On the motion under focal attraction in a rotating medium

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Abstract

New results are established here on the phase portraits and bifurcations of the kinematic model in system (1), first presented by H.K. Wilson in [3], and by him attributed to L. Markus (unpublished). A new, self-sufficient, study which extends that of [3] and allows an essential conclusion for the applicability of the model is reported here.

1 Introduction

Consider the following family of planar differential equations depending on three real parameters (ω, v, R), with $\omega \geq 0,\ v > 0,\ R > 0$,

$$\begin{array}{rcl} \dot{x} & = & -\omega y + v\frac{R-x}{\sqrt{(R-x)^2+y^2}}, \\ \dot{y} & = & \omega x - v\frac{y}{\sqrt{(R-x)^2+y^2}}. \end{array} \tag{1}$$

The *solutions*, also called *orbits*, of system (1) describe the motion of certain *entities* (such as particles or micro-organisms) attracted by a *focus* $F = (R, 0)$ (such as a light source or a magnetic pole) toward which they move with *velocity* v; the motion takes place in a *medium* (such as a fluid) which rotates with *angular velocity* ω around a fixed point, located at the origin $O = (0, 0)$.

The focal point $F = (R, 0)$, where system (1) is undefined, will be regarded as a *singular point*, outside of which it is analytic. A point at which the components of the system vanish will be referred to as an *equilibrium point*.

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According to Wilson [3], p. 297, this is a model suggested by L. Markus (see [3], p. viii,) for the motion of *phototropic platyhelminthes* —light-seeking flatworms— swimming in a liquid filling a shallow circular recipient with section

$$G_R = \{x^2 + y^2 \leq R^2\}.$$

No printed reference source containing Markus' suggestion is given in [3].

Definition 1. *Denote by $W^s(Q)$ the basin of attraction of an equilibrium or singular point Q of (1). This is the set of points p_0 such that $\varphi(t, p_0) \to Q$ as $t \to m_+(p_0)$, the right extreme of the maximal interval of $\varphi(t, p_0)$. The basin of repulsion, $W^u(Q)$, is defined analogously for $t \to m_-(p_0)$, referring to the left extreme of the maximal interval.*

Notice that for $Q = F$ the approach to F by orbits of (1) happens in finite time, that is, $m_+(p_0)$ or $m_-(p_0)$ is finite.

In [3], p. 298, using the Poincaré - Bendixson Theorem and the Bendixson Negative Criterium, it is proved the existence and uniqueness of an *equilibrium point* $P = P_{(\omega,v,R)}$ attracting all positively *complete* semi-orbits $\varphi(t, p_0)$ of system (1) starting at points $p_0 \in G_R \setminus F$. Complete means that $\varphi(t, p_0)$ is defined for all $t \in [0, \infty)$. Denote such set of points p_0 by $G_{R,+}$.

For the terms in ODE's not defined here, besides [3], the reader can profit from consulting Chicone [1], among other more recent up to date books.

Under the assumption $\omega > v/R$, P is

$$P_{(\omega,v,R)} = ((v/\omega)^2/R, \sqrt{R^2 - (v/\omega)^2}(v/\omega R)). \tag{2}$$

This corrects a misprint in [3], p. 298.

The theorem below improves the results on this subject outlined in [3]. An elementary proof will be given in section 2.

Theorem 1. *For all $\omega \geq 0$, the region $G_R \setminus F$ is positively invariant, in fact the radial component of system (1) is negative on the complement of the closed disk C of center $(R/2, 0)$ and radius $R/2$.*
It holds that

1. *For $0 \leq \omega \leq v/R$, F is a global attractor: $W^s(F) = \mathbb{R}^2 \setminus F$.*

2. *For $\omega > v/R$ there is a unique hyperbolic attracting equilibrium P located at (2) whose basin of attraction contains $W^u(F)$, the basin of repulsion of F, itself a regular analytic curve contained in $G_R \setminus F$.*

3. *Also, $W^s(F)$, the basin of attraction of F is a regular analytic curve disjoint from G_R.*

   In particular it holds that for $\omega > v/R$,

$$G_{R,+} = G_R \cap W^s(P) = G_R \setminus F.$$

2 Proof of Theorem 1

Performing the change of variables and parameter rescaling

$$x = R\bar{x},\ y = R\bar{y},\ t = \bar{t}R/v,\ \omega = \bar{\omega}v/R \tag{3}$$

and then removing the bars, obtain system (1) with $R = v = 1:$

$$\dot{x} = -\omega y + \frac{1-x}{\sqrt{(1-x)^2+y^2}},\quad \dot{y} = \omega x - \frac{y}{\sqrt{(1-x)^2+y^2}}. \tag{4}$$

Writing this equation in polar coordinates centered at F, given by

$$x = 1 - r\cos\theta,\ \ y = \ r\sin\theta, \tag{5}$$

and multiplying both components by r, which amounts to rescaling again the time, obtain

$$\dot{\theta} = -\omega(r - \cos\theta),\ \ \dot{r} = \ \ r(\omega\sin\theta - 1). \tag{6}$$

Derivation of $z = x^2 + y^2$ (which, in polar coordinates, is $z = 1 + r^2 - 2r\cos\theta$) in the direction of equation (4) (which, in polar coordinates, has the same direction field as (6)), gives $z' = -2r(r - \cos\theta)$.

Clearly z' is negative above the curve $r = \cos\theta$, which is the polar expression for the border of the disk C.

Calculation of the equilibrium of (6) on the plane $r > 0$ gives the point P whose polar coordinates (θ_P, r_P) satisfy

$$\sin\theta_P = 1/\omega,\ \ r_P = \cos\theta_P = \sqrt{1-(1/\omega)^2}.$$

The expression, (x_P, y_P), of P in cartesian coordinates follows:

$$x_P = 1 - \sqrt{1-(1/\omega)^2}\sqrt{1-(1/\omega)^2} = (1/\omega)^2,\ \ y_P = \sqrt{1-(1/\omega)^2}/\omega,$$

which in the original coordinates, before the change in (3), reproduces the expression in (2).

Calculation of the divergence, σ, and Jacobian, δ, of (6) gives

$$\sigma = -1,\ \ \delta = -\omega^2 + \omega\sin\theta + \omega^2 r\cos\theta + \omega^2\cos^2\theta.$$

Evaluation at P, i.e. at (θ_P, r_P), gives $\delta = \omega^2 - 1$. This shows that P is an attracting hyperbolic equilibrium point of *node* or *focus* type.

Calculation of the discriminant, $\sigma^2 - 4\delta$, for values of $\omega \geq 1$ gives that the transition of P from a node to a focus occurs at $\sqrt{5}/2$, that is at $\omega = \sqrt{5}v/2R$, in the original parameters.

This proves the first general assertion and item 1 in the theorem.

For $\omega > 0$, system (6) has two equilibria; one is a hyperbolic saddle at $S_- = (-\pi/2, 0)$ with unstable separatrix along the θ-axis and attracting eigenvector parallel to $(1, (2\omega + 1)/\omega)$.

The other equilibrium, located at $S_+ = (\pi/2, 0)$, is a hyperbolic node for $\omega < 1$. For $\omega > 1$, $S_+ = (\pi/2, 0)$ is a hyperbolic saddle with stable separatrix along the θ-axis and with repelling eigenvector, for positive eigenvalue $\omega - 1$, parallel to $(1, (-2\omega + 1)/\omega)$.

For the equilibrium at $(\pi/2, 0)$, the transition at $\omega = 1$, for increasing ω, gives a cubic *pitchfork* node to saddle bifurcation. For $\omega > 1$, bifurcate two attracting nodal equilibrium points, one to $r > 0$ and the other to $r < 0$, which capture the unstable separatrices of the saddle. Only the first one, located at P, is seen in the original (x, y)-plane.

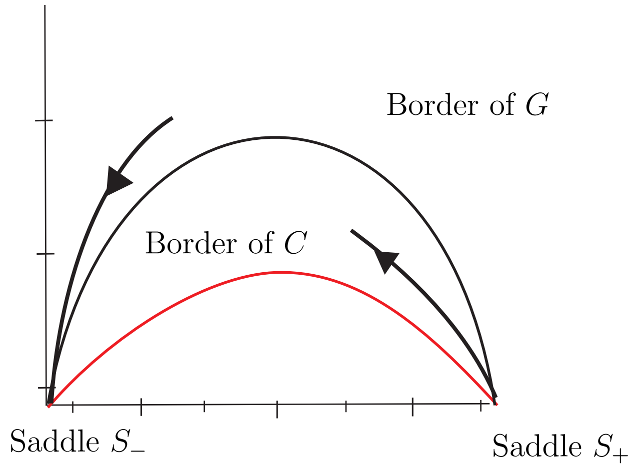

Figure 1: Borders of G and C and Separatrix Directions. Polar Coordinates.

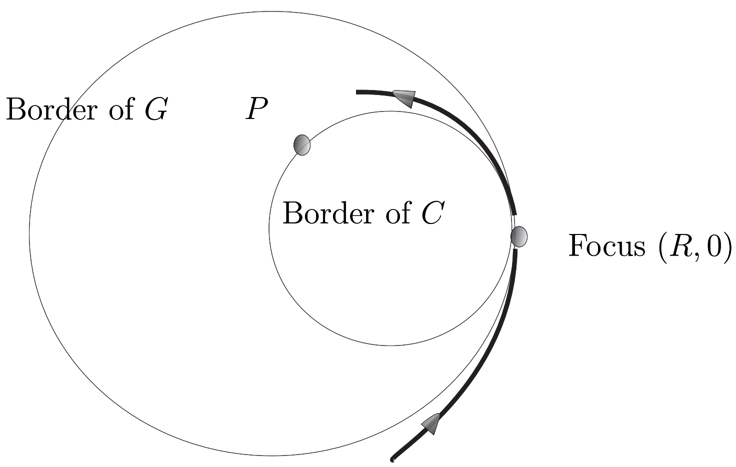

Figure 2: Borders of G and C, Equilibrium Point P and Separatrix Directions at Focus F.

Therefore, the slope of the unstable separatrix, at S_+, is $-2 + 1/\omega$ and that of the stable separatrix at S_- is $2 + 1/\omega$.

Comparison at S_- of the slope the stable separatrix with the slopes of the borders of G_R (for $R = 1$), given by $r = 2\cos\theta$, which is 2, and of C, given by $r = \cos\theta$, which is 1, leads to the location of $W^s(F)$ outside of G_R.

The location of $W^u(F)$ between G_R and C follows from similar comparison of slopes, taking into account that at S_+ the slopes of the borders of G_R (for $R = 1$) is -2, and of C is -1.

See Fig. 1, in polar coordinates, and Fig. 2, in the original coordinates.

By continuity, for any $\omega > 1$ (that is greater than v/R, in the original coordinates), the basin of repulsion of F is always contained in $W^s(P)$. This, for ω near 1, is a consequence of the structure of the local *saddle-node* bifurcation. For large ω this follows from Poincaré-Bendixson Theorem and the Bendixson Negative Criterium [3].

From the fact that z' is negative outside the disk C follows that $W^s(P)$ coincides with $\mathbb{R}^2 \setminus W^s(F)$.

The two main conclusions in items 2 and 3 are established.

3 Concluding Comments

Item 3 is essential for the purpose of the biological model in [3]. In fact, from the knowledge of the correct location, at (2), of the equilibrium attracting all entities moving with velocity v, their separation from similar ones also contained in the positive invariant set G_R but having different characteristic velocities and, therefore, clustering at different points of G_R, for large t. From this information the removal of the entities for study in isolation could be implemented practically. However, no laboratory experiment report where the model has actually been used is known by the author.

Both Wilson [3] and Sotomayor [2], p. 264, the only written sources for system (1), overlooked the property in item 3 of the theorem. Its consideration was proposed to the author toward 1995 by Dan Henry (1944 - 2002).

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