A new characterization of the generalized Hermite linear form

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Abstract

We show that the Generalized Hermite linear form $\mathcal{H}(\mu)$, which is symmetric D-semiclassical of class one, is the unique $\mathcal{D}_\theta$-Appell classical where $\mathcal{D}_\theta$ is the Dunkl operator.

1 Introduction and Preliminaries

The (MOPS) $\{\widetilde{H}_n^{(\mu)}\}_{n\geq 0}$ of generalized Hermite was introduced by G. Szegö (see [2]) who also gave the differential equation, for $n \geq 0$

$$x^2\widetilde{H}_{n+1}^{(\mu)''}(x) + 2x(\mu - x^2)\widetilde{H}_{n+1}^{(\mu)'}(x) + \Big(2(n+3)x^2 - \mu\Big(1+(-1)^n\Big)\Big)\widetilde{H}_{n+1}^{(\mu)}(x) = 0.$$

Some other characterizations such that the recurrence formula

$$\left\{\begin{array}{l} \widetilde{H}_0^{(\mu)}(x) = 1 \ , \ \widetilde{H}_1^{(\mu)}(x) = x, \\ \widetilde{H}_{n+2}^{(\mu)}(x) = x\widetilde{H}_{n+1}^{(\mu)}(x) - \dfrac{1}{2}\Big(n+1+\mu\Big(1+(-1)^n\Big)\Big)\widetilde{H}_n^{(\mu)}(x), \ n \geq 0 \end{array}\right. \tag{1.1}$$

the structure relation

$$x\widetilde{H}_{n+1}^{(\mu)'}(x) = -\mu\Big(1+(-1)^n\Big)\widetilde{H}_{n+1}^{(\mu)}(x) + \Big(n+1+\mu\Big(1+(-1)^n\Big)\Big)x\widetilde{H}_n^{(\mu)}(x), \ n \geq 0$$

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were recovered by T. S. Chihara in [2,3]. He also established a sort of Rodrigues' type formula by using the Kummer's transformation [3]. Also in [3], the same author showed that the generalized Hermite polynomials of the odd and even degrees are expressed in a simple manner through the classical Laguerre polynomials. Indeed, we have

$$\widetilde{H}_{2n}^{(\mu)}(x) = \widetilde{L}_n^{(\mu-\frac{1}{2})}(x^2)\ ;\ \widetilde{H}_{2n+1}^{(\mu)}(x) = x\widetilde{L}_n^{(\mu+\frac{1}{2})}(x^2)\ ,\ \mu \neq -n-\frac{1}{2}\ ,\ n \geq 0$$

where $\{\widetilde{L}_n^{(\alpha)}\}_{n\geq 0}$ is the D-classical (MOPS) of Laguerre ($\alpha \neq -n-1,\ n \geq 0$).

The generalized Hermite polynomials have been mentioned in connection with Gauss quadrature formulas in [12] and with the heat equation for Dunkl operator in [11]. This sequence appears as a solution of polynomials sequences having generating functions of the Brenke type in [4]. Moreover, all technique of the one dimensional Dunkl operator with respect to generalized Hermite polynomials was developed extensively in [10].

In [7] and from another point of view, P. Maroni observed that the linear form $\mathcal{H}(\mu)$ associated with the generalized Hermite polynomials is symmetric D-semiclassical of class one for $\mu \neq 0,\ \mu \neq -n-\frac{1}{2},\ n \geq 0$ (see also [1]) satisfying the functional equation

$$(x\mathcal{H}(\mu))' + \Big(2x^2 - 1 - 2\mu\Big)\mathcal{H}(\mu) = 0$$

from which he derived an integral representation and the moments

$$\langle \mathcal{H}(\mu), f\rangle = \frac{1}{\Gamma(\mu+\frac{1}{2})}\int_{-\infty}^{+\infty} |x|^{2\mu}\exp(-x^2)f(x)dx,\ f \in \mathcal{P},\ \Re\mu > -\frac{1}{2}$$

$$(\mathcal{H}(\mu))_{2n} = \frac{1}{2^{2n}}\frac{\Gamma(\mu+1)\Gamma(2n+2\mu+1)}{\Gamma(2\mu+1)\Gamma(n+\mu+1)}\ ;\ (\mathcal{H}(\mu))_{2n+1} = 0,\ n \geq 0.$$

In that work, it was proved that any polynomial $\widetilde{H}_{n+1}^{(\mu)}$ have simple zeros.

Lastly, it is an old result that the D-classical sequence of Hermite polynomials $\{\widetilde{H}_n^{(0)}\}_{n\geq 0}$ is the unique D-Appell classical one. So the aim of this contribution is to give another characterization of the generalized Hermite sequence based on the $\mathcal{D}_\theta$-Appell classical character where $\mathcal{D}_\theta$ is the Dunkl operator [5]. This first section contains preliminary results and notations used in the sequel. In the second section we determine all symmetric $\mathcal{D}_\theta$-Appell classical orthogonal polynomials; there's a unique solution, up to affine transformations, it is the sequence of generalized Hermite orthogonal polynomials.

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its topological dual. We denote by $\langle u, f\rangle$ the effect of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle\ ,\ n \geq 0$. Let us introduce some useful operations in $\mathcal{P}'$. For any linear form u, any $a \in \mathbb{C} - \{0\}$ and any $q \neq 1$, we let $Du = u'$, $h_a u$ and $H_q u$, be the linear forms defined by duality [6,8,9]

$$\langle u', f\rangle := -\langle u, f'\rangle\ ,\ f \in \mathcal{P},$$

$$\langle h_a u, f\rangle := \langle u, h_a f\rangle = \langle u, f(ax)\rangle\ ,\ u \in \mathcal{P}'\ ,\ f \in \mathcal{P},$$

and

$$\langle H_q u, f\rangle := -\langle u, H_q f\rangle\ ,\ f \in \mathcal{P}$$

where $(H_q f)(x) = \dfrac{f(qx) - f(x)}{(q-1)x}$.

The linear form u is called *regular* if we can associate with it a sequence of polynomials $\{P_n\}_{n\geq 0}$ such that $\langle u, P_m P_n\rangle = r_n \delta_{n,m}$, $n, m \geq 0$; $r_n \neq 0$, $n \geq 0$. The sequence $\{P_n\}_{n\geq 0}$ is then said orthogonal with respect to u. Therefore $\{P_n\}_{n\geq 0}$ is an (OPS) such that any polynomial can be supposed monic (MOPS). The (MOPS) $\{P_n\}_{n\geq 0}$ fulfils the recurrence relation

$$(1.2) \qquad \begin{cases} P_0(x) = 1 \ , \ P_1(x) = x - \beta_0 \ , \\ P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) - \gamma_{n+1} P_n(x) \ , \ \gamma_{n+1} \neq 0 \ , \ n \geq 0. \end{cases}$$

The (MOPS) $\{P_n\}_{n\geq 0}$ is symmetric if and only if $\beta_n = 0$, $n \geq 0$. Furthermore, the orthogonality is kept by shifting. In fact, let

$$(1.3) \qquad \{\widetilde{P}_n := a^{-n}(h_a P_n)\}_{n\geq 0} \ , \ a \neq 0,$$

then the recurrence elements $\widetilde{\beta}_n$, $\widetilde{\gamma}_{n+1}$, $n \geq 0$ of the sequence $\{\widetilde{P}_n\}_{n\geq 0}$ are

$$(1.4) \qquad \widetilde{\beta}_n = \frac{\beta_n}{a} \ , \ \widetilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, \ n \geq 0.$$

Lastly, let us recall the following result useful for our work [1]

Lemma 1.1. *Let $\{P_n\}_{n\geq 0}$ be a (MOPS) and $M(x,n)$, $N(x,n)$ two polynomials such that*

$$M(x,n) P_{n+1}(x) = N(x,n) P_n(x), \ n \geq 0.$$

Then, for any index n for which $\deg N(x,n) \leq n$, *we have*

$$N(x,n) = 0 \ \textit{and} \ M(x,n) = 0.$$

Let us introduce the Dunkl operator in $\mathcal{P}$ by [5]

$$\mathcal{D}_\theta = D + \theta H_{-1} : f \mapsto f'(x) + \theta \frac{f(-x) - f(x)}{-2x} \ , \ \theta \neq 0 \ , \ f \in \mathcal{P}.$$

We have $\mathcal{D}_\theta^{\mathrm{T}} = -D - \theta H_{-1}$ where $\mathcal{D}_\theta^{\mathrm{T}}$ denotes the transposed of $\mathcal{D}_\theta$. We can define $\mathcal{D}_\theta$ from $\mathcal{P}'$ to $\mathcal{P}'$ by $\mathcal{D}_\theta = -\mathcal{D}_\theta^{\mathrm{T}}$ so that

$$\langle \mathcal{D}_\theta u, f\rangle = -\langle u, \mathcal{D}_\theta f\rangle \ , \ u \in \mathcal{P}' \ , \ f \in \mathcal{P}.$$

In particular this yields

$$(\mathcal{D}_\theta u)_n = -\theta_n (u)_{n-1} \ , \ n \geq 0,$$

where $(u)_{-1} := 0$ and

$$(1.5) \qquad \theta_n = n + \theta \frac{1 - (-1)^n}{2} \ , \ n \geq 0.$$

In fact,

$$(1.6) \qquad \theta_{2n} = 2n \ , \ \theta_{2n+1} = 2n + 1 + \theta \ , \ n \geq 0.$$

It is easy to see that [6,9]

$$\mathcal{D}_\theta(fg) = (h_{-1}f)(\mathcal{D}_\theta g) + g(\mathcal{D}_\theta f) + (f - h_{-1}f)g' \ , \ f, g \in \mathcal{P}. \tag{1.7}$$

Now consider a (PS) $\{P_n\}_{n\geq 0}$ as above and let

$$P_n^{[1]}(x;\theta) = \frac{1}{\theta_{n+1}}(\mathcal{D}_\theta P_{n+1})(x) \ , \ \theta \neq -2n-1 \ , \ n \geq 0.$$

Definition 1.2. The sequence $\{P_n\}_{n\geq 0}$ is called $\mathcal{D}_\theta$-Appell classical if $P_n^{[1]}(.;\theta) = P_n$, $n \geq 0$ and $\{P_n\}_{n\geq 0}$ is orthogonal.

2 Determination of all symmetric $\mathcal{D}_\theta$-Appell classical orthogonal polynomials

Lemma 2.1. *Let $\{P_n\}_{n\geq 0}$ be a symmetric $\mathcal{D}_\theta$-Appell classical sequence. The following formulas hold*

(2.1)
$$\theta(h_{-1}P_{n+1})(x) = \Big\{\theta_{n+2} - \frac{\gamma_{n+1}}{\gamma_n}\theta_n - 1\Big\}P_{n+1}(x) + \Big(\frac{\gamma_{n+1}}{\gamma_n}\theta_n - \theta_{n+1}\Big)xP_n(x), \ n \geq 1,$$

$$\gamma_2 = \frac{2}{1+\theta}\gamma_1. \tag{2.2}$$

Proof. From (1.2) and the fact that $\{P_n\}_{n\geq 0}$ is symmetric we have

$$P_{n+2}(x) = xP_{n+1}(x) - \gamma_{n+1}P_n(x), \ \ n \geq 0. \tag{2.3}$$

Applying the operator $\mathcal{D}_\theta$ in (2.3), using (1.7) and in accordance of the $\mathcal{D}_\theta$-Appell classical character we obtain

(2.4)
$$\theta_{n+2}P_{n+1}(x) = -\theta_{n+1}xP_n(x) + (1+\theta)P_{n+1}(x) + 2xP'_{n+1}(x) - \gamma_{n+1}\theta_n P_{n-1}(x), \ n \geq 1.$$

From definition of the operator $\mathcal{D}_\theta$ and the recurrence relation in (1.2), formula (2.4) becomes

$$\theta_{n+2}P_{n+1}(x) = \theta_{n+1}xP_n(x) + P_{n+1}(x) + \\ +\theta(h_{-1}P_{n+1})(x) - \tfrac{\gamma_{n+1}}{\gamma_n}\theta_n\Big(xP_n - P_{n+1}(x)\Big), \ n \geq 1.$$

Consequently (2.1) is proved.

On the other hand, taking $n = 1$ in (2.1) and on account of $P_1(x) = x$ and $P_2(x) = x^2 - \gamma_1$, we get (2.2) after identification. ■

Now, we are able to give the system satisfied by γ_{n+1}, $n \geq 0$ written in terms of r_{n+1}, $n \geq 0$ where r_{n+1} is given by

$$r_{n+1} = \frac{\theta_{n+1}}{\gamma_{n+1}}, \ n \geq 0. \tag{2.5}$$

Proposition 2.2. *The sequence* $\{r_{n+1}\}_{n\geq 0}$ *fulfils the following system*

$$r_{n+1} = r_{n-1}, \qquad n \geq 2, \tag{2.6}$$

$$\frac{r_{n+1}}{r_{n+2}}\theta_{n+2} - \frac{r_{n-1}}{r_n}\theta_n = \theta_{n+3} - \theta_{n+1}, \quad n \geq 2, \tag{2.7}$$

$$\frac{r_1}{r_2} = 1. \tag{2.8}$$

Proof. Applying the dilatation h_{-1} for (2.3) and multiplying by θ, according to (2.1), we get successively

$$(h_{-1}P_{n+2})(x) = -x(h_{-1}P_{n+1})(x) - \gamma_{n+1}(h_{-1}P_n)(x), \ \ n \geq 0,$$

$$\theta(h_{-1}P_{n+2})(x) = -x\theta(h_{-1}P_{n+1})(x) - \gamma_{n+1}\theta(h_{-1}P_n)(x), \ \ n \geq 0,$$

$$\begin{aligned}&\Big(\theta_{n+3} - \tfrac{\gamma_{n+2}}{\gamma_{n+1}}\theta_{n+1} - 1\Big)P_{n+2}(x) + \Big(\tfrac{\gamma_{n+2}}{\gamma_{n+1}}\theta_{n+1} - \theta_{n+2}\Big)xP_{n+1}(x)\\ &= -x\Big\{\Big(\theta_{n+2} - \tfrac{\gamma_{n+1}}{\gamma_n}\theta_n - 1\Big)P_{n+1}(x) + \Big(\tfrac{\gamma_{n+1}}{\gamma_n}\theta_n - \theta_{n+1}\Big)xP_n(x)\Big\}\\ &\quad -\gamma_{n+1}\Big\{\Big(\theta_{n+1} - \tfrac{\gamma_n}{\gamma_{n-1}}\theta_{n-1} - 1\Big)P_n(x) + \Big(\tfrac{\gamma_n}{\gamma_{n-1}}\theta_{n-1} - \theta_n\Big)xP_{n-1}(x)\Big\},\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \geq 2.\end{aligned}$$

But from (2.3) another time we obtain

$$M(x,n)P_{n+1}(x) = N(x,n)P_n(x), \quad n \geq 2 \tag{2.9}$$

where for $n \geq 2$

$$M(x,n) = \Big(\theta_{n+3} - \gamma_{n+1}\frac{\theta_{n-1}}{\gamma_{n-1}} - 2\Big)x,$$

$$N(x,n) = \Big(\theta_{n+1} - \gamma_{n+1}\frac{\theta_{n-1}}{\gamma_{n-1}}\Big)x^2 + \gamma_{n+1}\Big(\theta_{n+3} - \theta_{n+1}\Big) - \gamma_{n+2}\theta_{n+1} + \gamma_{n+1}\gamma_n\frac{\theta_{n-1}}{\gamma_{n-1}}.$$

Next, according to Lemma 1.1., for $n \geq 2$, $M(x,n) = 0$, $N(x,n) = 0$, that is to say

$$\theta_{n+1} - \gamma_{n+1}\frac{\theta_{n-1}}{\gamma_{n-1}} = 0, \ \ n \geq 2 \tag{2.10}$$

$$\gamma_{n+1}\Big(\theta_{n+3} - \theta_{n+1}\Big) - \gamma_{n+2}\theta_{n+1} + \gamma_{n+1}\gamma_n\frac{\theta_{n-1}}{\gamma_{n-1}} = 0, \ \ n \geq 2. \tag{2.11}$$

According to (2.5) relations (2.10)-(2.11) give the desired results (2.6)-(2.7).
Also, from (2.5) and (1.6) we get

$$r_1 = \frac{1+\theta}{\gamma_1} \quad , \quad r_2 = \frac{2}{\gamma_2}.$$

Therefore, taking into account (2.2) we obtain (2.8). ■

Now we are going to solve the system (2.6)-(2.8).
By virtue of (2.6) and (1.5), (2.7) becomes

$$\frac{r_{n-1}}{r_n} = 1, \;\; n \geq 2.$$

Consequently

$$r_{n+1} = r_1, \;\; n \geq 0 \tag{2.12}$$

and (2.6), (2.8) are valid.
From (2.5) and (1.5) (2.12) give

$$\gamma_{n+1} = \frac{\gamma_1}{1+\theta}\Big(n+1+\theta\,\frac{1+(-1)^n}{2}\Big), \;\; n \geq 0. \tag{2.13}$$

Corollary 2.3. *The unique symmetric $\mathcal{D}_\theta$-Appell classical linear form, up to affine transformations, is the generalized Hermite $\mathcal{H}(\mu)$ $\Big(\mu \neq 0, \;\; \mu \neq -n-\frac{1}{2}, \;\; n \geq 0\Big)$.*

Proof. Let $\{P_n\}_{n\geq 0}$ be a symmetric $\mathcal{D}_\theta$-Appell classical sequence. By virtue of (2.13) and (2.2) we get

$$\begin{cases} \beta_n = 0, \;\; n \geq 0, \\ \gamma_{n+1} = \dfrac{\gamma_1}{1+\theta}\Big(n+1+\theta\,\dfrac{1+(-1)^n}{2}\Big), \;\; n \geq 0. \end{cases} \tag{2.14}$$

With the choice $a = \sqrt{\frac{2\gamma_1}{1+\theta}}$ in (1.3)-(1.4), and putting $\mu := \frac{\theta}{2}$ we are led to the following canonical case

$$\begin{cases} \widetilde{\beta}_n = 0, \;\; n \geq 0, \\ \widetilde{\gamma}_{n+1} = \frac{1}{2}\Big(n+1+\mu\big(1+(-1)^n\big)\Big), \;\; n \geq 0. \end{cases} \tag{2.15}$$

Thus (see (1.1))

$$\widetilde{P}_n = \widetilde{H}_n^{(\mu)}, \;\; \mu \neq 0, \;\; \mu \neq -n-\frac{1}{2}, \;\; n \geq 0. \qquad \blacksquare$$

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References

[1] J. Alaya and P. Maroni, *Symmetric Laguerre-Hahn forms of class $s=1$,* Int. Transf. and Spc. Funct. 4, (1996), pp. 301-320.

[2] T. S. Chihara, *An introduction to orthogonal polynomials,* Gordon and Breach, New York, 1978.

[3] T. S. Chihara, *Generalized Hermite polynomials,* Thesis, Purdue, 1955.

[4] T. S. Chihara, *Orthogonal polynomials with Brenke type generating function,* Duke. Math. J., 35(1968) pp. 505-518.

[5] C. F. Dunkl, *Differential-difference operators associated to reflection groups* Trans. Amer. Math. Soc. 311, (1989), pp. 167-183.

[6] L. Khériji and P. Maroni, *The* H_q- *Classical Orthogonal Polynomials* Acta Applicandae Mathematicae 71(2002), pp. 49-115.

[7] P. Maroni, *Sur la suite de polynômes orthogonaux associée à la forme* $u = \delta_c + \lambda\,(x-c)^{-1}\,L$. Period. Math. Hung. 21 (3), (1990), pp.223-248.

[8] P. Maroni, *Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques,* in: Orthogonal Polynomials and their applications, (C. Brezinski et al Editors.) IMACS, Ann. Comput. Appl. Math. 9 (Baltzer, Basel, 1991) pp. 95-130.

[9] P. Maroni, *Variations around classical orthogonal polynomials.Connected problems.* J. Comput. Appl. Math. 48 (1993), pp. 133-155.

[10] M. Rosenblum, *Generalized Hermite polynomials and the Bose-like oscillator calculus.* In: Operator Theory: Advances and Applications. Vol. 73, Basel: Birkhauser Verlag,(1994), pp. 369-396.

[11] M. Rösler *Generalized Hermite polynomials and the Heat equation for Dunkl operators* Comm. Math. Phys. 192(3), (1998), pp. 519-542.

[12] T. S. Shao, T. C. Chen and R. M. Frank *Tables of zeros and Gaussian weights of certain associated Laguerre polynomials and the related generalized Hermite polynomials* Math. Comp., XVIII(1964), pp. 598-616.

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