The orthogonal group of a Form Hilbert space

Hans A. Keller* Herminia Ochsenius †

Abstract

Form Hilbert spaces are constructed over fields that are complete in a non-archimedean valuation. They share with classical Hilbert spaces the basic property expressed by the Projection Theorem. However, there appear some remarkable geometric features which are unknown in Euclidean geometry. In fact, due to the so-called type condition there are only a few orthogonal straight lines containing vectors of the same length, so these non-archimedean spaces are utmost inhomogeneous.

In the paper we consider a typical Form Hilbert space $(E, <\,,\,>)$ and we show that this geometric feature has a strong impact on the group $\mathcal{O}(E)$ of all isometries $T : E \longrightarrow E$ and on the lattice $\mathcal{L}$ of all normal subgroups of $\mathcal{O}$. In particular, we describe some remarkable sublattices of $\mathcal{L}$ which have no analogue in the classical orthogonal groups.

1 Preliminaries

1.1 Form Hilbert spaces

In order to describe a non-archimedean normed vector space E over a valued field K we will introduce G, the range of the valuation $|\ |$ of K, and X, the range of the norm of the vectors in E.

The value group G is a multiplicative linearly ordered group. Of particular interest is the set $\{H_i : i \in I\}$ of its proper convex subgroups (recall that a subgroup H is convex if for all $h \in H$, $h > 1$, the interval $[h^{-1}, h] \subseteq H$). This set is linearly ordered by inclusion, therefore $\bigcup_{i\in I} H_i$ is a subgroup of G. We require that there

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exists a denumerable subset N of I such that for each $n \in N$ H_n is a proper convex subgroup of G and $G = \bigcup_{n \in N} H_n$.
The range of the norm function is a G-**module**, that is a linearly ordered set X together with an action of the group G on X which preserves both the order of G and the order of X, and such that for every $x, y \in X$ there is a $j \in G$ with $jx < y$, see [4]. A crucial object is the orbit Gx for $x \in X$, it is coinitial, and therefore also cofinal in X, but in general it is not evenly distributed over X.
For a vector $v \in E$ the set $G\|v\|$ will be called the **algebraic type** of v. It is clear that if two vectors v_1 and v_2 of E have different algebraic types, then v_1 and v_2 are (norm-)orthogonal. Therefore a set $\{v_i : i \in I\}$ such that $G\|v_i\| \neq G\|v_j\|$ whenever $i \neq j$ is a (norm-)orthogonal set.
Consider now a Banach vector space E over a valued field K with value group G. Let $\sqrt{G}$ be the subgroup of $\widetilde{G}$, the divisible hull of G, defined by $\sqrt{G} := \{\gamma \in \widetilde{G} : \gamma^2 \in G\}$. Since G is a subgroup of $\sqrt{G}$, under the multiplication $\sqrt{G}$ is a G-module. Let $< \ , \ >$ be an inner product in E (i.e. a symmetric bilinear form such that $\forall u \in E < u, u >= 0 \Longrightarrow u = 0$). We shall say that E is a **Form Hilbert space** (FHS) if the following conditions are true,

(i) the inner product induces a non-Archimedean norm on E by

$$\|x\| := \sqrt{| < x, x > |} \in \sqrt{G}$$

(ii) the Projection Theorem is valid in E.

Our specific frame of reference will be the canonical FHS constructed by H. Keller (1980).

1.2 Construction of a Form Hilbert space

In this section we shortly review the standard construction given in [3] and [2]. In contrast to these articles we prefer, for the present purpose, to write the value group multiplicatively.

1. The base field. For $i = 1, 2, \ldots$ let $G_i =< g_i >$ be an infinite cyclic group ordered by $g_i^r < g_i^s$ if and only if $r < s$. The value group G is the direct sum $G = \bigoplus_{i \in \mathbb{N}} G_i$. That is, $g \in G$ if and only if $g = (g_1^{n_1}, g_2^{n_2}, \ldots, g_r^{n_r}, 1, \ldots)$ for some $r \in \mathbb{N}$, $n_i \in \mathbb{Z}$. It is ordered antilexicographically, thus $g = (g_1^{n_1}, g_2^{n_2}, \ldots, g_r^{n_r}, 1, \ldots) > 1$ if $n_r > 0$.
Next, let $F = \mathbb{R}\left(X_i\right)_{i \in \mathbb{N}}$ be the field of all rational functions in the variables $X_1, X_2, \ldots$ with real coefficients. There is a uniquely determined Krull valuation $| \ | : F \longrightarrow G \cup \{0\}$ for which

(a) $| \ |$ is the trivial valuation on $\mathbb{R}$.

(b) $|X_n| = (1, \ldots, 1, g_n, 1, \ldots)$ for $n = 1, 2, \ldots$

To finish the construction of the base field we define K to be the completion of (F, v) by means of Cauchy sequences. Notice that the field K (with the extended valuation) is far from being algebraically complete.

Remark. Let A be the valuation ring and $k \cong \mathbb{R}$ the residual field of the valuation. We denote by $\Pi_0 : A \longrightarrow k$ the canonical projection.

2. The space. Let E be the space of all sequences $x = (\xi_i)_{i\in\mathbb{N}_0} \in K^{\mathbb{N}_0}$ for which the series $\sum_{i=0}^{\infty} \xi_i^2 X_i$ converges in the valuation topology, where $X_0 := 1$. Operations in E are of course componentwise. We define an inner product $< \, , \, >: E \times E \longrightarrow K$ by

$$< x, y >:= \sum_{i=0}^{\infty} \xi_i \eta_i X_i \quad \text{for } x = (\xi_i)_i \, , y = (\eta_i)_i \in E.$$

This symmetric bilinear form $< \, , \, >$ is anisotropic. As usual we say that $x, y \in E$ are (form-)orthogonal, $x \perp y$, if $< x, y >= 0$, and for a subspace $U \subset E$ we define its (form-)orthogonal complement by $U^{\perp} := \{x \in E : \, x \perp u \text{ for all } u \in U\}$.
Next the assignment

$$x \longmapsto \|x\| := \sqrt{| < x, x > |} \in \sqrt{G}$$

is a non-Archimedean norm (see [2], [4]).

The most important properties of the space $(E, < \, , \, >)$ thus constructed are summarized in the following result.

Theorem 1. *Let $(E, < \, , \, >)$ be as above. Then*

1. E is complete in the norm topology.

2. A subspace $U \subseteq E$ is topologically closed if and only if it is orthogonally closed, that is, $U = \mathrm{cl}(U)$ if and only if $U = U^{\perp\perp}$.

3. The Projection Theorem is valid in E : $U = \mathrm{cl}(U) \Rightarrow E = U \oplus U^{\perp}$.

Therefore E is a Form Hilbert space.

For a proof we refer to [3].

3. The canonical base. For $n = 0, 1, 2, \ldots$ we put $e_n := (0, \ldots, 0, 1, 0, \ldots,)$ where 1 is in position $n + 1$. Then $e_n \perp e_m$ for $n \neq m$, $< e_n, e_n >= X_n$ and since the algebraic types of the vectors e_i are all different, $\{e_0, e_1, \ldots, e_n, \ldots\}$ is a (norm-)orthogonal base, called the canonical base of $(E, < \, , \, >)$. Therefore every vector $x \in E$ can be written uniquely as

$$x = \sum_{i=0}^{\infty} \xi_i e_i.$$

1.3 The orthogonal group

Definition. Let E be a K vector space, $< \, , \, >$ be an inner product on E. A linear operator $T : E \longrightarrow E$ is called a (form-)isometry if $< Tx, Tx > = < x, x >$ for all $x \in E$.

Remark. In a Form Hilbert space a form-isometry preserves the norm, that is $\|Tx\| = \|x\|$. Hence it is also a norm-isometry.
From now on, the term isometry will be used instead of form-isometry.

Definition. The orthogonal group of a space E is the group $\mathcal{O}(E)$ of all isometries on E.

There is no lack of isometries. In any inner product space $(E, < \ , \ >$ over a field K with char $K \neq 2$ every vector $u \neq 0$ induces an isometry τ_u by

$$\tau_u(x) = x - 2\frac{< x, u >}{< u, u >}u$$

called the reflection with respect to the hyperplane $H_u := \{w \in E : w \perp u\}$. It is immediate that $\tau_u(u) = -u$, $\tau_u(x) = x$ for every $x \in H_u$. Therefore $(\tau_u)^2 = \tau_u$, hence τ_u is an involution.

The famous theorem of Cartan Dieudonné states that in an inner product space $(E, < \ , \ >)$ with $\dim E = n$, every isometry is a product of at most n hyperplane reflections. This is no longer true in infinite dimensional vector spaces. Indeed, since τ_u is the identity in the hyperplane H_u, the isometry $\sigma = -Id$, with Id the identity in E, cannot be written as a finite product of hyperplane reflections. The study of isometries when $\dim E = \infty$ turns out to be a difficult and challenging problem. A variety of outstanding results were obtained by H.Gross and his school in Zürich, above all in a purely algebraic setting of spaces of countable dimension.

1.4 Lattices, an overview

We have started an analysis of $\mathcal{O}(E)$ for the FHS space described above. We have identified and described some relevant sublattices of the lattice of normal subgroups of $\mathcal{O}(E)$. Therefore we give here a brief summary of the definitions and theorems we shall use, (see [1]).

Definition. Let P be a non-empty ordered set and $S \subseteq P$. An element $x \in P$ is an upper (respectively lower) bound of S if $s \leq x$ (respectively $x \leq s$) for all $s \in S$. The least upper bound of S, if it exists, is called the supremum of S and denoted by $\bigvee S$. Dually the infimum of S, $\bigwedge S$, is the greatest lower bound of S.

Remark. If $S = \{x, y\}$ then $x \vee y$ denotes its supremum and $x \wedge y$ its infimum.

Definition. Let L be a non-empty ordered set. If $x \vee y$ and $x \wedge y$ exist for all x, y in E, then L is called a **lattice**. If $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq L$, then L is called a **complete lattice**.

Theorem 2. *Let L be a non-empty ordered set that has a greatest element $\mathbf{1}$. If $\bigwedge S$ exists for every non-empty subset of L, then L is a complete lattice.*

Corollary 3. *Let $\mathcal{O}$ be a group and $\mathcal{L}$ be the set of all its normal subgroups ordered by inclusion. Then $\mathcal{L}$ is a complete lattice.*

Definition. A lattice L is called **distributive** if for all $x, y, z \in L$,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Equivalently, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in L$.

Definition. A lattice L is called **modular** if for all $x, y, z \in L$,

$$z \leq x \Longrightarrow x \wedge (y \vee z) = (x \wedge y) \vee z.$$

Theorem 4. *If L is a distributive (modular) lattice then every sublattice of L is also distributive (modular).*

Remark. The lattice of normal subgroups of a group $\mathcal{O}$ is modular, but in general not distributive.

Definition. A **Boolean algebra** is a distributive lattice B with a least element $\mathbf{0}_B$ and a greatest element $\mathbf{1}_B$ together with a map $B \longrightarrow B$, $a \longmapsto a'$ such that $a \vee a' = \mathbf{1}_B$ and $a \wedge a' = \mathbf{0}_B$ for all $a \in B$.

Remark. a' is called the **complement** of a. It is clear that $(a')' = a$.

Definition. Let L_1 and L_2 be two lattices. A map $\varphi : L_1 \longrightarrow L_2$ is said to be **order preserving** if for all x, y in L_1 we have that $x \leq y \Longrightarrow \varphi(x) \leq \varphi(y)$. If φ is an order preserving bijection and φ^{-1} is also an order preserving map, then φ is called an **isomorphism of lattices**.

2 The orthogonal group $\mathcal{O}$ of E

Let E be the FHS described in 1.2 and $\mathcal{O}$ its orthogonal group. We begin with a crucial result.

Theorem 5. *Let $T : E \longrightarrow E$ be an isometry and $T(e_k) = \sum_{i=0}^{\infty} \xi_{ik} e_i$ where $\xi_{ik} = < e_i, T(e_k) >< e_i, e_i >^{-1}$. Then $\Pi_0(\xi_{kk}) = \pm 1$.*

Proof. First we notice that $X_k =< e_k, e_k >=< T(e_k), T(e_k) >$. Since the types of the vectors $\{e_i : i \in \mathbb{N}_0\}$ are all different, $| < Te_k, Te_k > | = |\sum \xi_{ik}^2 X_i| = \max\{|\xi_{ik}^2 X_i| : i \in \mathbb{N}_0\}$, thus $|X_k| = |\xi_{kk}^2|\ |X_k| > |\xi_{ik}^2 X_i|$ for all $i \neq k$. In addition $|\xi_{kk}^2| = 1$, hence $1 > |\xi_{ik}^2 \frac{X_i}{X_k}|$ whenever $i \neq k$, hence $\Pi_0\left(\xi_{ik}^2 \frac{X_i}{X_k}\right) = 0$.

But $X_k =< T(e_k), T(e_k) >= \xi_{kk}^2 X_k + \sum_{i \neq k} \xi_{ik} X_i$, so that $1 = \xi_{kk}^2 + \sum_{i \neq k} \xi_{ik}^2 \frac{X_i}{X_k}$. Therefore $\Pi_0(\xi_{kk}^2) = 1$ and it follows that $\Pi_0(\xi_{kk}) = \pm 1$ as claimed.

Notation. From now on, if $T \in \mathcal{O}$ and $k \in \mathbb{N}_0$ we will write $T(k)$ for

$$T(k) := \Pi_0\left(\frac{< e_k, T(e_k) >}{< e_k, e_k >}\right).$$

Definition. For all $k \in \mathbb{N}_0$ we define $\mathcal{N}_k := \{T \in \mathcal{O} : T(k) = 1\}$.

Lemma 6. *Let $S, T \in \mathcal{O}$, $k \in \mathbb{N}_0$, then $(ST)(k) = S(k)T(k)$.*

Proof. Let $S(e_k) = \sum_i \eta_{ik} e_i$, $T(e_k) = \sum_i \xi_{ik} e_i$ and $(ST)(e_k) = \sum_i \alpha_{ik} e_i$. Write $S(\sum_{i \neq k} \xi_{ik} e_i) = \delta_k e_k + \sum_{i \neq k} \delta_i e_i$, and we have that

$$\alpha_{kk} e_k + \sum_{i \neq k} \alpha_{ik} e_i = \xi_{kk} \eta_{kk} e_k + \xi_{kk} \sum_{i \neq k} \eta_{ik} e_i + \delta_k e_k + \sum_{i \neq k} \delta_i e_i,$$

so

$$\alpha_{kk} = \xi_{kk} \eta_{kk} + \delta_k. \tag{1}$$

Now

$$|\delta_k^2 X_k| \le \max\{|\delta_k^2 X_k|, |\sum_{i\neq k}\delta_i^2 X_i|\} = |\delta_k^2 X_k + \sum_{i\neq k}\delta_i^2 X_i| = | < S(\sum_{i\neq k}\xi_{ik}e_i), S(\sum_{i\neq k}\xi_{ik}e_i) > |$$

but S is an isometry, hence $|\delta_k^2 X_k| \le | < \sum_{i\neq k}\xi_{ik}e_i, \sum_{i\neq k}\xi_{ik}e_i > | = |\sum_{i\neq k}\xi_{ik}^2 X_i|$ and by the proof of Theorem 5, $|\sum_{i\neq k}\xi_{ik}X_i| < |X_k|$. Hence $|\delta_k^2| < 1$ and $|\delta_k| < 1$, therefore $\Pi_0(\delta_k) = 0$.
From (1) we obtain now that $\Pi_0(\alpha_{kk}) = \Pi_0(\xi_{kk}\eta_{kk} + \delta_k) = \Pi_0(\xi_{kk})\Pi_0(\eta_{kk})$ and $(ST)(k) = S(k)T(k)$.

Theorem 7. *N_k is a normal subgroup of $\mathcal{O}$ such that $(\mathcal{O} : \mathcal{N}_k) = 2$.*

Proof. Define $\varphi : \mathcal{O} \longrightarrow \{1, -1\}$ by $\varphi(T) = T(k)$. Lemma 6 ensures that φ is a group homomorphism and clearly $\ker\varphi = \mathcal{N}_k$.

2.1 The sublattice generated by the subgroups $\mathcal{N}_k$

For each $A \subseteq \mathbb{N}_0$ we define

$$\mathcal{N}_A := \bigcap_{k\in A}\mathcal{N}_k$$

and $\mathcal{N} := \{\mathcal{N}_A : A \subseteq \mathbb{N}_0\}$.

Lemma 8. *$\mathcal{N}_{\mathbb{N}_0} \neq \{Id\}$, where Id is the identity mapping on E.*

Proof. For each $k \in \mathbb{N}_0$ we have that $\mathcal{O}/\mathcal{N}_k$ is an abelian group, hence $\mathcal{N}_k$ contains Ω, the commutator subgroup of $\mathcal{O}$. Therefore $\Omega \subseteq \bigcap_{k\in\mathbb{N}_0}\mathcal{N}_k = \mathcal{N}_{\mathbb{N}_0}$. But clearly $\Omega \neq \{Id\}$ since $\mathcal{O}$ is not an abelian group.

Theorem 9. *$\mathcal{N}$ is a sublattice of the lattice of normal subgroups of $\mathcal{O}$. It is a Boolean algebra with $0_{\mathcal{N}} = \mathcal{N}_{\mathbb{N}_0}$ and $1_{\mathcal{N}} = \mathcal{O}$.*

Proof. For every $A \subseteq \mathbb{N}_0$ we have that $\mathcal{N}_A$ is a normal subgroup of $\mathcal{O}$, since it is the intersection of a family of normal subgroups. We shall prove that for A, B subsets of $\mathbb{N}_0$ the subgroups $\mathcal{N}_A \wedge \mathcal{N}_B = \mathcal{N}_A \cap \mathcal{N}_B$ and $\mathcal{N}_A \vee \mathcal{N}_B = \mathcal{N}_A\mathcal{N}_B$ belong to $\mathcal{N}$.
First we have

$$\begin{aligned}\mathcal{N}_A \wedge \mathcal{N}_B &= \{T \in \mathcal{O} : T \in \mathcal{N}_A \text{ and } T \in \mathcal{N}_B\}\\ &= \{T \in \mathcal{O} : \forall i \in A\ (T(i) = 1) \text{ and } \forall j \in B\ (T(j) = 1)\}\\ &= \{T \in \mathcal{O} : \forall k \in A \cup B\ (T(k) = 1)\}\\ &= \mathcal{N}_{A\cup B} \in \mathcal{N}.\end{aligned}$$

Next we prove that $\mathcal{N}_A\mathcal{N}_B = \mathcal{N}_{A\cap B}$. Let $S \in \mathcal{N}_A$, $T \in \mathcal{N}_B$, then $S(k) = 1 = T(k)$ for all $k \in A \cap B$. By Lemma 6 $(ST)(k) = 1$. Therefore $\mathcal{N}_A\mathcal{N}_B \subseteq \mathcal{N}_{A\cap B}$.
Now let $R \in \mathcal{N}_{A\cap B}$ and put $C := \{k \in \mathbb{N}_0 : R \in \mathcal{N}_k\}$. We define $P \in \mathcal{O}$ by

$$P(e_i) = \begin{cases} -e_i & \text{if } i \in B \setminus C\\ e_i & \text{if } i \notin B \setminus C\end{cases}$$

It is readily seen that $P = P^{-1}$, moreover $P \in \mathcal{N}_A$. In fact, notice that $A \cap B \subseteq C$, from which it follows that $A \cap (B \setminus C) = \varnothing$, hence if $k \in A$ then $k \notin B \setminus C$ and therefore $P(e_k) = e_k$. On the other hand $PR \in \mathcal{N}_B$, since by construction $P(k) = R(k)$ for all $k \in B$. Thus $R = P^{-1}(PR) \in \mathcal{N}_A\mathcal{N}_B$ and $\mathcal{N}_{A\cap B} = \mathcal{N}_A\mathcal{N}_B \in \mathcal{N}$. Therefore $\mathcal{N}$ is a sublattice of the lattice of normal subgroups of $\mathcal{O}$ as claimed. Clearly the least element of $\mathcal{N}$ is $\mathcal{N}_{\mathbb{N}_0}$ and the greatest one is $\mathcal{O} = \mathcal{N}_\varnothing$. It is also clear that $\mathcal{N}$ is distributive.

Now we define for $\mathcal{N}_A \in \mathcal{N}$ its complement $\mathcal{N}_A{}'$ by $\mathcal{N}_A{}' = \mathcal{N}_{\mathbb{N}_0 \setminus A}$. It is obvious that $(\mathcal{N}_A{}')' = \mathcal{N}_A$, $\mathcal{N}_A \vee \mathcal{N}_A{}' = 1_\mathcal{N}$ and $\mathcal{N}_A \wedge \mathcal{N}_A{}' = 0_\mathcal{N}$. Therefore $\mathcal{N}$ is a Boolean algebra.

Lemma 10. *$\mathcal{N}$ is a complete lattice.*

Proof. Since $\mathcal{N}$ has a greatest element $\mathcal{N}_\varnothing = \mathcal{O}$, it is enough to prove that if $S = \{\mathcal{N}_{S_i} : S_i \subseteq \mathbb{N}_0,\ i \in I\}$ is a non-empty collection of elements in $\mathcal{N}$ then $\bigwedge S \in \mathcal{N}$. We contend that $\bigwedge S = \mathcal{N}_Z$ with $Z = \bigcup_{i\in I} S_i$.
In fact for every $T \in \mathcal{O}$ we have

$$T \in \mathcal{N}_Z \Leftrightarrow T(m) = 1 \text{ for all } m \in \textstyle\bigcup_{i\in I} S_i \Leftrightarrow T \in \mathcal{N}_{S_i} \text{ for all } i \in I \Leftrightarrow T \in \textstyle\bigcap_{i\in I} \mathcal{N}_{S_i}.$$

Corollary 11. *$\mathcal{N}$ is isomorphic to the Boolean algebra $(\mathcal{P}(\mathbb{N}), \subseteq_{op}) = \mathcal{P}$ where $A \subseteq_{op} B$ if and only if $A \supseteq B$.*

Proof. The map $\varphi : \mathcal{P} \longrightarrow \mathcal{N}$ defined by $\varphi(A) = \mathcal{N}_A$ is clearly surjective. It is also injective, because if $A, B \in \mathcal{P}$ with $A \neq B$ we may assume that $A \nsubseteq B$ and so there exists $m \in \mathbb{N}_0$ with $m \in A$ and $m \notin B$. The isometry I_m defined by $I_m(e_k) = e_k$ if $k \neq m$, $I_m(e_m) = -e_m$ belongs to $\mathcal{N}_B$ but not to $\mathcal{N}_A$. It is a direct verification that φ as well as φ^{-1} are lattice homomorphisms.

Remark. By the previous result the lattice of normal subgroups of the orthogonal group contains an isomorphic copy of $\mathcal{P}(\mathbb{N})$. The question as to whether it contains another another copy of this Boolean algebra is still open.

2.2 The lattice $\mathcal{N}^*$

The subgroup $J = \{Id, -Id\}$ is normal in $\mathcal{O}$ but does not coincide with any $\mathcal{N}_A$ when $A \subseteq \mathbb{N}_0$. We will study here the sublattice $\mathcal{N}^*$ generated by $\mathcal{N} \cup \{J\}$ in $\mathcal{L}$.
We denote by $\mathcal{N}_A^*$ the element $\mathcal{N}_A \vee J$. Explicitly

$$\mathcal{N}_A^* = \{T \in \mathcal{O} : \forall m \in A\ T(m) = 1\} \cup \{S \in \mathcal{O} : \forall m \in A\ S(m) = -1\}$$

.
The lattice $\mathcal{N}^*$ is modular since it is a sublattice of $\mathcal{L}$, but it is not a distributive lattice. In fact, for every $k \in \mathbb{N}$ $\mathcal{N}_k \wedge J = \mathcal{N}_k \cap J = \{Id\}$. Hence $(\mathcal{N}_1 \wedge J) \vee (\mathcal{N}_2 \wedge J) = \{Id\}$ but $(\mathcal{N}_1 \vee \mathcal{N}_2) \wedge J = \mathcal{N}_\phi \wedge J = \mathcal{O} \wedge J = J$.

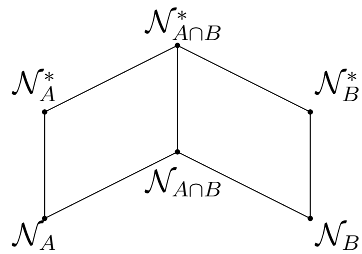
Lemma 12. *For all $A, B \subseteq \mathbb{N}_0$ we have $\mathcal{N}_A^* \vee \mathcal{N}_B = \mathcal{N}_A^* \vee \mathcal{N}_B^* = \mathcal{N}_{A\cap B}^*$.*

Proof.

$$\begin{array}{lllll} \mathcal{N}_A^* \vee \mathcal{N}_B & = & J \vee \mathcal{N}_A \vee \mathcal{N}_B & = & J \vee \mathcal{N}_{A\cap B} & = & \mathcal{N}_{A\cap B}^* \\ \mathcal{N}_A^* \vee \mathcal{N}_B^* & = & J \vee \mathcal{N}_A \vee J \vee \mathcal{N}_B & = & J \vee \mathcal{N}_A \vee \mathcal{N}_B & = & \mathcal{N}_{A\cap B}^* \end{array}$$

Remark. If $A \cap B = \varnothing$, $\mathcal{N}_\varnothing^* = \mathcal{N}_\varnothing = \mathcal{O}$.

The suprema are shown in the following diagram.



The computation of infima is more complex.

Let $A, B \subseteq \mathbb{N}_0$, $r, s \in \{0, 1, -1\}$. We introduce the following notation:

$$(A = r \wedge B = s) := \{T \in \mathcal{O} : \forall m \in A\ T(m) = r \text{ and } \forall m \in B\ T(m) = s\}$$

where $T(m) = 0$ means that no restriction is placed on $T(m)$. Then,

$$\begin{aligned} \mathcal{N}_A &= (A = 1 \wedge B = 0) \\ \mathcal{N}_B &= (A = 0 \wedge B = 1) \\ \mathcal{N}_{A\cup B} &= (A = 1 \wedge B = 1) = \mathcal{N}_A \cap \mathcal{N}_B \\ \mathcal{N}_A^* &= (A = 1 \wedge B = 0) \cup (A = -1 \wedge B = 0) \\ \mathcal{N}_B^* &= (A = 0 \wedge B = 1) \cup (A = 0 \wedge B = -1) \end{aligned}$$

We obtain by direct computation

$$\begin{aligned} &\mathcal{N}_A^* \cap \mathcal{N}_B^* \\ &= [(A = 1 \wedge B = 0) \cup (A = -1 \wedge B = 0)] \cap [(A = 0 \wedge B = 1) \cup (A = 0 \wedge B = -1)] \\ &= [(A = 1 \wedge B = 0) \cap (A = 0 \wedge B = 1)] \cup [(A = 1 \wedge B = 0) \cap (A = 0 \wedge B = -1)] \cup \\ &\quad [(A = -1 \wedge B = 0) \cap (A = 0 \wedge B = 1)] \cup [(A = -1 \wedge B = 0) \cap (A = 0 \wedge B = -1)]. \end{aligned}$$

Therefore

$$\mathcal{N}_A^* \ \cap\ \mathcal{N}_B^* =$$
$$(A = 1\ \wedge\ B = 1) \cup (A = 1\ \wedge\ B = -1) \cup (A = -1\ \wedge\ B = 1) \cup (A = -1\ \wedge\ B = -1).$$

In the same way we compute

$$\begin{aligned} &\mathcal{N}_A^* \cap \mathcal{N}_B \\ &= [(A = 1 \wedge B = 0) \cup (A = -1 \wedge B = 0)] \cap (A = 0 \wedge B = 1) \\ &= [(A = 1 \wedge B = 0) \cap (A = 0 \wedge B = 1)] \cup [(A = -1 \wedge B = 0) \cap (A = 0 \wedge B = 1)]. \end{aligned}$$

Therefore,

$$\mathcal{N}_A^* \cap \mathcal{N}_B = (A = 1 \wedge B = 1) \cup (A = -1 \wedge B = 1).$$

Remark. If $A \cap B \neq \varnothing$ then the values of A and B must be the same. Therefore,

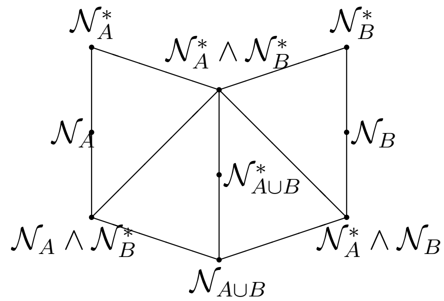
$$\mathcal{N}_A^* \cap \mathcal{N}_B^* = (A = 1 \wedge B = 1) \cup (A = -1 \wedge B = -1) = \mathcal{N}_{A\cup B}^*.$$

In addition,

$$\mathcal{N}_A^* \cap \mathcal{N}_B = \mathcal{N}_{A\cup B} = \mathcal{N}_A \cap \mathcal{N}_B = \mathcal{N}_A \cap \mathcal{N}_B^*.$$

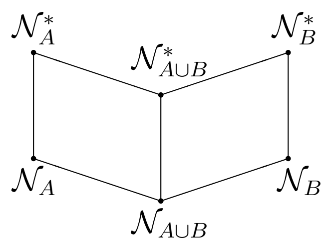
The following diagrams depict the situation.

1. If $A \cap B = \varnothing$,



   Note that there are elements in $\mathcal{L}^*$ that do not belong to $\{\mathcal{N}_C^* : C \subseteq \mathbb{N}_0\}$. They are $\mathcal{N}_A \wedge \mathcal{N}_B^*$, $\mathcal{N}_A^* \wedge \mathcal{N}_B$ and $\mathcal{N}_A^* \wedge \mathcal{N}_B^*$.

2. If $A \cap B \neq \varnothing$ the elements mentioned above vanish and the diagram collapses to the following one:



2.3 A characterization

Theorem 13. *For any $i \in \mathbb{N}_0$ let $C_i = \{1, -1\}$ be the multiplicative group of two elements and $\prod_{i\in\mathbb{N}_0} C_i$ their direct product. Then $\prod_{i\in\mathbb{N}_0} C_i \cong \mathcal{O}/\mathcal{N}_{\mathbb{N}_0}$.*

Proof. Let $\varphi : \mathcal{O} \longrightarrow \prod_{i\in\mathbb{N}_0} C_i$ be the map defined by $T \longmapsto (T(i))_{i\in\mathbb{N}_0}$. By Lemma 6 it is a group homomorphism. It is surjective, since for $\varepsilon = (\varepsilon_i) \in \prod C_i$ the involution T_ε defined by $T_\varepsilon(e_j) = \varepsilon_j$ is an isometry. Clearly $\ker \varphi = \mathcal{N}_{\mathbb{N}_0}$ so the First Isomorphism Theorem gives the result.

Definition. $\mathcal{L}_0 := [\mathcal{N}_{\mathbb{N}_0}, \mathcal{O}]$ is the interval of $\mathcal{L}$ of all the normal subgroups of $\mathcal{O}$ which contain $\mathcal{N}_{\mathbb{N}_0}$.

Corollary 14. *The (complete) lattice of all subgroups of $\prod_{i\in\mathbb{N}_0} C_i$ is isomorphic to the (complete) lattice $\mathcal{L}_0$.*

Proof. The map $B \longmapsto \varphi^{-1}(B) = \{T \in \mathcal{O} : \varphi(T) \in B\}$ is a bijection between the subgroups of $\prod C_i$ and the subgroups of $\mathcal{O}$ which contain $\mathcal{N}_{\mathbb{N}_0}$. This bijection is an order isomorphism, that is, if B_1, B_2 are subgroups of $\prod C_i$, $A_1 := \varphi^{-1}(B_1)$, $A_2 := \varphi^{-1}(B_2)$ then $A_1 \subseteq A_2 \Leftrightarrow B_1 \subseteq B_2$. Therefore φ is a lattice isomorphism.

Corollary 15. *Every subgroup of $\mathcal{O}$ which contains $\mathcal{N}_{\mathbb{N}_0}$ is a normal subgroup of $\mathcal{O}$.*

From the lattice $\mathcal{N}^*$ we remove the subgroup $J = \{1, -1\}$ as well as the subgroup $\{Id\}$. There remains then the interval $[\mathcal{N}_{\mathbb{N}_0}, \mathcal{O}]$ of $\mathcal{L}^*$. We shall call it the lattice $\mathcal{N}^{**}$.

$\mathcal{N}^{**}$ is a sublattice of $\mathcal{L}_0$. In fact, the latter is the completion of the former.

Theorem 16. *Let A be a subgroup of $\mathcal{O}$ such that $\mathcal{N}_{\mathbb{N}_0} \subseteq A$. Then A is the supremum of a family of elements of $\mathcal{N}^{**}$.*

Proof. Let $\varepsilon = (\varepsilon_i)_{i\in\mathbb{N}_0} \in \prod_{i\in\mathbb{N}_0} C_i$; set $A_\varepsilon = \{i : \varepsilon_i = 1\}$, $B_\varepsilon = \{i : \varepsilon_i = -1\}$. If $H_\varepsilon = \{\varepsilon, 1\} \leq \prod C_i$ then $\varphi^{-1}(H_\varepsilon) = \mathcal{N}_{A_\varepsilon} \wedge \mathcal{N}^*_{B_\varepsilon}$. In fact

$$\begin{aligned} T \in \mathcal{N}_{A_\varepsilon} \cap \mathcal{N}^*_{B_\varepsilon} &\Leftrightarrow T(i) = 1 \text{ for all } i \in A_\varepsilon \text{ and } T(i) = T(k) \text{ if } j, k \in B_\varepsilon \\ &\Leftrightarrow (T(i) = 1 \text{ for all } i \in \mathbb{N}_0) \text{ or } (T(i) = 1 \text{ if } i \in A_\varepsilon \\ &\qquad\qquad \text{and } T(i) = -1 \text{ if } i \in B_\varepsilon) \\ &\Leftrightarrow \varphi(T) = 1 \text{ or } \varphi(T) = \varepsilon \\ &\Leftrightarrow \varphi(T) \in H_\varepsilon. \end{aligned}$$

Now, let $A \leq \mathcal{O}$ with $\mathcal{N}_{\mathbb{N}_0} \subseteq A$ and $H = \varphi(A) \in \prod C_i$. Assume that $\{\alpha_k :\in k\}$ is a set of generators of H, then $H = \bigvee_{k\in K} H_k$ with $H_k = \{1, \alpha_k\}$. By Corollary 15, $A = \varphi^{-1}(\bigvee_{k\in K} H_k) = \bigvee_{k\in K} \varphi^{-1}(H_k) = \bigvee_{k\in K} (\mathcal{N}_{A_{\alpha_k}} \wedge \mathcal{N}^*_{B_{\alpha_k}})$ is the supremum of a family of elements of $\mathcal{N}^{**}$.

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H. Keller : Hochschule Technik+Architektur Luzern,
CH-6048 Horw, Switzerland.
email : hakeller@hta.fhz.ch

H. Ochsenius : Facultad de Matemáticas,
Pontificia Universidad Católica de Chile,
Casilla 306 - Correo 22, Santiago, Chile.
email : hochsen@mat.puc.cl