# Subharmonicity of Powers of Octonion-Valued Monogenic Functions and Some Applications

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**Abstract**

It is proven that for an octonion-valued monogenic function $f(\mathbf{x})$, $\mathbf{x} \in \mathbf{R}^8$, its powers $|f|^p$ are subharmonic for any $p \geq 6/7$. This implies, in particular, Hadamard's three circles and three lines theorems and a Phragmén-Lindelöf theorem for monogenic functions.

## 1 Introduction

Let $\sum_{j=0}^{7} x_j e_j$ be a generic octonion, where $e_0 \equiv 1, e_1, \ldots, e_7$ are the basis octonion (imaginary) units; we identify it with a vector $\mathbf{x} = (x_0, \ldots, x_7) \in \mathbf{R}^8$. In notation we follow [3]. Let

$$f(\mathbf{x}) = \sum_{j=0}^{7} e_j f_j(\mathbf{x})$$

be an octonion-valued left-monogenic function in a domain $\Omega \subset \mathbf{R}^8$, where $f_0(\mathbf{x}), \ldots,$ $f_7(\mathbf{x})$ are real-valued $\mathcal{C}^1$ functions. That means $Df = 0$, where $D = \sum_{k=0}^{7} e_k \frac{\partial}{\partial x_k}$ is the Dirac (or Cauchy-Riemann) operator. It is known that all the components $f_0, \ldots, f_7$ of a left-monogenic function are harmonic functions, that is, $\Delta f_0 = \cdots = \Delta f_7 = 0$ in $\Omega$. The equation $Df = 0$ is a system of eight first-order linear partial differential equations with constant coefficients. It can be written as a matrix

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equation

$$\begin{bmatrix}
\frac{\partial}{\partial x_0} & -\frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_4} & -\frac{\partial}{\partial x_5} & -\frac{\partial}{\partial x_6} & -\frac{\partial}{\partial x_7} \\
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_0} & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_7} & \frac{\partial}{\partial x_6} & -\frac{\partial}{\partial x_5} & \frac{\partial}{\partial x_4} \\
\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_0} & -\frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_6} & -\frac{\partial}{\partial x_7} & \frac{\partial}{\partial x_4} & \frac{\partial}{\partial x_5} \\
\frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_0} & \frac{\partial}{\partial x_5} & -\frac{\partial}{\partial x_4} & -\frac{\partial}{\partial x_7} & \frac{\partial}{\partial x_6} \\
\frac{\partial}{\partial x_4} & \frac{\partial}{\partial x_7} & \frac{\partial}{\partial x_6} & -\frac{\partial}{\partial x_5} & \frac{\partial}{\partial x_0} & \frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_1} \\
\frac{\partial}{\partial x_5} & -\frac{\partial}{\partial x_6} & \frac{\partial}{\partial x_7} & \frac{\partial}{\partial x_4} & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_0} & \frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} \\
\frac{\partial}{\partial x_6} & \frac{\partial}{\partial x_5} & -\frac{\partial}{\partial x_4} & \frac{\partial}{\partial x_7} & \frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_0} & -\frac{\partial}{\partial x_3} \\
\frac{\partial}{\partial x_7} & -\frac{\partial}{\partial x_4} & -\frac{\partial}{\partial x_5} & -\frac{\partial}{\partial x_6} & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_0}
\end{bmatrix} F = 0,$$

where $F = [f_0, \ldots, f_7]^T$ is the unknown column vector-function. It can also be rewritten as a matrix equation

$$\sum_{j=0}^{7} A_j \frac{\partial F}{\partial x_j} = 0. \tag{1.1}$$

Here $A_0$ is the identity matrix of order 8 and $8 \times 8$ antisymmetric matrices $A_1 - A_7$ are given by

$$A_1 = \begin{bmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}$$

$$A_2 = \begin{bmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}$$

$$A_3 = \begin{bmatrix}
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It should be mentioned that $det(A_0) = \cdots = det(A_7) = 1$. The octonion multiplication table can be written in various ways, for example, the table in [2, p. 150] is different from one in [3]. These different tables result in different systems (1.1), though all these systems are clearly equivalent. Solutions of the system $[f]D = 0$ are called right-monogenic functions; functions, which are both left- and right-monogenic, are called monogenic.

Systems (1.1), where each component $f_j$, $0 \leq j \leq 7$, is harmonic, are called the generalized Cauchy-Riemann systems (GCR) - see Stein and Weiss [10, pp. 260-262]. Systems $\sum_{j=0}^{7} A_j \frac{\partial F}{\partial x_j} + BF = 0$, where $B$ is also a constant matrix, were considered by Evgrafov [5]. Stein and Weiss have proved that for any GCR system there exists a nonnegative index $p_0 < 1$ such that $|F|^p$ is a subharmonic function for all $p \geq p_0$.

It is known (ibid, p. 262) that for the M. Riesz system in $\mathbf{R}^n$

$$\frac{\partial u_1}{\partial x_1} + \cdots + \frac{\partial u_n}{\partial x_n} = 0,$$

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i},\ i, j = 1, \ldots, n,\ i \neq j,$$

the exact value of $p_0$ is $(n-2)/(n-1)$. Our first goal is to prove that the same is valid for system (1.1) in $\mathbf{R}^8$, that is for system (1.1)

$$p_0 = \left.\frac{n-2}{n-1}\right|_{n=8} = 6/7.$$

The question on the precise value of $p_0$ for the system (1.1) has been risen by Li and Peng [7].

*Our result leads to a natural question whether the conclusion* $p_0 = \frac{n-2}{n-1}$ *is valid for any generalized Cauchy-Riemann system.*

The theorem on the subharmonicity of powers allows us to transplant many properties of subharmonic functions to monogenic functions. Thus, we extend Hadamard's three circles and three lines theorems to octonion-valued monogenic functions and prove a simple Phragmén-Lindelöf theorem.

## 2 Subharmonicity of powers of monogenic functions

**Theorem 2.1.** *For any solution $F$ of system (1.1) and for each $p \geq p_0 = 6/7$ the function $s(\mathbf{x}) = |F(\mathbf{x})|^p$ is subharmonic in the corresponding domain and this value of $p_0$ cannot be decreased. Moreover, the same conclusion holds true for right-monogenic and monogenic functions.*

**Proof.** As in [10, Theorem 4.9], the problem can be reduced to the following one:

Find the smallest value of $\alpha \geq 0$ such that

$$\max_{|v|=1} \sum_{j=0}^{7} (u^{(j)} \cdot v)^2 \leq \alpha \sum_{j=0}^{7} |u^{(j)}|^2,$$

where $u^{(j)},\ j = 0, 1, \ldots, 7$, are 8-columns satisfying the equation $\sum_{j=0}^{7} A_j u^{(j)} = 0$; without loss of generality we assume $\sum_{j=0}^{7} |u^{(j)}|^2 = 1$.

However, the reasoning in [10, p. 262] does not work for system (1.1) since unlike

M. Riesz' system, the relevant matrix

$$M = \begin{bmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & -\alpha_5 & -\alpha_6 & -\alpha_7 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 & -\alpha_7 & \alpha_6 & -\alpha_5 & \alpha_4 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 & -\alpha_6 & -\alpha_7 & \alpha_4 & \alpha_5 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 & \alpha_5 & -\alpha_4 & -\alpha_7 & \alpha_6 \\ \alpha_4 & \alpha_7 & \alpha_6 & -\alpha_5 & \alpha_0 & \alpha_3 & -\alpha_2 & -\alpha_1 \\ \alpha_5 & -\alpha_6 & \alpha_7 & \alpha_4 & -\alpha_3 & \alpha_0 & \alpha_1 & -\alpha_2 \\ \alpha_6 & \alpha_5 & -\alpha_4 & \alpha_7 & \alpha_2 & -\alpha_1 & \alpha_0 & -\alpha_3 \\ \alpha_7 & -\alpha_4 & -\alpha_5 & -\alpha_6 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_0 \end{bmatrix}$$

is not symmetric and has nonzero trace; indeed, it has complex eigenvalues

$$\lambda_{\pm} = \alpha_0 \pm \imath\sqrt{\alpha_1^2 + \cdots + \alpha_7^2}$$

each of multiplicity 4.

Therefore, we straightforwardly consider an extremal problem

$$g_0(\mathbf{y}) \equiv \sum_{j=0}^{7} \left( \sum_{i=0}^{7} u_{j,i} v_i \right)^2 \to \mathit{maximum},$$

where $\mathbf{y} = (u_{0,0}, \ldots, u_{7,7}, v_0, \ldots, v_7) \in \mathbf{R}^{72}$, subject to ten constrains. Two of the latter are the normalization conditions,

$$g_1(\mathbf{y}) \equiv \sum_{i=0}^{7} v_i^2 - 1 = 0$$

and

$$g_2(\mathbf{y}) \equiv \sum_{j,i=0}^{7} u_{j,i}^2 - 1 = 0.$$

The other eight constrains correspond to the differential equations comprising system (1.1):

$$\begin{aligned}
g_3(\mathbf{y}) &\equiv u_{0,0} - u_{1,1} - u_{2,2} - u_{3,3} - u_{4,4} - u_{5,5} - u_{6,6} - u_{7,7} = 0 \\
g_4(\mathbf{y}) &\equiv u_{0,1} + u_{1,0} + u_{2,3} - u_{3,2} + u_{4,7} - u_{5,6} + u_{6,5} - u_{7,4} = 0 \\
g_5(\mathbf{y}) &\equiv u_{0,2} - u_{1,3} + u_{2,0} + u_{3,1} + u_{4,6} + u_{5,7} - u_{6,4} - u_{7,5} = 0 \\
g_6(\mathbf{y}) &\equiv u_{0,3} + u_{1,2} - u_{2,1} + u_{3,0} - u_{4,5} + u_{5,4} + u_{6,7} - u_{7,6} = 0 \\
g_7(\mathbf{y}) &\equiv u_{0,4} - u_{1,7} - u_{2,6} + u_{3,5} + u_{4,0} - u_{5,3} + u_{6,2} + u_{7,1} = 0 \\
g_8(\mathbf{y}) &\equiv u_{0,5} + u_{1,6} - u_{2,7} - u_{3,4} + u_{4,3} + u_{5,0} - u_{6,1} + u_{7,2} = 0 \\
g_9(\mathbf{y}) &\equiv u_{0,6} - u_{1,5} + u_{2,4} - u_{3,7} - u_{4,2} + u_{5,1} + u_{6,0} + u_{7,3} = 0 \\
g_{10}(\mathbf{y}) &\equiv u_{0,7} + u_{1,4} + u_{2,5} + u_{3,6} - u_{4,1} - u_{5,2} - u_{6,3} + u_{7,0} = 0.
\end{aligned}$$

We apply the method of Lagrange multipliers. The resulting system of 72 cubic equations in 72 indeterminates $u_{0,0}, \ldots, u_{7,7}, v_0, \ldots, v_7$ with two quadratic and eight linear constraints and with ten additional parameters (the Lagrangian multipliers) looks (almost) intractable. However it has many symmetries and can be reduced to a cubic equation with rational roots for the values of the objective function $g_0$ at the critical points, yielding three extremal values:

**(A)** $g_0 = 0$, which is the obvious global minimum, since $g_0 \geq 0$;

**(B)** an extraneous value $g_0 = 1$, which is the maximum value of $g_0$ if the linear constraints $g_3 - g_{10}$ are not taken into consideration;

**(C)** the maximum value $\alpha = g_0 = 7/8$, which we are looking for.

Thus as in [10], the critical exponent is $p_0 = 2 - 1/\alpha = 6/7$. To show that this value cannot be decreased, we can use the function $h(\mathbf{x}) = |\mathbf{x}|^{-6}/(-6)$ [10], since $|\nabla h|$ is monogenic [7].

The same argument and the same example work for the right-monogenic functions. Therefore, for the monogenic functions the precise value of the exponent is $p_0 = 6/7$ as well. ∎

Theorem 2.1 and the theorem Coifman and Weiss [4] yield the following result, where $F^\lambda(\mathbf{x}) = [f_0, f_1, \ldots, f_6, \lambda f_7],\ \lambda \geq 0$.

**Corollary 2.1.** *Together with* $|F(\mathbf{x})|^p$ *the functions*

$$8|F^0(\mathbf{x})|^p - (p-1)|F(\mathbf{x})|^p,\ 1 < p \leq 2,$$

*and*

$$8p(p-1)|F^0(\mathbf{x})|^2|F^\lambda(\mathbf{x})|^{p-2} - |F^\lambda(\mathbf{x})|^p,\ 2 \leq p < \infty,$$

*where* $0 < \lambda^2 \leq \min\{1;\ (32(p-2))^{-1}\}$, *are subharmonic.* ∎

**Remark 2.1.** *The same argument also holds true for quaternion-valued monogenic functions in* $\mathbf{R}^4$ *giving the exponent* $p_0 = 2/3$ *– this has already been shown in [7].* ∎

## 3 Applications

As an application of Theorem 2.1, we extend the Hadamard three circles theorem to our case.

**Theorem 3.1.** *Let $f(\mathbf{x})$ be an octonion-valued monogenic function in*

$$\{\mathbf{x} \in \mathbf{R}^8 \big| \, r_1 < |\mathbf{x}| < r_2, \ 0 \leq r_1 < r_2 \leq \infty\}.$$

*Denote*

$$M_q(f,t) = \left( \frac{1}{\sigma_8 t^7} \int_{S(t)} |f(\mathbf{x})|^q d\sigma \right)^{1/q},$$

*where $d\sigma$ is the surface measure on the sphere $S(t)$ of radius $t$ centered at the origin and $\sigma_8$ is the area of the unit sphere in $\mathbf{R}^8$. Then $M_q(|f|^{6/7}, t)$ and $\log M_q(\exp\{|f|^{6/7}\}, t)$ are convex functions of $t^{-6}$ on $(r_1, r_2)$ for any $q \geq 1$. Setting $q = 7/6$ and denoting $\mu(t) = \left(\int_{S(t)} |f| d\sigma\right)^{6/7}$, the first conclusion can be stated as the inequality*

$$(t_2^6 - t_1^6)\mu(t) \leq (t_2^6 - t^6)\mu(t_1) + (t^6 - t_1^6)\mu(t_2)$$

*for all $r_1 < t_1 \leq t \leq t_2 < r_2$.*

The conclusion follows immediately from Theorem 2.1 and [1, Theorem 3.5.7 (i)]. ■

All other results on convexity such as in [1, Sect. 3.5] can also be reformulated for the monogenic functions. We only state the following three lines theorem.

**Theorem 3.2.** *Let $f(y,t)$ be monogenic and upper-bounded on the strip $\mathbf{R}^7 \times (0,1)$. Then*

$1^o$ *$F(t) \equiv \sup_{y \in \mathbf{R}^7} |f(y,t)|$ is convex on $(0,1)$. Moreover,*

$$F(t) \leq (1 - t^{6/7})^{7/6} F(0^+) + tF(1^-). \tag{3.1}$$

$2^o$ *$\int_{\mathbf{R}^7} |f(y,t)|^{6/7} dy$ is convex for $0 < t < 1$, provided that this function is locally bounded on $(0,1)$.* ■

**Remark 3.1.** *Since $(1 - t^{6/7})^{7/6} < 1 - t$ for $0 < t < 1$, (3.1) is stronger than the inequality $F(t) \leq (1-t)F(0) + tF(1)$ – Cf. [8], where another version of the three lines theorem is given.* ■

We also state a Phragmén-Lindelöf theorem for octonion-valued monogenic functions. The limiting growth in this statement is not likely precise; this question is open.

Let $D$ be a domain on the unit sphere $S^7 \subset \mathbf{R}^8$, whose complement with respect to $S^7$ is not a polar set. Then $\lambda(D)$, the smallest eigenvalue of the Dirichlet problem for the Laplace-Beltrami operator in $D$, is positive. Let $\varphi(D)$ be the corresponding (positive) eigenfunction normed in $L^2(D)$ and $\alpha(D)$ the characteristic constant of the domain $D$, i.e., the positive root of the quadratic equation $\alpha(\alpha+6) = \lambda(D)$. For example, if $D$ is a half-sphere in $\mathbf{R}^8$, then $\lambda = 7$ and $\alpha = 1$.

The domain $D$ generates a cone

$$K^D = \left\{ \mathbf{x} \in \mathbf{R}^8 \middle| \, \mathbf{x}/|\mathbf{x}| \in D, \; 0 < |\mathbf{x}| < \infty \right\}.$$

Applying Theorem 2.1 and a Phragmén-Lindelöf theorem from [6], we arrive at the following result.

**Theorem 3.3.** *Let $f$ be an octonion-valued monogenic function in $K^D$. If*

$$\liminf_{|\mathbf{x}|\to\infty} |\mathbf{x}|^{-\alpha(D)} \int_D |f(\mathbf{x})|^{6/7} \varphi(\mathbf{x}/|\mathbf{x}|) d\sigma = 0 \tag{3.2}$$

*and $|f|\Big|_{\partial K^D} \le A$, where $A$ is a constant, then $|f(\mathbf{x})| \le A$ in $K^D$.* ■

We note finally that (3.2) can be replaced by a more transparent but less precise condition

$$|f(\mathbf{x})| = \bar{\bar{o}} \left( |\mathbf{x}|^{(7/6)\alpha(D)} \right), \; |\mathbf{x}| \to \infty.$$

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