $(*)$, we get $r_k(x) \le \max_n |g_n(x)| q_k(y_n) = q_k(x) \le p_{s_k}(x)$, and
$p_k(x) \le \max_n |g_n(x)| p_k(y_n) \le \max_n |g_n(x)| p_{s_k}(f_{t_n}(y_n) x_{t_n}) = r_{s_k}(x)$.

Thus (r_k) is a base of norms in $\mathcal{P}(E)$. Clearly, (y_n) is 1-orthogonal with respect to (r_k) and

$$\frac{r_k(y_n)}{r_{k+1}(y_n)} = \frac{p_k(f_{t_n}(y_n) x_{t_n})}{p_{k+1}(f_{t_n}(y_n) x_{t_n})} = \frac{p_k(x_{t_n})}{p_{k+1}(x_{t_n})} \quad \text{for all } k, n \in \mathbb{N}.$$

Since the basis (x_n) is regular with respect to (p_k) and the sequence $(t_{\sigma(n)})$ is non-decreasing, the basis $(y_{\sigma(n)})$ is regular with respect to (r_k). ∎

By Propositions 3 and 4, we obtain our main result.

Theorem 5. *Any metrizable lcs with a regular orthogonal basis possesses the quasi-equivalence property.*

Corollary 6. *The power series spaces $A_1(a)$ and $A_\infty(a)$ have the quasi-equivalence property. In particular, the space $A_1(\mathbb{K})$ of all analytic functions in the unit ball of $\mathbb{K}$, and the space $A_\infty(\mathbb{K})$ of all entire functions in $\mathbb{K}$ have this property.*

By the closed graph theorem (see [9], Theorem 2.49), we get

Remark 7. *For orthogonal bases $(x_n), (y_n)$ in a Fréchet space E the following conditions are equivalent:*

(1) *the bases (x_n) and (y_n) are equivalent;*

(2) *for any $(\beta_n) \subset \mathbb{K}$ the sequence $(\beta_n x_n)$ is convergent to 0 in E if and only if the sequence $(\beta_n y_n)$ is convergent to 0 in E;*

(3) *for any $(\beta_n) \subset \mathbb{K}$ the series $\sum_{n=1}^\infty \beta_n x_n$ is convergent in E if and only if the series $\sum_{n=1}^\infty \beta_n y_n$ is convergent in E.*

Using Remark 7, we shall prove the following

Proposition 8. *The Fréchet spaces: $\mathbb{K}^{\mathbb{N}}, c_0, c_0 \times \mathbb{K}^{\mathbb{N}}$ and $c_0^{\mathbb{N}}$ possess the quasi-equivalence property.*

Proof. (A). First, we show that any two orthogonal bases $(x_n), (y_n)$ in $\mathbb{K}^{\mathbb{N}}$ are equivalent. Assume that (x_n) is 1-orthogonal with respect to a base (p_k) in $\mathcal{P}(\mathbb{K}^{\mathbb{N}})$. Let $k \in \mathbb{N}$. Since the space $\mathbb{K}^{\mathbb{N}}$ is of finite type, $\dim(\mathbb{K}^{\mathbb{N}}/\ker p_k) < \infty$. It is easy to see that $\ker p_k$ is the closed linear span of $\{x_n : p_k(x_n) = 0\}$. Hence the set $\{x_n : p_k(x_n) > 0\}$ is finite for any $k \in \mathbb{N}$. This follows that for any $(\beta_n) \subset \mathbb{K}$, the sequence $(\beta_n x_n)$ is convergent to 0 in $\mathbb{K}^{\mathbb{N}}$. Similarly, for any $(\beta_n) \subset \mathbb{K}$, the sequence $(\beta_n y_n)$ is convergent to 0 in $\mathbb{K}^{\mathbb{N}}$. By Remark 7, the bases $(x_n), (y_n)$ are equivalent.

(B). Any two orthogonal bases $(x_n), (y_n)$ in c_0 are semi equivalent. Indeed, set $||(\beta_n)|| = \max_n |\beta_n|$ for $(\beta_n) \in c_0$. Let $\beta \in \mathbb{K}$ with $|\beta| > 1$. Then there exists a sequence $(\beta_n) \subset (\mathbb{K} \setminus \{0\})$ such that $||y_n|| \le ||\beta_n x_n|| < |\beta| ||y_n||$ for any $n \in \mathbb{N}$. Using Remark 7, we infer that the bases $(\beta_n x_n)$ and (y_n) are equivalent. Hence (x_n) and (y_n) are semi equivalent.

(C). Now, we prove that any two orthogonal bases (x_n) and (y_n) in $c_0 \times \mathbb{K}^{\mathbb{N}}$ are quasi-equivalent. Set $r_k((\alpha_n), (\beta_n)) = k \max\{\max_n |\alpha_n|, \max_{1 \le n \le k} |\beta_n|\}$ for all $k \in \mathbb{N}$ and $((\alpha_n), (\beta_n)) \in c_0 \times \mathbb{K}^{\mathbb{N}}$. Of course, (r_k) is a base in $\mathcal{P}(c_0 \times \mathbb{K}^{\mathbb{N}})$.

Assume that (x_n) is 1-orthogonal with respect to a base (p_k) in $\mathcal{P}(c_0 \times \mathbb{K}^{\mathbb{N}})$. Clearly, there exist $m, k \in \mathbb{N}$ with $r_1 \le p_m \le r_k$. Since $\ker r_k \subset \ker p_m \subset \ker r_1 \subset c_0 \times \mathbb{K}^{\mathbb{N}}$, then $(c_0 \times \mathbb{K}^{\mathbb{N}}/\ker p_m)$ is an infinite-dimensional quotient space of the Banach space $(c_0 \times \mathbb{K}^{\mathbb{N}}/\ker r_k)$ of countable type. Thus $(c_0 \times \mathbb{K}^{\mathbb{N}}/\ker p_m)$ is an infinite-dimensional Banach space of countable type, so it is isomorphic to c_0. Put $M = \{n \in \mathbb{N} : p_m(x_n) > 0\}$. Denote by X_0 and X_1 the closed linear spans of $\{x_n : n \in M\}$ and $\{x_n : n \in (\mathbb{N} \setminus M)\}$, respectively. Since $c_0 \times \mathbb{K}^{\mathbb{N}}$ is isomorphic to $X_0 \times X_1$ and $X_1 = \ker p_m$, then X_0 is isomorphic to c_0 and $\dim X_1 = \infty$. For any $s \ge m$ there exists $t \in \mathbb{N}$ with $r_1 \le p_m \le p_s \le r_t$; then $\dim(\ker p_m/\ker p_s) \le$

$\dim(\ker r_1/\ker r_t) < \infty$. Thus X_1 is an infinite-dimensional Fréchet space of finite type, so it is isomorphic to $\mathbb{K}^{\mathbb{N}}$.

Similarly, there exists $L \subset \mathbb{N}$ such that the closed linear span Y_0 of $\{y_n : n \in L\}$ is isomorphic to c_0 and the closed linear span Y_1 of $\{y_n : n \in (\mathbb{N} \setminus L)\}$ is isomorphic to $\mathbb{K}^{\mathbb{N}}$.

Let σ be a permutation of $\mathbb{N}$ with $\sigma(L) = M$. Since any two orthogonal bases in c_0 are semi equivalent, there is a sequence $(\alpha_n)_{n\in L} \subset (\mathbb{K} \setminus \{0\})$ such that the orthogonal basis $(y_n)_{n\in L}$ in Y_0 is equivalent to the orthogonal basis $(\alpha_n x_{\sigma(n)})_{n\in L}$ in X_0. Let α_n be equal to the unit of $\mathbb{K}$ for all $n \in (\mathbb{N} \setminus L)$. Then the orthogonal basis $(y_n)_{n\in(\mathbb{N}\setminus L)}$ in Y_1 is equivalent to the orthogonal basis $(\alpha_n x_{\sigma(n)})_{n\in(\mathbb{N}\setminus L)}$ in X_1. Using Remark 7 we obtain that the orthogonal bases $(y_n), (\alpha_n x_{\sigma(n)})$ in $c_0 \times \mathbb{K}^{\mathbb{N}}$ are equivalent, so the bases (x_n) and (y_n) are quasi-equivalent.

(D). Finally, we prove that any two orthogonal bases $(x_n), (y_n)$ in $c_0^{\mathbb{N}}$ are quasi-equivalent. Put $||\beta|| = \max_i |\beta_i|$ for $\beta = (\beta_i) \in c_0$, and $r_k((\alpha_n)) = k \max_{1\le n\le k} ||\alpha_n||$ for all $k \in \mathbb{N}$ and $(\alpha_n) \in c_0^{\mathbb{N}}$. Clearly, (r_k) is a base in $\mathcal{P}(c_0^{\mathbb{N}})$.

Assume that (x_n) is 1-orthogonal with respect to a base (p_k) in $\mathcal{P}(c_0^{\mathbb{N}})$. Denote by p_0 the zero seminorm on $c_0^{\mathbb{N}}$. For any $m \in \mathbb{N}$ there exist $s, k \in \mathbb{N}$ such that $p_{m-1} \le r_s \le r_{s+1} \le p_k$; then $\dim(\ker p_{m-1}/\ker p_k) \ge \dim(\ker r_s/\ker r_{s+1}) = \infty$. Thus without loss of generality we can assume that $\dim(\ker p_{m-1}/\ker p_m) = \infty$ for any $m \in \mathbb{N}$.

Let $m \in \mathbb{N}$ and $t \in \mathbb{N}$ with $p_m \le r_t$. Since $\ker r_t \subset \ker p_m \subset \ker p_{m-1} \subset c_0^{\mathbb{N}}$, then $(\ker p_{m-1}/\ker p_m)$ is a quotient space of $(\ker p_{m-1}/\ker r_t)$ and $(\ker p_{m-1}/\ker r_t)$ is a closed subspace of the Banach space $(c_0^{\mathbb{N}}/\ker r_t)$ of countable type. Thus $(\ker p_{m-1}/\ker p_m)$ is an infinite-dimensional Banach space of countable type, so it is isomorphic to c_0.

Similarly, there exists a base (q_k) in $\mathcal{P}(c_0^{\mathbb{N}})$ such that (y_n) is 1-orthogonal with respect to (q_k) and the quotient space $(\ker q_{m-1}/\ker q_m)$ is isomorphic to c_0 for any $m \in \mathbb{N}$ (we set $q_0 = p_0$).

Let $k \in \mathbb{N}$. Put $N_k = \{n \in \mathbb{N} : x_n \in (\ker p_{k-1} \setminus \ker p_k)\}$ and $M_k = \{n \in \mathbb{N} : y_n \in (\ker q_{k-1} \setminus \ker q_k)\}$. Denote by X_k and Y_k the closed linear span of $\{x_n : n \in N_k\}$ and $\{y_n : n \in M_k\}$, respectively. Clearly, $\ker p_{k-1}$ is isomorphic to $X_k \times \ker p_k$ and $\ker q_{k-1}$ is isomorphic to $Y_k \times \ker q_k$. Hence X_k and Y_k are isomorphic to c_0. Of course, $\bigcup_{k=1}^{\infty} N_k = \bigcup_{k=1}^{\infty} M_k = \mathbb{N}, N_i \cap N_j = \emptyset = M_i \cap M_j$ for all $i, j \in \mathbb{N}$ with $i \neq j$, and the sets N_k, M_k are infinite for any $k \in \mathbb{N}$. Thus there exists a permutation σ of $\mathbb{N}$ such that $\sigma(M_k) = N_k$ for any $k \in \mathbb{N}$. Since any two orthogonal bases in c_0 are semi equivalent, then there exists a sequence $(\alpha_n) \subset (\mathbb{K} \setminus \{0\})$ such that the orthogonal basis $(\alpha_n x_{\sigma(n)})_{n\in M_k}$ in X_k is equivalent to the orthogonal basis $(y_n)_{n\in M_k}$ in Y_k for any $k \in \mathbb{N}$.

We shall prove that the orthogonal bases $(y_n), (\alpha_n x_{\sigma(n)})$ in $c_0^{\mathbb{N}}$ are equivalent. Let $(\beta_n) \subset \mathbb{K}$. Assume that $\beta_n y_n \to 0$ in $c_0^{\mathbb{N}}$. Then $\lim_{n\in M_k} \beta_n y_n = 0$ in $c_0^{\mathbb{N}}$ for any $k \in \mathbb{N}$. By Remark 7, $\lim_{n\in M_k} \beta_n \alpha_n x_{\sigma(n)} = 0$ in $c_0^{\mathbb{N}}$ for any $k \in \mathbb{N}$. We show that $\beta_n \alpha_n x_{\sigma(n)} \to 0$ in $c_0^{\mathbb{N}}$. Suppose, by contradiction, that there exists a neighborhood U of 0 in $c_0^{\mathbb{N}}$ and an increasing sequence $(s_n) \subset \mathbb{N}$ such that $\beta_{s_n} \alpha_{s_n} x_{\sigma(s_n)} \in (c_0^{\mathbb{N}} \setminus U)$. Then for any $k \in \mathbb{N}$ the set $\{n \in \mathbb{N} : \sigma(s_n) \in N_k\}$ is finite. Thus for every $k \in \mathbb{N}$ there is $n_k \in \mathbb{N}$ with $\{\sigma(s_n) : n > n_k\} \subset \bigcup_{i=k+1}^{\infty} N_i$. Hence $p_k(\beta_{s_n} \alpha_{s_n} x_{\sigma(s_n)}) = 0$ for all $k, n \in \mathbb{N}$ with $n > n_k$. It means that $\beta_{s_n} \alpha_{s_n} x_{\sigma(s_n)} \to_n 0$ in $c_0^{\mathbb{N}}$, a contradiction. Thus we have proved that $\beta_n \alpha_n x_{\sigma(n)} \to 0$ in $c_0^{\mathbb{N}}$. Similarly, assuming that $\beta_n \alpha_n x_{\sigma(n)} \to 0$

in $c_0^{\mathbb{N}}$, we get $\beta_n y_n \to 0$ in $c_0^{\mathbb{N}}$. By Remark 7, the orthogonal bases $(\alpha_n x_{\sigma(n)})$, (y_n) are equivalent; so the bases (x_n) and (y_n) are quasi-equivalent. ■

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