Existence results for degenerate quasilinear elliptic equations in weighted Sobolev spaces

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Abstract

In this paper we are interested in the existence of solutions for Dirichlet problem associated to the degenerate quasilinear elliptic equations

$$-\operatorname{div}\left[v(x)\,\mathcal{A}(x,u,\nabla u)\right] + \omega(x)\mathcal{A}_0(x,u(x)) = f_0 - \sum_{j=1}^{n} D_j f_j, \text{ on } \Omega$$

in the setting of the weighted Sobolev spaces $\mathrm{W}_0^{1,p}(\Omega,\omega,v)$.

1 Introduction

In this paper we prove the existence of (weak) solutions in the weighted Sobolev spaces $\mathrm{W}_0^{1,p}(\Omega,\omega,v)$ (see Definition 2.3) for the Dirichlet problem

$$(P)\begin{cases} Lu(x) &= f_0(x) - \sum_{j=1}^{n} D_j f_j(x), \text{ on } \Omega \\ u(x) &= 0, \text{ on } \partial\Omega \end{cases}$$

where L is the partial differential operator

$$Lu(x) = -\sum_{j=1}^{n} D_j\left[v(x)\mathcal{A}_j(x,u(x),\nabla u(x))\right] + \omega(x)\,\mathcal{A}_0(x,u(x)) \qquad (1.1)$$

Received by the editors August 2008.
Communicated by P. Godin.
2000 *Mathematics Subject Classification* : 37J70, 35J60.
Key words and phrases : degenerate quasilinear elliptic equations, weighted Sobolev spaces.

where $D_j = \partial/\partial x_j$, Ω is a bounded open set in $\mathbb{R}^n$, ω and v are weight functions (i.e., locally integrable non-negative functions on $\mathbb{R}^n$), and the functions $\mathcal{A}_j : \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ $(j = 1, ..., n)$ (with $\mathcal{A}(x,\eta,\xi) = (\mathcal{A}_1(x,\eta,\xi), ..., \mathcal{A}_n(x,\eta,\xi)))$ and $\mathcal{A}_0 : \Omega\times\mathbb{R}\to\mathbb{R}$ satisfy the following conditions:

(H1) $x \mapsto \mathcal{A}_j(x,\eta,\xi)$ is measurable in Ω for all $(\eta,\xi) \in \mathbb{R}\times\mathbb{R}^n$ $(j = 1, 2..., n)$
$(\eta,\xi) \mapsto \mathcal{A}_j(x,\eta,\xi)$ is continuous in $\mathbb{R}\times\mathbb{R}^n$ for almost all $x \in \Omega$, and

$x \mapsto \mathcal{A}_0(x,\eta)$ is measurable in Ω for all $\eta \in \mathbb{R}$
$\eta \mapsto \mathcal{A}_0(x,\eta)$ is continuous in $\mathbb{R}$ for almost all $x \in \Omega$.
(H2) There are $\lambda > 0$ and functions $h_0, h_1, h_2, \tilde{h}_1$ and $\tilde{h}_2$, with $h_0 \in L^{p'}(\Omega, v)$, $h_2 \in L^1(\Omega, v)$, $h_1 v/\omega \in L^{p'}(\Omega,\omega)$, $\tilde{h}_1 \in L^{p'}(\Omega,\omega)$ and $\tilde{h}_2 \in L^1(\Omega,\omega)$ such that

$$\mathcal{A}(x,\eta,\xi).\xi \geq \lambda|\xi|^p - h_0(x)|\xi| - h_1(x)|\eta| - h_2(x)$$

(where $\mathcal{A}(x,\eta,\xi).\xi$ denotes the usual inner product in $\mathbb{R}^n$) and

$$\mathcal{A}_0(x,\eta)\eta \geq \lambda|\eta|^p - \tilde{h}_1(x)|\eta| - \tilde{h}_2(x)$$

with $1 < p < \infty$, and we denote by p' the real number such that $1/p + 1/p' = 1$ (that is , p' is the conjugate exponent to p).
(H3) There are positive functions $K_1, K_2, h_3, h_4, \tilde{h}_3$ and $\tilde{h}_4$, with $h_3, \tilde{h}_3, h_4$ and $\tilde{h}_4 \in L^\infty(\Omega)$, $K_1 \in L^{p'}(\Omega, v)$ and $K_2 \in L^{p'}(\Omega,\omega)$ such that

$$|\mathcal{A}(x,\eta,\xi)| \leq K_1(x) + h_3(x)|\eta|^{\theta(p-1)} + h_4(x)|\xi|^{p-1},$$
$$|\mathcal{A}_0(x,\eta)| \leq K_2(x) + \tilde{h}_3(x)|\eta|^{p-1},$$

where θ as in Theorem 2.6.
(H4) For $(\eta,\xi), (\eta',\xi') \in \mathbb{R}\times\mathbb{R}^n$ and $x \in \Omega$, the function

$$t \mapsto v(x)\mathcal{A}(x,\eta' + t\eta, \xi' + t\xi).\xi + \omega(x)\mathcal{A}_0(x,\eta' + t\eta)\eta$$

is monotone increasing as a function of $t \in [0,1]$.

Remark 1.1 Note that the condition (H4) holds if

$$[\mathcal{A}(x,\eta,\xi) - \mathcal{A}(x,\eta',\xi')].(\xi - \xi') \geq 0 \text{ and } (\mathcal{A}_0(x,\eta) - \mathcal{A}_0(x,\eta'))(\eta - \eta') \geq 0$$

whenever $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$, η, $\eta' \in \mathbb{R}$, $\eta \neq \eta'$ (see Proposition 25.6 in [17]). ∎

By a *weight*, we shall mean a locally integrable function v on $\mathbb{R}^n$ such that $v(x) > 0$ for a.e. $x \in \mathbb{R}^n$. Every weight v gives rise to a measure on the measurable subsets on $\mathbb{R}^n$ through integration. This measure will also be denoted by v. Thus, $v(E) = \int_E v(x)\,dx$ for measurable sets $E \subset \mathbb{R}^n$.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [5], [6],[7] and [10]).

A class of weights, which is particularly well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [13]). These classes have found many useful applications in harmonic analysis (see [14]). Another reason for studying A_p-weights is the fact that powers of distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [11]). There are, in fact, many interesting examples of weights (see [10] for p-admissible weights).

Equations like (1.1) have been studied by many authors in the non-degenerate case (i.e. with $\omega(x) = v(x) \equiv 1$) (see e.g. [4] and the references therein).

The degenerate case with different conditions have been studied by many authors. In [2] Drábek, Kufner and Mustonen proved that under certain condition the Dirichlet problem associated with the equation $-\operatorname{div}(a(x,u,\nabla u) = h$, $h \in [W_0^{1,2}(\Omega,\omega)]^*$, has at least one solution u in $W_0^{1,p}(\Omega,\omega)$. See also [16].

The purpose in this paper, is to prove the same results for the degenerate nonlinear elliptic equations

$$-\operatorname{div}(v(x)\,\mathcal{A}(x,u(x),\nabla u(x)) + \omega(x)\,\mathcal{A}_0(x,u(x)) = f_0(x) - \sum_{j=1}^{n} D_j f_j(x).$$

When $\omega = v \equiv 1$ (the non weighted case) existence results for problem (P) have been shown in [1].

The main result of this article is given in the next theorem, which is proved in the section 3.

Theorem 1.2 *Assume that conditions (H1)-(H4) hold. Let ω and v be weights. If $v \le \omega$, $\omega \in A_p$, $v \in A_p$, $1 < p < \infty$, $f_0/\omega \in L^{p'}(\Omega,\omega)$ and $f_j/v \in L^{p'}(\Omega,v)$ $(j = 1,...,n)$, then problem (P) has a solution $u \in W_0^{1,p}(\Omega,\omega,v)$.*

The basic idea is to reduce the problem (P) to an operator equation $Au = T$ and apply the theorem below.

Theorem 1.3 *Let $A : X \to X^*$ be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space X. Then for each $T \in X^*$ the equation $Au = T$ has a solution $u \in X$.*
Proof. See Theorem 26.A in [17]. ∎

Remark 1.4 Let X be a Banach space and let $A : X \to X^*$ be an operator (where X^* denotes the dual space of X).

(i) A is called monotone iff

$$\langle Au - Av, u - v\rangle \ge 0$$

for all $u, v \in X$ (where $\langle f, u\rangle$ denotes the value of the linear functional $f \in X^*$ at point $u \in X$).

(ii) A is called coercive iff

$$\lim_{\|u\|\to\infty} \frac{\langle Au, u\rangle}{\|u\|} = +\infty.$$

(iii) A is said to be hemicontinuous iff the real function

$$t \mapsto \langle A(u_1 + tu_2), u_3\rangle$$

is continuous on $[0,1]$ for all $u_1, u_2, u_3 \in X$ (see [17] for more informations about monotone, coercive and hemicontinuous operators.)

2 Definitions and basic results

Let v be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < v(x) < \infty$ almost everywhere. We say that v belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that v is an A_p-weight, if there is a constant $C = C_{p,v}$ such that

$$\left(\frac{1}{|B|}\int_B v(x)\,dx\right)\left(\frac{1}{|B|}\int_B v^{1/(1-p)}(x)\,dx\right)^{p-1} \le C$$

for all balls $B \subset \mathbb{R}^n$, where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \le p$, then $A_q \subset A_p$ (see [9],[10] or [14] for more informations about A_p-weights). The weight v satisfies the doubling condition if $v(2B) \le C\,v(B)$, for all balls $B \subset \mathbb{R}^n$, where $v(B) = \int_B v(x)dx$ and $2B$ denotes the ball with the same center as B which is twice as large. If $v \in A_p$, then v is doubling (see Corollary 15.7 in [10]).

As an example of A_p-weight, the function $v(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is in A_p if and only if $-n < \alpha < n(p-1)$ (see Corollary 4.4, Chapter IX in [14]).

If $v \in A_p$, then

$$\left(\frac{|E|}{|B|}\right)^p \le C\,\frac{v(E)}{v(B)}$$

whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B (see 15.5 *strong doubling property* in [10]). Therefore, if $v(E) = 0$ then $|E| = 0$.

Definition 2.1 Let v be a weight, and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. For $1 < p < \infty$, we define $L^p(\Omega, v)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega,v)} = \left(\int_\Omega |f(x)|^p v(x)\,dx\right)^{1/p} < \infty.$$

Remark 2.2 If $v \in A_p$, $1 < p < \infty$, then since $v^{-1/(p-1)}$ is locally integrable, we have

$$L^p(\Omega, v) \subset L^1_{\mathrm{loc}}(\Omega)$$

for every open set Ω (see Remark 1.2.4 in [15]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega, v)$. We also have that the dual space of $L^p(\Omega, v)$ is the space $[L^p(\Omega, v)]^* = L^{p'}(\Omega, v^{1-p'})$. ■

Definition 2.3 Let $\Omega \subset \mathbb{R}^n$ be bounded open set, and let ω and v be A_p-weights, $1 < p < \infty$. We define the weighted Sobolev space $W^{1,p}(\Omega, \omega, v)$ as the set of functions $u \in L^p(\Omega, \omega)$ with weak derivatives $D_j u \in L^p(\Omega, v)$ $(j = 1, .., n)$. The norm of u in $W^{1,p}(\Omega, \omega, v)$ is defined by

$$\|u\|_{W^{1,p}(\Omega,\omega,v)} = \left(\int_\Omega |u(x)|^p \omega(x)\, dx + \sum_{j=1}^n \int_\Omega |D_j u(x)|^p v(x)\, dx \right)^{1/p}. \tag{2.1}$$

The space $W_0^{1,p}(\Omega, \omega, v)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p}(\Omega,\omega,v)}$.

Equipped by this norm, $W^{1,p}(\Omega, \omega, v)$ and $W_0^{1,p}(\Omega, \omega, v)$ are reflevixe Banach spaces (see [12] for more informations about the spaces $W^{1,p}(\Omega, \omega, v)$). The dual space of $W_0^{1,p}(\Omega, \omega, v)$ is the space $[W_0^{1,p}(\Omega, \omega, v)]^* = \{T = f_0 - \operatorname{div} f \;:\; f = (f_1, ..., f_n),\ f_0/\omega \in L^{p'}(\Omega, \omega) \text{ and } f_j/v \in L^{p'}(\Omega, v),\ j = 1, ..., n\}$.

Remark 2.4 (i) If $v \in A_p$, $1 < p < \infty$, then $C^\infty(\Omega)$ is dense in $W^{1,p}(\Omega, v) = W^{1,p}(\Omega, v, v)$ (see Corollary 2.1.6 in [15]).
(ii) If $v \leq \omega$ then $W_0^{1,p}(\Omega, \omega) \subset W_0^{1,p}(\Omega, \omega, v) \subset W_0^{1,p}(\Omega, v)$. ■

For a general theory of weighted Sobolev spaces $W^{1,p}(\Omega, v)$ with $v \in A_p$ see [10], [13] and [15]. For informations about weighted Sobolev spaces with others weights see [18]. And for informations about weighted Sobolev spaces $W^{k,p}(\Omega, \omega)$, where $\omega = \{\omega_\alpha(x),\ |\alpha| \leq k\}$ describes the family of weight functions ω_α, see [3].

It is evident that the weights ω and v which satisfy $C_1 \leq v(x) \leq \omega(x) \leq C_2$ (C_1 and $C_2 > 0$ constants) for $x \in \Omega$, gives nothing new (the space $\mathrm{W}^{1,p}(\Omega, \omega, v)$ is then identical with the classical Sobolev space $\mathrm{W}^{1,p}(\Omega)$). Consequently, we shall interested above all in such weight functions ω and v which either vanish somewhere in $\bar{\Omega}$ or increase to infinity (or both).

In this paper we use the following two theorems.

Theorem 2.5 *Let $v \in A_p$, $1 < p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m \to u$ in $L^p(\Omega, v)$ then there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi \in L^p(\Omega, v)$ such that*
(i) $u_{m_k}(x) \to u(x)$, $m_k \to \infty$, v-a.e. on Ω;
(ii) $|u_{m_k}(x)| \leq \Phi(x)$, v-a.e. on Ω.
Proof. The proof of this theorem follows the lines of Theorem 2.8.1 in [8]. ■

Theorem 2.6 (The Weighted Sobolev Inequality)*Let Ω be an open bounded set in $\mathbb{R}^n$ $(n \geq 2)$ and $v \in A_p$ $(1 < p < \infty)$. There exist constants C_Ω and δ positive such that for all $u \in W_0^{1,p}(\Omega, v)$ and all θ satisfying $1 \leq \theta \leq n/(n-1) + \delta$,*

$$\|u\|_{L^{\theta p}(\Omega,v)} \leq C_\Omega \|\nabla u\|_{L^p(\Omega,v)}. \tag{2.2}$$

Proof. Its suffices to prove the inequalities for functions $u \in C_0^\infty(\Omega)$ (see Theorem 1.3 in [6]). To extend the estimates (2.2) to arbitrary $u \in W_0^{1,p}(\Omega, v)$, we let $\{u_m\}$ be a sequence of $C_0^\infty(\Omega)$ functions tending to u in $W_0^{1,p}(\Omega, v)$. Applying the estimates (2.2) to differences $u_{m_1} - u_{m_2}$, we see that $\{u_m\}$ will be a Cauchy sequence in $L^{\theta p}(\Omega, v)$. Consequently the limit functions u will lie in the desired spaces and satisfy (2.2). ∎

Definition 2.7 Let $1 < p < \infty$, ω and v A_p-weights, with $v \leq \omega$. We say that an element $u \in W_0^{1,p}(\Omega, \omega, v)$ is a (weak) solution of problem (P) if

$$\begin{aligned} & \sum_{j=1}^{n} \int_\Omega v(x)\, \mathcal{A}(x, u, \nabla u).\nabla \varphi\, dx + \int_\Omega \mathcal{A}_0(x, u)\, \varphi\, \omega\, dx \\ = & \int_\Omega f_0 \varphi\, dx + \sum_{j=1}^{n} \int_\Omega f_j D_j \varphi\, dx, \end{aligned}$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega, v)$.

3 Proof of theorem 1.2

Step 1. We define

$$\begin{aligned} & B : W_0^{1,p}(\Omega, \omega, v) \times W_0^{1,p}(\Omega, \omega, v) \to \mathbb{R} \\ & B(u, \varphi) = \sum_{j=1}^{n} \int_\Omega v\, \mathcal{A}_j(x, u, \nabla u) D_j \varphi\, dx + \int_\Omega \mathcal{A}_0(x, u)\, \varphi\, \omega\, dx \end{aligned}$$

and

$$\begin{aligned} & T : W_0^{1,p}(\Omega, \omega, v) \to \mathbb{R} \\ & T(\varphi) = \int_\Omega f_0\, \varphi\, dx + \sum_{j=1}^{n} \int_\Omega f_j D_j \varphi\, dx. \end{aligned}$$

Then $u \in W_0^{1,p}(\Omega, \omega, v)$ is a (weak) solution to problem (P) if

$$B(u, \varphi) = T(\varphi), \ \text{ for all } \varphi \in W_0^{1,p}(\Omega, \omega, v).$$

Using (H3), $f_0/\omega \in L^{p'}(\Omega, \omega)$, $f_j/v \in L^{p'}(\Omega, v)$ $(j = 1, 2..., n)$, Remark 2.4(ii) and Theorem 2.6, we have that

$$|T(\varphi)| \leq C \left(\|f_0/\omega\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^{n} \|f_j/v\|_{L^{p'}(\Omega,v)} \right) \|\varphi\|_{W_0^{1,p}(\Omega,\omega,v)}, \tag{3.1}$$

and

$$\begin{aligned}
|B(u,\varphi)| \leq & \int_\Omega |\mathcal{A}(x,u,\nabla u)||\nabla\varphi| v\,dx + \int_\Omega |\mathcal{A}_0(x,u)||\varphi|\,\omega\,dx \\
\leq & \int_\Omega (K_1 + h_3|u|^{\theta(p-1)} + h_4|\nabla u|^{p-1})|\nabla\varphi| v\,dx + \int_\Omega (K_2 + \tilde{h}_3|u|^{p-1})|\varphi|\,\omega\,dx \\
\leq & C\Bigg(\|K_1\|_{L^{p'}(\Omega,v)} + \|h_3\|_{L^\infty(\Omega)}\|u\|^{\theta(p-1)}_{W^{1,p}_0(\Omega,\omega,v)} + \|h_4\|_{L^\infty(\Omega)}\|u\|^{p-1}_{W^{1,p}_0(\Omega,\omega,v)} \\
& + \|K_2\|_{L^{p'}(\Omega,\omega)} + \|\tilde{h}_3\|_{L^\infty(\Omega)}\|u\|^{p-1}_{W^{1,p}_0(\Omega,\omega,v)}\Bigg)\|\varphi\|_{W^{1,p}_0(\Omega,\omega,v)}. \qquad (3.2)
\end{aligned}$$

Since $B(u,.)$ is linear for each $u \in W^{1,p}_0(\Omega,\omega,v)$, there exists a linear and continuous operator $A : W^{1,p}_0(\Omega,\omega,v) \to [W^{1,p}_0(\Omega,\omega,v)]^*$ such that

$$\langle Au,\varphi\rangle = B(u,\varphi), \ \text{ for all}\, u,\varphi \in W^{1,p}_0(\Omega,\omega,v)$$

and

$$\begin{aligned}
\|Au\|_* \ \leq \ & C\Bigg(\|K_1\|_{L^{p'}(\Omega,v)} + \|h_3\|_{L^\infty(\Omega)}\|u\|^{\theta(p-1)}_{W^{1,p}_0(\Omega,\omega,v)} + \|h_4\|_{L^\infty(\Omega)}\|u\|^{p-1}_{W^{1,p}_0(\Omega,\omega,v)} \\
+ \ & \|K_2\|_{L^{p'}(\Omega,\omega)} + \|\tilde{h}_3\|_{L^\infty(\Omega)}\|u\|^{p-1}_{W^{1,p}_0(\Omega,\omega,v)}\Bigg)
\end{aligned}$$

where $\|.\|_*$ denotes the norm in $[W^{1,p}_0(\Omega,\omega,v)]^*$. Consequently, problem (P) is equivalent to the operator equation $Au = T, \ u \in W^{1,p}_0(\Omega,\omega,v)$.

Step 2. The operator $A : W^{1,p}_0(\Omega,\omega,v) \to [W^{1,p}_0(\Omega,\omega,v)]^*$ is continuous. In fact, we define the operators

$$\begin{aligned}
& F_j : W^{1,p}_0(\Omega,\omega,v) \to L^{p'}(\Omega,v), \ (j = 1,...,n) \\
& (F_j u)(x) = \mathcal{A}_j(x,u(x),\nabla u(x))
\end{aligned}$$

and

$$\begin{aligned}
& G : W^{1,p}_0(\Omega,\omega,v) \to L^{p'}(\Omega,\omega) \\
& (Gu)(x) = \mathcal{A}_0(x,u(x)).
\end{aligned}$$

We have that the operators F_j and G are bounded and continuous. In fact,

(i) Using (H3), Remark 2.4(ii) and Theorem 2.6, we obtain

$$\begin{aligned}
\|F_j u\|^{p'}_{L^{p'}(\Omega,v)} &= \int_\Omega |F_j u(x)|^{p'} v\,dx = \int_\Omega |\mathcal{A}_j(x,u,\nabla u)|^{p'} v\,dx \\
&\leq \int_\Omega \Big(K_1 + h_3|u|^{\theta(p-1)} + h_4|\nabla u|^{p-1}\Big)^{p'} v\,dx \\
&\leq C \int_\Omega \Big[(K_1^{p'} + h_3^{p'}|u|^{\theta p} + h_4^{p'}|\nabla u|^p)v\Big]\,dx \\
&= C\Big[\int_\Omega K_1^{p'} v\,dx + \int_\Omega h_3^{p'}|u|^{\theta p} v\,dx + \int_\Omega h_4^{p'}|\nabla u|^p v\,dx\Big] \\
&\leq C\Big[\|K_1\|^{p'}_{L^{p'}(\Omega,v)} + \|h_3\|^{p'}_{L^\infty(\Omega)} C_\Omega^{\theta p}\|\nabla u\|^{\theta p}_{L^p(\Omega,v)} \\
&+ \|h_4\|^{p'}_{L^\infty(\Omega)}\|\nabla u\|^p_{L^p(\Omega,v)}\Big] \\
&\leq C\Big[\|K_1\|^{p'}_{L^{p'}(\Omega,v)} + \|h_3\|^{p'}_{L^\infty(\Omega)} C_\Omega^{\theta p}\|u\|^{\theta p}_{W_0^{1,p}(\Omega,\omega,v)} \\
&+ \|h_4\|^{p'}_{L^\infty(\Omega)}\|u\|^p_{W_0^{1,p}(\Omega,\omega,v)}\Big].
\end{aligned} \tag{3.3}$$

Analogously, we have

$$\begin{aligned}
\|Gu\|^{p'}_{L^{p'}(\Omega,\omega)} &= \int_\Omega |\mathcal{A}_0(x,u)|^{p'}\omega\,dx \\
&\leq \int_\Omega \Big(K_2 + \tilde{h}_3|u|^{p-1}\Big)^{p'}\omega\,dx \\
&\leq C\Big(\|K_2\|^{p'}_{L^{p'}(\Omega,\omega)} + \|\tilde{h}_3\|^{p'}_{L^\infty}\|u\|^p_{W_0^{1,p}(\Omega,\omega,v)}\Big).
\end{aligned}$$

(ii) Let $u_m \to u$ in $W_0^{1,p}(\Omega,\omega,v)$ as $m\to\infty$. We need to show that $F_j u_m \to F_j u$ in $L^{p'}(\Omega,v)$ and $Gu_m \to Gu$ in $L^{p'}(\Omega,\omega)$.
If $u_m \to u$ in $W_0^{1,p}(\Omega,\omega,v)$, then $u_m \to u$ in $L^p(\Omega,\omega)$ and $\nabla u_m \to \nabla u$ in $(L^p(\Omega,v))^n$. Using Theorem 2.5, there exist a subsequence $\{u_{m_k}\}$, functions $\Phi_1 \in L^p(\Omega,\omega)$ and $\Phi_2 \in L^p(\Omega,v)$ such that

$$\begin{aligned}
&u_{m_k}(x) \to u(x),\ \ \omega-\text{a.e. in } \Omega \\
&|u_{m_k}(x)| \leq \Phi_1(x)\ \ \omega-\text{a.e in } \Omega \\
&\nabla u_{m_k}(x) \to \nabla u(x),\ \ v-\text{a.e. in } \Omega \\
&|\nabla u_{m_k}(x)| \leq \Phi_2(x),\ \ v-\text{a.e. in } \Omega.
\end{aligned}$$

Hence, using (H3), we obtain

$$
\begin{aligned}
&\|F_j u_{m_k} - F_j u\|^{p'}_{L^{p'}(\Omega,v)} = \int_\Omega |F_j u_{m_k}(x) - F_j u(x)|^{p'} v\,dx \\
= & \int_\Omega |\mathcal{A}_j(x, u_{m_k}, \nabla u_{m_k}) - \mathcal{A}_j(x, u, \nabla u)|^{p'} v\,dx \\
\leq & \ C \int_\Omega \Big(|\mathcal{A}_j(x, u_{m_k}, \nabla u_{m_k})|^{p'} + |\mathcal{A}_j(x, u, \nabla u)|^{p'} \Big) v\,dx \\
\leq & \ C \Big[\int_\Omega \Big(K_1 + h_3 |u_{m_k}|^{\theta(p-1)} + h_4 |\nabla u_{m_k}|^{p-1} \Big)^{p'} v\,dx \\
+ & \int_\Omega \Big(K_1 + h_3 |u|^{\theta(p-1)} + h_4 |\nabla u|^{p-1} \Big)^{p'} v\,dx \Big] \\
\leq & \ C \Big[\int_\Omega K_1^{p'} v\,dx + \|h_3\|^{p'}_{L^\infty(\Omega)} C_\Omega^{\theta p} \Big(\|\nabla u_{m_k}\|^{\theta p}_{L^p(\Omega,v)} + \|\nabla u\|^{\theta p}_{L^p(\Omega,v)} \Big) \\
+ & \ \|h_4\|^{p'}_{L^\infty(\Omega)} \Big(\int_\Omega |\nabla u_{m_k}|^p v\,dx + \int_\Omega |\nabla u|^p v\,dx \Big) \Big] \\
\leq & \ C \Big[\int_\Omega K_1^{p'} v\,dx + \|h_3\|^{p'}_{L^\infty(\Omega)} C_\Omega^{\theta p} \|\Phi_2\|^{\theta p}_{L^p(\Omega,v)} + \|h_4\|^{p'}_{L^\infty(\Omega)} \int_\Omega \Phi_2^p v\,dx \Big] \\
= & \ C \int_\Omega \Big(K_1^{p'} + \frac{\|h_3\|^{p'}_{L^\infty(\Omega)} C_\Omega^{\theta p} \|\Phi_2\|^{\theta p}_{L^p(\Omega,v)}}{v(\Omega)} + \|h_4\|^{p'}_{L^\infty(\Omega)} \Phi_2^p \Big) v\,dx \\
= & \ C \int_\Omega g_1(x)\, v(x)\,dx
\end{aligned}
$$

where $0 < v(\Omega) = \int_\Omega v\,dx < \infty$, $g_1 \in L^1(\Omega, v)$ with

$$
g_1(x) = K_1^{p'}(x) + \|h_4\|^{p'}_{L^\infty(\Omega)} \Phi_2^p(x) + \frac{\|h_3\|^{p'}_{L^\infty(\Omega)} C_\Omega^{\theta p} \|\Phi_2\|^{\theta p}_{L^p(\Omega,v)}}{v(\Omega)}.
$$

Analogously, we have

$$
\begin{aligned}
&\|G u_{m_k} - G u\|^{p'}_{L^{p'}(\Omega,\omega)} = \int_\Omega |G u_{m_k}(x) - G u(x)|^{p'} \omega\,dx \\
\leq & \ C \int_\Omega (|\mathcal{A}_0(x, u_{m_k})|^{p'} + |\mathcal{A}_0(x, u)|^{p'})\, \omega dx \\
\leq & \ C \int_\Omega (K_2 + \tilde{h}_3 |u_{m_k}|^{p-1})^{p'} \omega\,dx + \int_\Omega (K_2 + \tilde{h}_3 |u|^{p-1})^{p'} \omega\,dx \\
\leq & \ C \Big[\int_\Omega K_2^{p'} \omega\,dx + \|\tilde{h}_3\|^{p'}_{L^\infty(\Omega)} \Big(\int_\Omega |u_{m_k}|^p \omega\,dx + \int_\Omega |u|^p \omega\,dx \Big) \Big] \\
\leq & \ C \Big(\int_\Omega (K_2^{p'} + \|\tilde{h}_3\|^{p'}_{L^\infty(\Omega)} \Phi_1^p)\, \omega\,dx \Big) \\
= & \ C \int_\Omega g_2\, \omega\,dx,
\end{aligned}
$$

where $g_2(x) = K_2^{p'}(x) + \|\tilde{h}_3\|^{p'}_{L^\infty(\Omega)} \Phi_1^p(x)$ and $g_2 \in L^1(\Omega, \omega)$.

By condition (H1), we have, as $k \to \infty$

$$\begin{aligned} F_j u_{m_k}(x) &= \mathcal{A}_j(x, u_{m_k}(x), \nabla u_{m_k}(x)) \to \mathcal{A}_j(x, u(x), \nabla u(x)) = F_j u(x), \\ G u_{m_k}(x) &= \mathcal{A}_0(x, u_{m_k}(x)) \to \mathcal{A}_0(x, u(x)) = G u(x), \end{aligned}$$

for almost all $x \in \Omega$. Therefore, by Dominated Convergence Theorem, we obtain

$$\|F_j u_{m_k} - F_j u\|_{L^{p'}(\Omega,v)} \to 0,$$

and

$$\|G u_{m_k} - G u\|_{L^{p'}(\Omega,\omega)} \to 0,$$

that is, $F_j u_{m_k} \to F_j u$ in $L^{p'}(\Omega, v)$ and $G u_{m_k} \to G u$ in $L^{p'}(\Omega, \omega)$. By Convergence principle in Banach spaces, we have

$$F_j u_m \to F_j u \ \text{ in } \ L^{p'}(\Omega, v), \tag{3.4}$$

and

$$G u_m \to G u \ \text{ in } \ L^{p'}(\Omega, \omega). \tag{3.5}$$

Finally, let $u_m \to u$ in $W_0^{1,p}(\Omega, \omega, v)$ as $m \to \infty$. We have

$$\begin{aligned} &|B(u_m, \varphi) - B(u, \varphi)| \\ &\leq \sum_{j=1}^{n} \int_\Omega |\mathcal{A}_j(x, u_m, \nabla u_m) - \mathcal{A}_j(x, u, \nabla u)||D_j\varphi|\, v\, dx \\ &+ \int_\Omega |\mathcal{A}_0(x, u_m) - \mathcal{A}_0(x, u)||\varphi|\, \omega\, dx \\ &= \sum_{j=1}^{n} \int_\Omega |F_j u_m - F_j u||D_j\varphi| v\, dx + \int_\Omega |G u_m - G u||\varphi|\, \omega\, dx \\ &\leq \left(\sum_{j=1}^{n} \|F_j u_m - F_j u\|_{L^{p'}(\Omega,v)} + \|G u_m - G u\|_{L^{p'}(\Omega,\omega)} \right) \|\varphi\|_{W_0^{1,p}(\Omega,\omega,v)} \end{aligned}$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega, v)$. Hence,

$$\|A u_m - A u\|_* \leq \|G u_m - G u\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^{n} \|F_j u_m - F_j u\|_{L^{p'}(\Omega,v)}.$$

Therefore, by (3.4) and (3.5), $\|A u_m - A u\|_* \to 0$ as $m \to \infty$, that is, the operator A is continuous.

Step 3. The operator A is monotone. In fact, by Proposition 25.6 in [17] the operator $A : W_0^{1,p}(\Omega, \omega, v) \to [W_0^{1,p}(\Omega, \omega, v)]^*$ is monotone if and only if the function $h(t) = \langle A(u + t\,\varphi), \varphi \rangle$, $t \in [0, 1]$, is increasing for all $u, \varphi \in W_0^{1,p}(\Omega, \omega, v)$. We have,

$$\begin{aligned} h(t) &= \langle A(u+t\,\varphi),\varphi\rangle = B(u+t\,\varphi,\varphi)\\ &= \int_\Omega \Big(v\mathcal{A}(x,u+t\,\varphi,\nabla(u+t\,\varphi)).\nabla\varphi + \omega\,\mathcal{A}_0(x,u+t\,\varphi))\varphi\Big)\,dx \end{aligned}$$

is monotone increasing as a functions of $t\in[0,1]$ by (H4).

Step 4. We need to show that the operator A is coercive. We have, using (H2),

$$\begin{aligned} &\langle Au,u\rangle = B(u,u)\\ &= \int_\Omega v\mathcal{A}(x,u,\nabla u).\nabla u\,dx + \int_\Omega \mathcal{A}_0(x,u)u\,\omega\,dx\\ &\geq \int_\Omega \Big(\lambda\,|\nabla u|^p - h_0(x)|\nabla u| - h_1(x)|u| - h_2(x)\Big)\,v\,dx\\ &+ \int_\Omega \Big(\lambda|u|^p - \tilde h_1|u| - \tilde h_2\Big)\,\omega\,dx\\ &\geq \lambda\|u\|^p_{W_0^{1,p}(\Omega,\omega,v)} - (\|h_0\|_{L^{p'}(\Omega,v)} + \|h_1 v/\omega\|_{L^{p'}(\Omega,\omega)})\|u\|_{W_0^{1,p}(\Omega,\omega,v)}\\ &- \|\tilde h_1\|_{L^{p'}(\Omega,\omega)}\|u\|_{W_0^{1,p}(\Omega,\omega,v)} - \|h_2\|_{L^1(\Omega,v)} - \|\tilde h_2\|_{L^1(\Omega,\omega)}. \end{aligned}$$

Hence since $p>1$, we have

$$\begin{aligned} \frac{\langle Au,u\rangle}{\|u\|_{W_0^{1,p}(\Omega,\omega,v)}} \geq\ & \lambda\|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega,v)} - \|h_0\|_{L^{p'}(\Omega,v)} - \|h_1 v/\omega\|_{L^{p'}(\Omega,\omega)}\\ & - \|\tilde h_1\|_{L^{p'}(\Omega,\omega)} - \frac{\|h_2\|_{L^1(\Omega,v)}}{\|u\|_{W_0^{1,p}(\Omega,\omega,v)}} - \frac{\|\tilde h_2\|_{L^1(\Omega,\omega)}}{\|u\|_{W_0^{1,p}(\Omega,\omega,v)}}. \end{aligned}$$

Therefore,

$$\frac{\langle Au,u\rangle}{\|u\|_{W_0^{1,p}(\Omega,\omega,v)}} \to \infty \ \text{ as } \ \|u\|_{W_0^{1,p}(\Omega,\omega,v)} \to \infty,$$

that is, A is coercive.
Therefore, by Theorem 1.3, the operator equation $Au = T$ has a solution $u\in W_0^{1,p}(\Omega,\omega,v)$ and it is a solution for problem (P).

Example Let $\Omega = \{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\}$, $1<p<\infty$ and consider the weight functions $v(x,y) = (x^2+y^2)^{-1/3p}$ and $\omega(x,y) = (x^2+y^2)^{-1/2p}$ (ω, $v\in A_p$), the functions $\mathcal{A}_i : \Omega\times\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}$ $(i=1,2)$ and $\mathcal{A}_0 : \Omega\times\mathbb{R}\to\mathbb{R}$ defined by

$$\begin{aligned} \mathcal{A}_i((x,y),\eta,\xi) &= h_4(x,y)|\xi_i|^{p-1}\mathrm{sgn}(\xi_i),\\ \mathcal{A}_0((x,y),\eta) &= |\eta|^{p-1}\mathrm{sgn}(\eta)(2-\cos^2(xy)), \end{aligned}$$

where $h_4(x,y) = 2e^{x^2+y^2}$. It is easy to show that the functions $\mathcal{A}_i((x,y),\eta,\xi)$ $(i=1,2)$ and $\mathcal{A}_0((x,y),\eta)$ satisfy the conditions (H1), (H2) and (H3). By Remark 1.1, the conditions (H4) is satisfied, because if $\xi = (\xi_1,\xi_2)$, $\xi' = (\xi_1',\xi_2') \in \mathbb{R}^2$, η, $\eta' \in \mathbb{R}$ ($\xi \neq \xi'$ and $\eta \neq \eta'$) we have

$$\begin{aligned}
&[\mathcal{A}((x,y),\eta,\xi) - \mathcal{A}((x,y),\eta',\xi')].(\xi - \xi') \\
&= h_4(x,y)[(|\xi_1|^{p-1}\mathrm{sgn}(\xi_1) - |\xi_1'|^{p-1}\mathrm{sgn}(\xi_1'))(\xi_1 - \xi_1') \\
&+(|\xi_2|^{p-1}\mathrm{sgn}(\xi_2) - |\xi_2'|^{p-1}\mathrm{sgn}(\xi_2'))(\xi_2 - \xi_2')] > 0
\end{aligned}$$

and

$$\begin{aligned}
&[\mathcal{A}_0((x,y),\eta) - \mathcal{A}_0((x,y),\eta')](\eta - \eta') \\
&= (2 - \cos^2(xy))(|\eta|^{p-1}\,\mathrm{sgn}(\eta) - |\eta'|^{p-1}\,\mathrm{sgn}(\eta'))(\eta - \eta') > 0.
\end{aligned}$$

Let us consider the partial differential operator

$$\begin{aligned}
Lu(x,y) &= -\mathrm{div}\Big[v(x,y)\mathcal{A}((x,y),u,\nabla u)\Big] + \omega(x,y)\mathcal{A}_0((x,y),u) \\
&= -\frac{\partial}{\partial x}\Big[v(x,y)\mathcal{A}_1((x,y),u,\nabla u)\Big] - \frac{\partial}{\partial y}\Big[v(x,y)\mathcal{A}_2((x,y),u,\nabla u)\Big] \\
&+ \omega(x,y)\,\mathcal{A}_0((x,y),u).
\end{aligned}$$

Therefore, by Theorem 1.2, the problem

$$\begin{aligned}
&Lu(x,y) = \frac{\cos(xy)}{(x^2+y^2)^{\alpha}} - \frac{\partial}{\partial x}\left(\frac{\sin(xy)}{(x^2+y^2)^{\beta}}\right) - \frac{\partial}{\partial y}\left(\frac{\sin(xy)}{(x^2+y^2)^{\beta}}\right), \text{ on } \Omega \\
&u(x,y) = 0, \text{ on } \partial\Omega
\end{aligned}$$

where $\alpha < (2p - p' - 1)/2pp'$ and $\beta < (3p - p' - 1)/3pp'$, has a solution $u \in W_0^{1,2}(\Omega,\omega,v)$.

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