Polynomial decay for coupled Schrödinger equations with variable coefficients and damped by one Dirichlet boundary feedback

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Abstract

The main purpose of this paper is to study, under a suitable geometric conditions, the indirect boundary stabilization for coupled Schrödinger equations with variable coefficients and one Dirichlet boundary feedback. The polynomial energy decay rate for smooth solutions is obtained by the combination of the Riemannian geometry method in $[Ya1]$ and the ideas of I. Lasiecka and R. Triggiani in $[LT3]$.

1 Introduction

Let Ω be a bounded open domain of class C^2 in $\mathbb{R}^n (n \in \mathbb{N}^*)$ and let $\{\Gamma_0, \Gamma_1\}$ be a partition of the boundary Γ verifying $\Gamma_1 \neq \varnothing$. Denote by $v = (v_1, ..., v_n)$ the outward unit normal vector to Γ. Let $T > 0$ and put $Q = \Omega \times]0, T[$, $\Sigma_l = \Gamma_l \times]0, T[$ $(l = 0, 1)$.

$$\mathcal{A}y = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial y}{\partial x_j} \right)$$

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is a second order differential operator with real coefficients $a_{ij} = a_{ji}$ of class $C^\infty(\overline{\Omega})$ and there exists $a_0 > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\zeta_i\zeta_j \geq a_0 \sum_{i=1}^{n} \zeta_i^2,$$

for all $x \in \Omega$, $\zeta = (\zeta_1, \zeta_2, ..., \zeta_n)^\tau \in \mathbb{R}^n$.

Let $\frac{\partial y}{\partial v_{\mathcal{A}}} = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial y}{\partial x_j} v_i$ be the co normal derivative with respect to $\mathcal{A}$.

We are concerned with the coupled *complex valued* Schrödinger equations with *variable coefficients* and forcing term u in the *Dirichlet boundary condition*. This control function u is acting on one end only (no damping acting on z on Σ_1):

$$\begin{cases} \mathrm{i}y_t + \mathcal{A}y + az = 0 \text{ in } Q, \\ \mathrm{i}z_t + \mathcal{A}z + ay = 0 \text{ in } Q, \\ y = 0 \text{ on } \Sigma_0,\ y = u \text{ on } \Sigma_1 \text{ and } z = 0 \text{ on } \Sigma, \\ y(0) = y_0 \text{ and } z(0) = z_0 \text{ in } \Omega. \end{cases} \tag{1}$$

where a is a positive constant (the coupling coefficient).

The uniform stabilization and controllability of one Schrödinger equation with *constant coefficients* have been studied by many authors (see for examples [$LT3$, MZ, Mac]). Recently, P. F. Yao has introduced the Riemann geometric method to study the problem of exact controllability of *real valued* wave, Euler-Bernoulli and Shalow Shells equations with *variable coefficients* (see [$Ya1$, $Ya2$, $Ya3$]). In [NP], the authors have used this approach to establish observability estimates for *vector-valued* Maxwell's system with *variable coefficients.* Using this approach, several papers were devoted to the stability of *variable* systems (see [$FF2$, GY]). More recently, another question has been considered by F. Alabau, A. Beyrath, P. Cannarsa and V. Komornick [Ala, ACK, Bey]: the problem of indirect boundary and internal stabilization of coupled *real valued* hyperbolic systems. They have proved that the feedback of the first equation is sufficient to stabilize polynomially the total system.

In this paper, we prove that we can use a successful combination of two key ingredients to obtain the polynomial energy decay rate for smooth solutions of system (1), with a suitable choice of the Dirichlet control function u:

(1) The Riemann geometric approach, recently, developed by P. F. Yao in order to reduce the original variable coefficients principal part problem to a problem on an appropriate Riemann manifold, where the principal part is the Laplacian. Using this geometry on $\mathbb{R}^n$, we construct, in our paper, a main geometry on $\mathbb{C}^n$ (see section 2 below).

(2) The ideas of I. Lasiecka and R. Triggiani [$LT3$] used to obtain the *direct* stabilization of *one* Schrödinger equation with *Dirichlet boundary feedback*.

2 Metric on $\mathbb{C}^n$.

Let $\langle .,. \rangle_g$ and $\|.\|_g$ be the Riemannian inner product and norm on the tangent space $\mathbb{R}^n_x = \mathbb{R}^n$ generated by the principal part $\mathcal{A}$ (see $[Ya1]$).

We can construct an inner product on $\mathbb{C}^n_x = \mathbb{C}^n$ (we take the same symbol) by for all $Z_1, Z_2 \in \mathbb{C}^n$

$$\begin{aligned}\langle Z_1, Z_2 \rangle_g &= \langle \operatorname{Re} Z_1, \operatorname{Re} Z_2 \rangle_g + \langle \operatorname{Im} Z_1, \operatorname{Im} Z_2 \rangle_g \\ &\quad + \mathrm{i}\left(\langle \operatorname{Im} Z_1, \operatorname{Re} Z_2 \rangle_g - \langle \operatorname{Re} Z_1, \operatorname{Im} Z_2 \rangle_g \right),\end{aligned}$$

so the norm is

$$\|Z\|_g^2 = \langle Z, Z \rangle_g = \|\operatorname{Re} Z\|_g^2 + \|\operatorname{Im} Z\|_g^2,$$

for all $Z \in \mathbb{C}^n$.

Let f be a *complex valued* function and h be a vector field on $\mathbb{R}^n$. We put

$$h(f) := h(\operatorname{Re} f) + \mathrm{i} h(\operatorname{Im} f)$$

and

$$\nabla_g f := \nabla_g \operatorname{Re} f + \mathrm{i} \nabla_g \operatorname{Im} f,$$

where the gradient of a *real valued* function k in the Riemannian metric is defined, via Riesz representation theorem, by

$$X(k) = \langle \nabla_g k, X \rangle_g,$$

where X is any vector field on the manifold $\mathbb{R}^n$ (see $[Ya1]$).

We shall need the following formulas:

Lemma 1. *1/ Let f_1, f_2 be a complex valued functions in $H^2(\Omega)$. Then*

$$\int_\Omega (\mathcal{A} f_1) \overline{f_2} = \int_\Omega \langle \nabla_g f_1, \nabla_g f_2 \rangle_g - \int_\Gamma \frac{\partial f_1}{\partial v_{\mathcal{A}}} \overline{f_2}.$$

2/ Let f be a complex valued function in $C^1(\overline{\Omega})$ and h be a vector field on $\mathbb{R}^n$. Then

$$\begin{aligned}&\operatorname{Re} \langle \nabla_g f, \nabla_g (h(f)) \rangle_g = \\ &Dh(\nabla_g \operatorname{Re} f, \nabla_g \operatorname{Re} f) + Dh(\nabla_g \operatorname{Im} f, \nabla_g \operatorname{Im} f) \\ &+ \frac{1}{2} h\left(\|\nabla_g f\|_g^2 \right),\end{aligned} \tag{2}$$

where Dh is the covariant differential of h.

Proof. 1/

$$\begin{aligned}\int_\Omega (\mathcal{A} f_1) \overline{f_2} &= \int_\Omega \mathcal{A}(\operatorname{Re} f_1) \operatorname{Re} f_2 + \int_\Omega \mathcal{A}(\operatorname{Im} f_1) \operatorname{Im} f_2 \\ &\quad + \mathrm{i}\left(\int_\Omega \mathcal{A}(\operatorname{Im} f_1) \operatorname{Re} f_2 - \int_\Omega \mathcal{A}(\operatorname{Re} f_1) \operatorname{Im} f_2 \right),\end{aligned}$$

using Green's formula in $[Ya1]$ concerned with the *real valued* functions, we find

$$\begin{aligned}
&\int_\Omega \mathcal{A}f_1\overline{f_2} \\
=\ & \int_\Omega \left(\left\langle \nabla_g \operatorname{Re} f_1, \nabla_g \operatorname{Re} f_2\right\rangle_g + \left\langle \nabla_g \operatorname{Im} f_1, \nabla_g \operatorname{Im} f_2\right\rangle_g\right) \\
&+\mathrm{i}\int_\Omega \left(\left\langle \nabla_g \operatorname{Im} f_1, \nabla_g \operatorname{Re} f_2\right\rangle_g - \left\langle \nabla_g \operatorname{Re} f_1, \nabla_g \operatorname{Im} f_2\right\rangle_g\right) \\
&-\int_\Gamma \left(\frac{\partial \operatorname{Re} f_1}{\partial v_{\mathcal{A}}} \operatorname{Re} f_2 + \frac{\partial \operatorname{Im} f_1}{\partial v_{\mathcal{A}}} \operatorname{Im} f_2\right) \\
&-\mathrm{i}\int_\Gamma \left(\frac{\partial \operatorname{Im} f_1}{\partial v_{\mathcal{A}}} \operatorname{Re} f_2 - \frac{\partial \operatorname{Re} f_1}{\partial v_{\mathcal{A}}} \operatorname{Im} f_2\right),
\end{aligned}$$

so

$$\begin{aligned}
&\int_\Omega \mathcal{A}f_1\overline{f_2} \\
=\ & \int_\Omega \left(\left\langle \operatorname{Re}\left(\nabla_g f_1\right), \operatorname{Re}\left(\nabla_g f_2\right)\right\rangle_g + \left\langle \operatorname{Im}\left(\nabla_g f_1\right), \operatorname{Im}\left(\nabla_g f_2\right)\right\rangle_g\right) \\
&+\mathrm{i}\int_\Omega \left(\left\langle \operatorname{Im}\left(\nabla_g f_1\right), \operatorname{Re}\left(\nabla_g f_2\right)\right\rangle_g - \left\langle \operatorname{Re}\left(\nabla_g f_1\right), \operatorname{Im}\left(\nabla_g f_2\right)\right\rangle_g\right) \\
&-\int_\Gamma \left(\operatorname{Re}\left(\frac{\partial f_1}{\partial v_{\mathcal{A}}}\right) \operatorname{Re} f_2 - \operatorname{Im}\left(\frac{\partial f_1}{\partial v_{\mathcal{A}}}\right) \operatorname{Im} \overline{f_2}\right) \\
&-\mathrm{i}\int_\Gamma \left(\operatorname{Im}\left(\frac{\partial f_1}{\partial v_{\mathcal{A}}}\right) \operatorname{Re} f_2 + \operatorname{Re}\left(\frac{\partial f_1}{\partial v_{\mathcal{A}}}\right) \operatorname{Im} \overline{f_2}\right),
\end{aligned}$$

which implies the desired formula.

2/ It is sufficient to see that

$$\begin{aligned}
\operatorname{Re}\left\langle \nabla_g f, \nabla_g \left(h\left(f\right)\right)\right\rangle_g &= \left\langle \nabla_g \operatorname{Re} f, \nabla_g \left(h\left(\operatorname{Re} f\right)\right)\right\rangle_g \\
&\quad + \left\langle \nabla_g \operatorname{Im} f, \nabla_g \left(h\left(\operatorname{Im} f\right)\right)\right\rangle_g,
\end{aligned}$$

so, (2) is obtained by lemma 2.1 in $[Ya1]$. ■

3 Geometric assumptions and notations.

Geometric assumptions.

Assume that there exists a real vector field $h \in \left[C^1\left(\overline{\Omega}\right)\right]^n$ on Riemannian manifold $\mathbb{R}^n$, a constant $m_0 > 0$ such that

$$Dh\left(X, X\right) \geq m_0 \left\|X\right\|_g^2, \text{ for all } X \in \mathbb{R}_x^n \tag{3}$$

and

$$2m_0 > \alpha C_h, \tag{4}$$

where $C_h = \sup_{x \in \Omega} \left\| \nabla_g \left(div_0 h \right) \right\|_g$ and α is the positive constant verifying $\int_\Omega |f|^2 \leq \alpha^2 \int_\Omega \left\| \nabla_g f \right\|_g^2$ for all $f \in H_0^1(\Omega)$.

We assume that

$$\Gamma_0 = \{ x \in \Gamma : h.v \leq 0 \}.$$

Let us give an example of vector field h satisfying (3) and (4).

Example. Noting that, in $[FF1]$ the authors have proved that the geometric condition (3), derived by P. F. Yao in term of the Riemannian geometry method, is equivalent with the following analytical condition given by A. Wyler $[Wyl]$ for the boundary stabilization of wave equations with variable coefficients:

$$\begin{aligned} &\left(p_{ij}\right) \text{ is uniformly positive definite matrix in } \Omega, \text{ where} \\ p_{ij} &= p_{ji} = \sum_{l=1}^{n} a_{il} \frac{\partial h_j}{\partial x_l} + \sum_{l=1}^{n} a_{jl} \frac{\partial h_i}{\partial x_l} - \nabla_0 a_{ij}.h. \end{aligned}$$

If $h(x) = \sum_{i=1}^{n} h_i(x) \frac{\partial}{\partial x_i}$ is a vector field on $\mathbb{R}^n$ such that $div_0 h$ is constant, thus, (4) is verified. If $\left(a_{ij}\right)$ is a matrix defined by

$$a_{ij}\left(x_1, x_2, ..., x_n\right) = \begin{cases} 0 : i \neq j, \\ f_i\left(x_i\right) : i = j, \end{cases}$$

where, for all $i = 1, ..., n$, $f_i : \mathbb{R} \to \mathbb{R}$ is a function of class C^∞ satisfying the condition

$$\min f_i > 0$$

and

$$\begin{aligned} &\min \left(2 f_i \frac{\partial h_i}{\partial xi} - \frac{\partial f_i}{\partial xi} h_i \right) > 0, \ i = j, \\ &f_i \frac{\partial h_j}{\partial xi} + f_j \frac{\partial h_i}{\partial x_j} = 0, \qquad i \neq j, \end{aligned}$$

then $\left(p_{ij}\right)$ is an uniformly positive definite matrix in Ω.

As an example of such vector and matrix we can take $h_i = x_i - x_i^0$, $x_0 \in \mathbb{R}^n$ and $f_i(x_i) = \left(x_i - x_i^0\right)^2 + \beta$ where $\beta \in \mathbb{R}_+^*$.

Remark 2. *We note that assumption (4) has been used in* $[LT2]$ *to study the exact controllability of wave equation. It is needed to absorb the lower order term with respect to the energy in (13).*

Notations.

Let A be the positive self adjoint operator on $L^2(\Omega)$ defined by

$$Af = \mathcal{A}f \text{ and } D(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

The following space identification are know (with equivalent norms)

$$\begin{aligned} D\left(A^{\frac{1}{2}}\right) &= H_0^1(\Omega), \left(D\left(A^{\frac{1}{2}}\right)\right)' = H^{-1}(\Omega). \\ \|f\|_{D\left(A^{\frac{1}{2}}\right)} &= \left\| A^{\frac{1}{2}} f \right\|_{L^2(\Omega)}, \|f\|_{\left(D\left(A^{\frac{1}{2}}\right)\right)'} = \left\| A^{-\frac{1}{2}} f \right\|_{L^2(\Omega)}. \end{aligned}$$

Let us introduce the operator $D : L^2(\Gamma) \to L^2(\Omega)$ defined by

$$f = Dw \Longleftrightarrow (Af = 0, f_{/\Gamma_0} = 0, f_{/\Gamma_1} = w)$$

and his adjoint D^* by

$$(Dw, f)_{L^2(\Omega)} = (w, D^* f)_{L^2(\Gamma)},$$

$w \in L^2(\Gamma)$ and $f \in L^2(\Omega)$.

We have (see $[LLT]$), for all $f \in D(A)$

$$D^* A f = \begin{cases} 0 \text{ on } \Gamma_0, \\ -\frac{\partial f}{\partial v_{\mathcal{A}}} \text{ on } \Gamma_1. \end{cases}$$

In all this paper, C is a generic positive constant which do not depend on the initial data and it may change from line to line.

4 The closed loop system: Choice of the Dirichlet control function u.

Using the techniques of $[LT1, LT3]$, problem (1) can be written in abstract form as

$$\frac{d}{dt}\begin{pmatrix} y \\ z \end{pmatrix} = \mathrm{i}\begin{pmatrix} A & a \\ a & A \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} - \begin{pmatrix} \mathrm{i}ADu \\ 0 \end{pmatrix}. \tag{5}$$

If we take $u = F(y) = -\mathrm{i}D^* y$ then (5) is rewritten as

$$\frac{d}{dt}\begin{pmatrix} y \\ z \end{pmatrix} = A_F \begin{pmatrix} y \\ z \end{pmatrix}, \tag{6}$$

where

$$A_F = \mathrm{i}\begin{pmatrix} A + \mathrm{i}ADD^* & a \\ a & A \end{pmatrix},$$

with domain

$$D(A_F) = \left\{ \begin{array}{c} \begin{pmatrix} y \\ z \end{pmatrix} \in \left(D\left(A^{\frac{1}{2}}\right)\right)' \times \left(D\left(A^{\frac{1}{2}}\right)\right)' : \\ A_F \begin{pmatrix} y \\ z \end{pmatrix} \in \left(D\left(A^{\frac{1}{2}}\right)\right)' \times \left(D\left(A^{\frac{1}{2}}\right)\right)' \end{array} \right\}.$$

This choice of u makes the operator A_F dissipative on $\left(D\left(A^{\frac{1}{2}}\right)\right)' \times \left(D\left(A^{\frac{1}{2}}\right)\right)'$. Indeed,

$$\begin{aligned} & \operatorname{Re}\left(A_F \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix}\right)_{\left(D\left(A^{\frac{1}{2}}\right)\right)' \times \left(D\left(A^{\frac{1}{2}}\right)\right)'} \\ = & -\|D^* y\|^2_{L^2(\Gamma)} \leq 0. \end{aligned}$$

Remark 3. *The operator A_F with domain $D(A_F)$ is a maximal dissipative operator in $\left(D\left(A^{\frac{1}{2}}\right)\right)^{'} \times \left(D\left(A^{\frac{1}{2}}\right)\right)^{'}$. We shall here omit details of the proof and refer to similar situation for other dynamic in $[BT]$.*

Let $N \geq 1$. If we use theorem VII4 and theorem VII5 in $[Bre]$, we have that if $(y_0, z_0) \in D\left(A_F^N\right)$ the system (6) has a unique solution $(y,z) \in C^{N-j}([0,+\infty); D\left(A_F^j\right))$ for $j = 0, ..., N$.

Here $D\left(A_F^1\right) = D\left(A_F\right)$ and $D\left(A_F^N\right) = \left\{ \begin{array}{c} (u_1,u_2) \in D\left(A_F^{N-1}\right) : \\ A_F\left(u_1,u_2\right) \in D\left(A_F^{N-1}\right) \end{array} \right\}$ for $N \geq 2$.

5 Indirect boundary stabilization result.

We define the total energy E of (6) by

$$\begin{aligned} E(t) &= \frac{1}{2}\left\|(y,z)\right\|^2_{\left(D\left(A^{\frac{1}{2}}\right)\right)^{'}\times\left(D\left(A^{\frac{1}{2}}\right)\right)^{'}} \\ &= \frac{1}{2}\left\|A^{-\frac{1}{2}}y\right\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|A^{-\frac{1}{2}}z\right\|^2_{L^2(\Omega)}. \end{aligned}$$

By the dissipation of the operator A_F, we can see that E is a decreasing function

$$\frac{dE(t)}{dt} = -\left\|D^* y\right\|^2_{L^2(\Gamma)} \leq 0.$$

We have

Theorem 4. *Let $N \geq 1$. For any initial data $(y_0, z_0) \in D\left(A_F^N\right)$, the energy E of the solution of the closed loop dynamics (1), with the choice of $u = -\mathrm{i}D^* y$ inserted in the boundary condition, decays polynomially:*

$$E\left(y(t), z(t)\right) \leq \frac{C}{t^N} \sum_{p=0}^{p=N} E\left(y^{(p)}(0), z^{(p)}(0)\right),$$

for all $t > 0$.

Proof. **Step 1.** Change of variable.

Motivated by the techniques of $[LT3]$, we introduce a new variables p and q by setting

$$p = A^{-1}y \text{ and } q = A^{-1}z,$$

where $(y_0, z_0) \in D(A_F)$, then, by (6), we obtain the system

$$\left\{ \begin{array}{c} ip_t + \mathcal{A}p + G + aq = 0 \text{ in } Q, \\ iq_t + \mathcal{A}q + ap = 0 \text{ in } Q, \\ p = q = 0 \text{ on } \Sigma, \\ p(0) = p_0 \text{ and } q(0) = q_0 \text{ in } \Omega, \end{array} \right.$$

where $G = \mathrm{i}DD^{*}Ap$. On the other hand, we have $E(t) = E_1(t) + E_2(t)$, where

$$\begin{aligned} E_1(t) &= \frac{1}{2}\left\| A^{-\frac{1}{2}} y \right\|^2_{L^2(\Omega)} \\ &= \frac{1}{2}\left\| A^{\frac{1}{2}} p \right\|^2_{L^2(\Omega)} = \frac{1}{2}\int_{\Omega} \|\nabla_g p\|^2_g \end{aligned}$$

and

$$\begin{aligned} E_2(t) &= \frac{1}{2}\left\| A^{-\frac{1}{2}} z \right\|^2_{L^2(\Omega)} \\ &= \frac{1}{2}\left\| A^{\frac{1}{2}} q \right\|^2_{L^2(\Omega)} = \frac{1}{2}\int_{\Omega} \|\nabla_g q\|^2_g . \end{aligned}$$

We can see that

$$\begin{aligned} \int_{\Sigma_1} \left| \frac{\partial p}{\partial v_{\mathcal{A}}} \right|^2 &= \|D^{*}y\|^2_{L^2(\Sigma)} \\ &= -\int_0^T \frac{dE(t)}{dt} \le E(0) \end{aligned}$$

and

$$\begin{aligned} \int_{Q} |G|^2 &= \int_{Q} |DD^{*}Ap|^2 \\ &\le C\int_{\Sigma_1} \left| \frac{\partial p}{\partial v_{\mathcal{A}}} \right|^2 \le CE(0) . \end{aligned}$$

Step 2. In this step we shall estimate the term $\int_0^T E_1(t)$.

We have

$$\begin{aligned} 0 &= \operatorname{Re}\int_Q (\mathrm{i}p_t + \mathcal{A}p + G + aq)\,(2h(\overline{p}) + div_0 h\overline{p} + \overline{q}) \\ &\quad - \operatorname{Re}\int_Q (-i\overline{q}_t + \mathcal{A}\overline{q} + a\overline{p})\, p \\ &= \operatorname{Re}\mathrm{i}\int_Q (p_t\overline{q} + p\overline{q}_t) + \operatorname{Re}\int_Q (\mathcal{A}p\overline{q} - \mathcal{A}\overline{q}p) \\ &\quad + \operatorname{Re}\int_Q \mathcal{A}p\,(2h(\overline{p}) + div_0 h\overline{p}) \\ &\quad - \operatorname{Im}\int_Q p_t\,(2h(\overline{p}) + div_0 h\overline{p}) \\ &\quad + \operatorname{Re}\int_Q G\,(2h(\overline{p}) + div_0 h\overline{p} + \overline{q}) \\ &\quad + \operatorname{Re}\int_Q aq\,(2h(\overline{p}) + div_0 h\overline{p}) \\ &\quad - \int_Q a|p|^2 + \int_Q a|q|^2 . \end{aligned} \tag{7}$$

But

$$\operatorname{Re} i \int_Q (p_t\overline{q} + p\overline{q}_t) = \operatorname{Re} i \int_Q (p\overline{q})_t \tag{8}$$
$$= -\operatorname{Im} \int_\Omega p\overline{q}\big|_0^T$$

$$\operatorname{Re} \int_Q (\mathcal{A}p\overline{q} - \mathcal{A}\overline{q}p) = 0 \tag{9}$$

$$\begin{aligned}
& \operatorname{Re} \int_Q \mathcal{A}p\,(2h(\overline{p}) + div_0 h\overline{p}) \\
= \; & \operatorname{Re} \int_Q \langle \nabla_g p, \nabla_g (2h(p) + div_0 hp) \rangle_g \\
& -2\operatorname{Re} \int_\Sigma \frac{\partial \overline{p}}{\partial v_{\mathcal{A}}} h(p) \\
= \; & 2\int_Q Dh(\nabla_g \operatorname{Re} p, \nabla_g \operatorname{Re} p) \\
& +2\int_Q Dh(\nabla_g \operatorname{Im} p, \nabla_g \operatorname{Im} p) \\
& +\operatorname{Re} \int_Q \langle \nabla_g p, \nabla_g (div_0 h) \rangle_g \overline{p} \\
& -2\operatorname{Re} \int_\Sigma \frac{\partial \overline{p}}{\partial v_{\mathcal{A}}} h(p) + \int_\Sigma h.v \, \|\nabla_g p\|_g^2
\end{aligned}$$

Since, $\operatorname{Re} p = \operatorname{Im} p = 0$ on Γ, then we have $[Ya1]$

$$\begin{aligned}
h(\operatorname{Re} p) &= \frac{h.v}{\|v_{\mathcal{A}}(x)\|_g^2} \frac{\partial \operatorname{Re} p}{\partial v_{\mathcal{A}}}, \\
\|\nabla_g \operatorname{Re} p\|_g^2 &= \frac{1}{\|v_{\mathcal{A}}(x)\|_g^2} \left(\frac{\partial \operatorname{Re} p}{\partial v_{\mathcal{A}}} \right)^2
\end{aligned}$$

and

$$\begin{aligned}
h(\operatorname{Im} p) &= \frac{h.v}{\|v_{\mathcal{A}}(x)\|_g^2} \frac{\partial \operatorname{Im} p}{\partial v_{\mathcal{A}}}, \\
\|\nabla_g \operatorname{Im} p\|_g^2 &= \frac{1}{\|v_{\mathcal{A}}(x)\|_g^2} \left(\frac{\partial \operatorname{Im} p}{\partial v_{\mathcal{A}}} \right)^2.
\end{aligned}$$

So

$$h(p) = \frac{h.v}{\|v_{\mathcal{A}}(x)\|_g^2} \frac{\partial p}{\partial v_{\mathcal{A}}}$$

and

$$\|\nabla_g p\|_g^2 = \frac{1}{\|v_{\mathcal{A}}(x)\|_g^2} \left| \frac{\partial p}{\partial v_{\mathcal{A}}} \right|^2,$$

then

$$\begin{aligned}
&\operatorname{Re}\int_Q \mathcal{A}p\left(2h\left(\overline{p}\right)+div_0 h\overline{p}\right)\\
= &\ 2\int_Q Dh\left(\nabla_g \operatorname{Re} p, \nabla_g \operatorname{Re} p\right)\\
&+2\int_Q Dh\left(\nabla_g \operatorname{Im} p, \nabla_g \operatorname{Im} p\right)\\
&+\operatorname{Re}\int_Q \left\langle \nabla_g p, \nabla_g\left(div_0 h\right)\right\rangle_g \overline{p} \qquad (10)\\
&-\int_\Sigma \frac{h.v}{\left\|v_{\mathcal{A}}(x)\right\|_g^2}\left|\frac{\partial p}{\partial v_{\mathcal{A}}}\right|^2.
\end{aligned}$$

We have

$$\begin{aligned}
&\int_Q p_t h\left(\overline{p}\right)\\
= &\int_\Omega ph\left(\overline{p}\right)\Big|_0^T-\int_Q ph\left(\overline{p_t}\right)\\
= &\int_\Omega ph\left(\overline{p}\right)\Big|_0^T-\int_Q ph.\nabla_0\overline{p_t}\\
= &\int_\Omega ph\left(\overline{p}\right)\Big|_0^T+\int_Q \overline{p_t}div_0\left(hp\right)\\
= &\int_\Omega ph\left(\overline{p}\right)\Big|_0^T+\int_Q \overline{p_t}div_0 hp+\int_Q \overline{p_t}h\left(p\right).
\end{aligned}$$

Then

$$\operatorname{Im}\int_Q p_t\left(2h\left(\overline{p}\right)+div_0 h\overline{p}\right)=\operatorname{Im}\int_\Omega ph\left(\overline{p}\right)\Big|_0^T. \qquad (11)$$

Replacing (8)-(11) in (7), we find

$$\begin{aligned}
&2\int_Q Dh\left(\nabla_g \operatorname{Re} p, \nabla_g \operatorname{Re} p\right) \qquad (12)\\
&+2\int_Q Dh\left(\nabla_g \operatorname{Im} p, \nabla_g \operatorname{Im} p\right)\\
= &\ I_\Omega+I_\Sigma+I_Q,
\end{aligned}$$

where

$$I_\Omega=\operatorname{Im}\int_\Omega p\left(h\left(\overline{p}\right)+\overline{q}\right)\Big|_0^T\leq CE\left(0\right),$$

$$\begin{aligned}
I_\Sigma &= \int_\Sigma \frac{h.v}{\left\|v_{\mathcal{A}}(x)\right\|_g^2}\left|\frac{\partial p}{\partial v_{\mathcal{A}}}\right|^2\\
&= \int_{\Sigma_0} \frac{h.v}{\left\|v_{\mathcal{A}}(x)\right\|_g^2}\left|\frac{\partial p}{\partial v_{\mathcal{A}}}\right|^2\\
&\quad+\int_{\Sigma_1} \frac{h.v}{\left\|v_{\mathcal{A}}(x)\right\|_g^2}\left|\frac{\partial p}{\partial v_{\mathcal{A}}}\right|^2\\
&\leq CE\left(0\right),
\end{aligned}$$

$$\begin{aligned} I_Q &= -\operatorname{Re}\int_Q \left\langle \nabla_g p, \nabla_g \left(div_0 h\right)\right\rangle_g \overline{p} \\ &\quad -\operatorname{Re}\int_Q G\left(2h\left(\overline{p}\right) + div_0 h\overline{p} + \overline{q}\right) \\ &\quad -\operatorname{Re}\int_Q aq\left(2h\left(\overline{p}\right) + div_0 h\overline{p}\right) \\ &\quad +\int_Q a\left|p\right|^2 - \int_Q a\left|q\right|^2 . \end{aligned}$$

First, we can see that

$$\begin{aligned} &-\operatorname{Re}\int_Q \left\langle \nabla_g p, \nabla_g \left(div_0 h\right)\right\rangle_g \overline{p} \\ \leq &\int_Q \left\|\nabla_g p\right\|_g \left\|\nabla_g \left(div_0 h\right)\right\|_g \left|p\right| \\ \leq &\frac{C_h \alpha}{2}\int_Q \left\|\nabla_g p\right\|_g^2 + \frac{C_h}{2\alpha}\int_Q \left|p\right|^2 , \end{aligned}$$

then, for all $\eta > 0$,

$$\begin{aligned} I_Q \leq &\frac{C_h \alpha}{2}\int_Q \left\|\nabla_g p\right\|_g^2 + \frac{C_h}{2\alpha}\int_Q \left|p\right|^2 + CE\left(0\right) \\ &+\eta\left(C\int_0^T E_1\left(t\right) + \int_Q \left|q\right|^2\right) \\ &-\frac{a}{2}\int_Q \left|q\right|^2 + Ca\int_0^T E_1\left(t\right) . \end{aligned} \tag{13}$$

Replace the majorities of I_Ω, $I_{\Sigma_l}\left(l = 0, 1\right)$ and I_Q in (12), use (3) and (4), choose η and a sufficiently small, we find

$$\int_0^T E_1\left(t\right) \leq CE\left(0\right) . \tag{14}$$

Step 3. In this step we shall estimate the term $\int_0^T E\left(t\right)$.

We have

$$\begin{aligned} 0 &= \operatorname{Re}\int_Q \left(-i\overline{q}_t + \mathcal{A}\overline{q} + a\overline{p}\right) p \\ &\quad -\operatorname{Re}\int_Q \left(ip_t + \mathcal{A}p + G + aq\right)\overline{q} \\ &= \operatorname{Im}\int_\Omega p\overline{q}\big|_0^T + \int_Q a\left|p\right|^2 \\ &\quad -\operatorname{Re}\int_Q G\overline{q} - \int_Q a\left|q\right|^2 , \end{aligned}$$

so

$$\int_Q \left|q\right|^2 \leq CE\left(0\right) . \tag{15}$$

If we use this inequality with the derivatives, we obtain

$$\int_Q \left|q_t\right|^2 \leq CE\left(y_t\left(0\right), z_t\left(0\right)\right) . \tag{16}$$

On the other hand, we have

$$\begin{aligned} 0 &= \operatorname{Re} \int_Q (iq_t + \mathcal{A}q + ap)\, \overline{q} \\ &= -\operatorname{Im} \int_Q q_t \overline{q} + \int_Q \left\| \nabla_g q \right\|_g^2 \\ &\quad + \operatorname{Re} \int_Q ap\overline{q}, \end{aligned}$$

then

$$\int_Q \left\| \nabla_g q \right\|_g^2 = \operatorname{Im} \int_Q q_t \overline{q} - \operatorname{Re} \int_Q ap\overline{q}.$$

If we use (14), (15) and (16) we find

$$\int_0^T E_2(t) \leq C\left(E\left(y(0), z(0)\right) + E\left(y_t(0), z_t(0)\right)\right).$$

Finally, we have

$$\begin{aligned} \int_0^T E(t) &= \int_0^T E_1(t) + \int_0^T E_2(t) \\ &\leq C\left(E\left(y(0), z(0)\right) + E\left(y_t(0), z_t(0)\right)\right). \end{aligned}$$

Since E is a decreasing function we find

$$TE\left(y(T), z(T)\right) \leq C\left(E\left(y(0), z(0)\right) + E\left(y_t(0), z_t(0)\right)\right).$$

This imply the polynomial decay of the energy for $(y_0, z_0) \in D(A_F)$.

By induction on $N \geq 1$, we obtain

$$E\left(y(t), z(t)\right) \leq \frac{C}{t^N} \sum_{p=0}^{p=N} E\left(y^{(p)}(0), z^{(p)}(0)\right),$$

for all $t > 0$ and $(y_0, z_0) \in D\left(A_F^N\right)$. ∎

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