Poisson Integral Representation of some Eigenfunctions of Landau Hamiltonian on the Hyperbolic Disc

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Abstract

We characterize some eigenfunctions of Landau Hamiltonian on the hyperbolic disc which are Poisson integrals of square integrable functions at the disc boundary.

1 Introduction

In this Letter, we will be concerned with the second order differential operator in the complex unit disc $\mathbb{D} = \{z \in \mathbf{C}, |z| < 1\}$:

$$\Delta_B := 4\left(1-|z|^2\right)\left(\left(1-|z|^2\right)\frac{\partial^2}{\partial z\partial\overline{z}} + Bz\frac{\partial}{\partial z} - B\overline{z}\frac{\partial}{\partial\overline{z}} + B^2\right)$$

acting in the space $C^\infty(\mathbb{D},\mathbf{C})$ of complex$-$valued $C^\infty-$functions. This operator is obtained from the operator

$$H_B := y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 2iBy\frac{\partial}{\partial x}$$

in the complex upper half plane $\mathbb{H}^2 = \{w = x + iy, \mathbf{C}, x \in \mathbf{R}, y > 0\}$ by

$$\Delta_B f(z) = 4\left(\frac{\overline{w}-i}{w+i}\right)^{-B} H_B\left(\frac{\overline{w}-i}{w+i}\right)^{B} f(\mathcal{C}(w)),$$

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where $f \in C^{\infty}(\mathbb{D},\mathbf{C})$ and $z = \mathcal{C}(w) \in \mathbb{D}$ is the image of $w \in \mathbb{H}^2$ under the Cayley transform : $w \to \mathcal{C}(w) = (w-i)(w+i)^{-1}$.

In physics, the operator H_B represents the Hamiltonian of a uniform magnetic field on $\mathbb{H}^2$ of magnitude proportional to $|B|, B \in \mathbf{R}$. The latter being the *Curl* of the vector potential represented by the $1-form$: $\omega_B = By^{-1}dx$ in the Landau gauge (see [1] and references therein). If $B = 0$, Δ_0 is the Lobachevsky Laplacian on the unit disc $\mathbb{D}$ endowed with the metric $ds^2 = \left(1-|z|^2\right)^{-2}(dx^2+dy^2)$. For $B \neq 0$, we will call Δ_B the Landau Hamiltonian on $\mathbb{D}$.

In [4] , p.582, H.O. Kim and E.G. Kwon have established a necessary and sufficient condition for some eigenfunctions of the Bergman Laplacian on the unit ball of $\mathbf{C}^n$ to be represented by a Poisson integral of square integrable functions at the ball boundary.

Here, we deal with an analogous question in the context of the unit disc $\mathbb{D}$ and for the Landau Hamiltonian Δ_B with the associated Poisson integral transform defined for a C^{∞} function φ on the boundary $\mathbb{T} = \partial\mathbb{D}$ (*see* [2] , p.308) by

$$P_B^{\alpha}[\varphi](z) := \int_{\mathbb{T}} \exp(\alpha Log P(z,\zeta)) \exp(2iB \arg(1-\overline{z}\zeta)) \varphi(\zeta)\, d\sigma(\zeta),$$

where

$$P(z,\zeta) = \frac{1-|z|^2}{\left|1-z\overline{\zeta}\right|^2}, \quad (z,\zeta) \in \mathbb{D}\times\mathbb{T}$$

being the Poisson-Szegö kernel of the unit disc $\mathbb{D}$, $\alpha \in \mathbf{C}$, $Log P(z,\zeta)$ is the principal branch and $d\sigma$ denotes the measure area on $\mathbb{T}$.

We precisely characterize eigenfunctions of Δ_B in $C^{\infty}(\mathbb{D},\mathbf{C})$ with eigenvalues $\mu(\alpha) := 4\alpha(\alpha-1)$, which are Poisson integrals of functions of $L^2(\mathbb{T}, d\sigma)$ in the case when the parameter $\alpha \in \mathbf{C}$ satisfies $\operatorname{Re}\alpha \neq \frac{1}{2}$ and $\alpha \neq |B| - m$, $m \in \mathbf{Z}_+$.

The organization of this Letter is as follows. In section 2, we establish series expansion of eigenfunctions of Δ_B in $C^{\infty}(\mathbb{D},\mathbf{C})$, and we discuss some spectral properties of this operator. Section 3 deals with some required properties of the Poisson integral transform P_B^{α} as its action on spherical harmonics of $\mathbb{T}$ and its injectiveness . In section 4, we give the precise statement of our announced result and we establish its proof.

2 Eigenfunctions of Δ_B

In this section, we shall give the general form of eigenfunctions of Δ_B. For this we have to fix some notations. Let $\alpha \in \mathbf{C}$ be a fixed complex number and let $\mathcal{E}_{\alpha,B}$ denote the space of all eigenfunctions f of Δ_B associated with the eigenvalue $4\alpha(\alpha-1)$. Since the differential operator Δ_B is elliptic on $\mathbb{D}$, therefore the eigenfunctions f are in $C^{\infty}(\mathbb{D},\mathbf{C})$. i.e., $\mathcal{E}_{\alpha,B} = \{f \in \mathbf{C}^{\infty}(\mathbb{D},\mathbf{C}), \Delta_B f = 4\alpha(\alpha-1) f\}$. In the following, we give series expansion in $C^{\infty}(\mathbb{D},\mathbf{C})$ of any function in $\mathcal{E}_{\alpha,B}$.

Proposition 2.1. *For every eigenfunction $f \in \mathcal{E}_{\alpha,B}$ there exists a family of complex numbers $(c_{B,\alpha,k})_{k\in\mathbf{Z}}$ such that*

$$f\left(\rho e^{i\theta}\right) = \left(1-\rho^2\right)^\alpha \sum_{k\in\mathbf{Z}} c_{B,\alpha,k}\ {}_2F_1\left(\alpha+B+\frac{|k|+k}{2}, \alpha-B+\frac{|k|-k}{2}, 1+|k|, \rho^2\right)\rho^{|k|}e^{ik\theta}$$

in $C^\infty(\mathbb{D},\mathbf{C})$, $\rho e^{i\theta}\in D$, $0\leq\rho<1, 0\leq\theta\leq 2\pi$.

Proof. Let $f\in\mathcal{E}_{\alpha,B}$. Then f satisfies the equation

$$\Delta_B f = 4\alpha(\alpha-1) f. \tag{2.1}$$

Since f is C^∞ on $\mathbb{D}$, it can be expanded into its Fourier series as

$$f\left(\rho e^{i\theta}\right) = \sum_{k\in\mathbb{Z}} \gamma_k(\rho)\, e^{ik\theta}, \quad 0\leq\rho<1, 0\leq\theta\leq 2\pi \tag{2.2}$$

where $\rho\to\gamma_k(\rho)$ is C^∞ on $[0,1[$ for each $k\in\mathbf{Z}$. Writing Δ_B into polar coordinates (ρ,θ) :

$$\Delta_B = \left(1-\rho^2\right)\frac{\partial^2}{\partial\rho^2}+\left(1-\rho^2\right)^2\frac{1}{\rho}\frac{\partial}{\partial\rho}+\left(1-\rho^2\right)^2\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}+4iB\left(1-\rho^2\right)\frac{\partial}{\partial\theta}+4B^2\left(1-\rho^2\right)$$

and inserting the expansion (2.2) of $f\left(\rho e^{i\theta}\right)$ in Eq. (2.1), we obtain that every Fourier coefficient $\gamma_k(\rho)$ satisfies the second order differential equation :

$$\rho^2\left(1-\rho^2\right)^2\gamma_k''(\rho)+\left(1-\rho^2\right)^2\rho\gamma_k'(\rho)$$
$$+\left[4\alpha(1-\alpha)\rho^2+4B^2\rho^2\left(1-\rho^2\right)-k^2\left(1-\rho^2\right)^2-4kB\rho^2\left(1-\rho^2\right)\right]\gamma_k(\rho)=0. \tag{2.3}$$

Observe that $\rho=0$ is a singular point and that the characteristic polynomial is $X^2-|k|^2$ whose zeros are $|k|$ and -$|k|$. Then, every solution of this equation is a linear combination of two functions $u_1(\rho), u_2(\rho)$ whose behaviour at $\rho=0$ is respectively like $\rho^{|k|}$ and $\rho^{-|k|}$. Since $\gamma_k(\rho)$ is bounded near zero , we shall look for regular solution of Eq.(2.3) in the form $\gamma_k(\rho)=\rho^{|k|}h_k(\rho^2)$ with $h_k\in\mathbf{C}^\infty([0,1[)$ .We reduce Eq.(2.3) into a standard hypergeometric equation ([3] , $p.1045-1046.$) , by making the change of function $h_k(\rho^2)=(1-\rho^2)^\alpha\,\Psi_k(\rho^2)$. After calculations, we find that $\Psi_k(\rho^2)$ is given , up to a multiplicative constant, by

$${}_2F_1\left(\alpha+B+\frac{1}{2}(|k|+k), \alpha-B+\frac{1}{2}(|k|-k), 1+|k|, \rho^2\right)$$

Consequently, there exists a family of complex numbers $(c_{B,\alpha,k})_{k\in\mathbf{Z}}$ such that

$$f\left(\rho e^{i\theta}\right) = \left(1-\rho^2\right)^\alpha \sum_{k\in\mathbf{Z}} c_{B,\alpha,k}\ {}_2F_1\left(\alpha+B+\frac{|k|+k}{2}, \alpha-B+\frac{|k|-k}{2}, 1+|k|, \rho^2\right)\rho^{|k|}e^{ik\theta}$$

Remark 2.1. One can also consider the operator Δ_B acting in the weighted Hilbert space $\mathcal{H}:= L^2\left(\mathbb{D}, \left(1-|z|^2\right)^{-2} d\nu(z)\right)$, where $d\nu(z)$ being the Lebesgue measure on $\mathbb{D}$. Therefore, general spectral properties of the operator Δ_B acting in $\mathcal{H}$ are similar to those of the operator H_B acting in $L^2\left(\mathbb{H}^2, y^{-2}dx\wedge dy\right)$. Namely, Δ_B is an essentially self-adjoint operator in the Hilbert space $\mathcal{H}$. The spectrum of Δ_B in $\mathcal{H}$ consists of two parts : (i) an absolutely continuous spectrum $]-\infty, 0]$ which corresponds to scattering states, (ii) a point spectrum consisting of a finite number of infinitely degenerate eigenvalues given by $e_m = (|B|-m)(|B|-m-1)$, $0\leq m < |B|-1/2$ when $|B|>1/2$, which correspond to bound states.

3 The integral transform P_B^α

Let us write the integral transform P_B^α associated with Δ_B as

$$P_B^\alpha[\varphi](z) = \int\limits_{\mathbb{T}} \left(\frac{1-|z|^2}{\left|1-z\overline{\zeta}\right|^2}\right)^\alpha \exp\left(2iB\arg\left(1-\overline{z}\zeta\right)\right)\varphi(\zeta)\, d\sigma(\zeta) \tag{3.1}$$

for every continuous function φ on $\mathbb{T}$. At first, one can use direct calculations to establish the following :

Proposition 3.1. *Let $B\in\mathbf{R}$ and $\alpha\in\mathbf{C}$. Then, $P_B^\alpha[\varphi]\in\mathcal{E}_{\alpha,B}$ for every $\varphi\in L^2(\mathbb{T}, d\sigma)$.*

Now, since functions of $L^2(\mathbb{T}, d\sigma)$ can be expanded into series in the basis of spherical harmonics $\{Y_k\}$ of $\mathbb{T}$: $\zeta\to Y_k(\zeta)=\zeta^k, k\in\mathbf{Z}$, we need then to compute the action of P_B^α on these functions $\{Y_k\}$. This is given by the following:

Lemma 3.1. *Let $B\in\mathbf{R}$, $\alpha\in\mathbf{C}$ and $k\in\mathbf{Z}$. Then we have*

$$P_B^\alpha[Y_k](z) = \lambda_k^{\alpha,B}\Phi_k^{\alpha,B}(|z|)\exp(ik\arg z), \qquad z\in\mathbb{D},$$

where

$$\lambda_k^{\alpha,B} := \frac{2\pi\Gamma\left(\alpha+B+\frac{1}{2}(|k|+k)\right)\Gamma\left(\alpha-B+\frac{1}{2}(|k|-k)\right)}{\Gamma(1+|k|)\Gamma(\alpha+B)\Gamma(\alpha-B)} \tag{3.2}$$

and

$$\Phi_k^{\alpha,B}(|z|) := |z|^{|k|}\left(1-|z|^2\right)^\alpha {}_2F_1\left(\alpha+B+\frac{|k|+k}{2}, \alpha-B+\frac{|k|-k}{2}, 1+|k|, |z|^2\right) \tag{3.3}$$

Proof. By (3.1), the action of P_B^α on Y_k can be written as

$$P_B^\alpha[Y_k](z) = \left(1-|z|^2\right)^\alpha \int\limits_{\mathbb{T}} \left(1-\overline{z}\zeta\right)^{-(\alpha-B)}\left(1-z\overline{\zeta}\right)^{-(\alpha+B)}\zeta^k d\sigma(\zeta). \tag{3.4}$$

Making use of the binomial formula

$$(1-x)^{-a} = \sum_{0\leq p<\infty}\frac{\Gamma(a+p)}{\Gamma(a)}\frac{x^p}{\Gamma(1+p)}, \tag{3.5}$$

then, (3.4) transforms to

$$P_B^{\alpha}[Y_k](z) = \left(1-|z|^2\right)^{\alpha} \sum_{0\le j,k<+\infty} \frac{\Gamma(\alpha-B+j)}{\Gamma(\alpha-B)} \frac{\Gamma(\alpha+B+l)}{\Gamma(\alpha+B)} \frac{\overline{z}^j z^l}{j!l!} \int_{\mathbb{T}} \zeta^{k+j}\overline{\zeta}^l d\sigma(\zeta) \tag{3.6}$$

But since

$$\int_{\mathbb{T}} \zeta^{k+j}\overline{\zeta}^l d\sigma(\zeta) = 2\pi\delta_{k+j,l}$$

we set $j = n+\frac{1}{2}(|k|-k)$ and $k = n+\frac{1}{2}(|k|+k)$, therefore the double sum in (3.6) reduces to

$$P_B^{\alpha}[Y_k](z) =$$
$$2\pi\left(1-|z|^2\right)^{\alpha}|z|^k \sum_{0\le n<+\infty} \frac{\Gamma(\alpha-B+n+\frac{1}{2}(|k|-k))}{\Gamma(\alpha-B)} \frac{\Gamma(\alpha+B+n+\frac{1}{2}(|k|+k))}{\Gamma(\alpha+B)}$$
$$\times \frac{1}{\Gamma(n+\frac{1}{2}(|k|-k)+1)\Gamma(n+\frac{1}{2}(|k|+k)+1)} \left(|z|^2\right)^n e^{ik\arg z}$$

Recalling the series of the hypergeometric function

$${}_2F_1(a,b,c,x) = \sum_{0\le n<+\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{\Gamma(b+n)}{\Gamma(b)} \frac{\Gamma(c)}{\Gamma(c+n)} \frac{x^n}{n!}$$

(see [3], $p.$1039), we obtain the result.

Proposition 3.2. *The Poisson transform P_B^{α} is injective if and only if $\alpha \neq |B|-m,\ m\in\mathbf{Z}_+$.*

Proof. Let $\varphi \in L^2(\mathbb{T},d\sigma)$ be such that $P_B^{\alpha}[\varphi]=0$. Expanding φ into its Fourier series as : $\varphi(z) = \sum_{k\in\mathbf{Z}} c_k\zeta^k, \zeta\in\mathbb{T}$, $c_k\in\mathbf{C}$ with $\sum_{k\in\mathbf{Z}}|c_k|^2 < +\infty$ then, we can write :

$$P_B^{\alpha}[\varphi](z) = \sum_{k\in\mathbf{Z}} c_k\lambda_k^{\alpha,B}\Phi_k^{\alpha,B}(|z|)\exp(ik\arg z) = 0 \tag{3.6}$$

where $\lambda_k^{\alpha,B}$ and $\Phi_k^{\alpha,B}(|z|)$ are given in (3.2) and (3.3). Now, since $\Phi_k^{\alpha,B}(|z|)$ is a nonvanishing term, then equality (3.6) is equivalent to $\lambda_k^{\alpha,B}=0$ if $c_k\neq 0$. Thus, a necessary and sufficient condition for P_B^{α} to be injective is that $\alpha+B$ and $\alpha-B$ avoid poles of the Gamma function. i.e., $\alpha\neq|B|-m, m\in\mathbf{Z}_+$.

Remark 3.1. If $\alpha=\alpha_m := |B|-m$, $m\in\mathbf{Z}_+$, the integral transform P_B^{α} is noninjective and yet we still have $P_B^{\alpha_m}[\varphi]\in\mathcal{E}_{\alpha_m,B}$, for all $\varphi\in L^2(\mathbb{T},d\sigma)$. In this case it would be of interest to characterize all those functions in $L^2(\mathbb{T},d\sigma)$ which are mapped via $P_B^{\alpha_m}$ into the space $\mathcal{E}_{\alpha_m,B}\cap\mathcal{H}$ of bound states associated with a hyperbolic Landau level in $\mathbb{D}$ when $|B|>\frac{1}{2}$. For instance, images of spherical harmonics $(Y_k)_{k\in\mathbf{Z}}$ under $P_B^{\alpha_m}$ belong to $\mathcal{E}_{\alpha_m,B}\cap\mathcal{H}$. This is due to the fact that the hypergeometric function arising in the expression of $P_B^{\alpha_m}[Y_k](z)$ is always a polynomial function in the variable $|z|^2, z\in\mathbb{D}$, therefore one can easily establish that the norm $\|P_B^{\alpha_m}[Y_k]\|_{\mathcal{H}}$ is finite.

4 A characterization theorem

In this section, we shall establish the following characterization theorem

Theorem 4.1. *Let* $\alpha \in \mathbf{C}$ *with* $Re\alpha \neq \frac{1}{2}$ *and* $\alpha \neq |B| - m$, $m \in \mathbf{Z}_+$. *Then, a function* $f : D \to C$ *satisfies* $f = P_B^\alpha [\varphi]$ *for a certain* $\varphi \in L^2(\mathbb{T},\ d\sigma)$ *if and only if* $\Delta_B f = \mu(\alpha) f$ *and*

$$\mathcal{N}(f) := \sup_{0 \leq \rho < 1} \left(\left(1 - \rho^2\right)^{|1 - 2\operatorname{Re}\alpha| - 1} \int_{\mathbb{T}} |f(\rho\omega)|^2 \, d\sigma(\omega) \right) < +\infty$$

Proof. We deal the case $\operatorname{Re}\alpha < \frac{1}{2}$. Let $f : \mathbb{D} \to \mathbf{C}$ be such that $f = P_B^\alpha [\varphi]$ with $\varphi \in L^2(\mathbb{T}, d\sigma)$. By proposition 3.1, we have that $\Delta_B f = \mu(\alpha) f$. Next, to prove that the quantity $\mathcal{N}(f)$ is finite, we start by the inequality

$$|f(z)| \leq \int_{\mathbb{T}} \left(\frac{1 - |z|^2}{\left|1 - z\overline{\zeta}\right|^2} \right)^{\operatorname{Re}\alpha} |\varphi(\zeta)| \, d\sigma(\zeta). \tag{4.1}$$

Set $z = \rho\omega$ where $\rho \in [0, 1[$ and $\omega \in \mathbb{T}$ are polar coordinates, then we can write inequality (4.1) as

$$|f(\rho\omega)| \leq (\phi_{\rho,\alpha} * |\varphi|)(\omega) \tag{4.2}$$

where the convolution is taken in $\mathbb{T}$ and

$$\phi_{\rho,\alpha}(\zeta) := \left(\frac{1 - \rho^2}{|1 - \rho\zeta|^2} \right)^{\operatorname{Re}\alpha}, \zeta \in \mathbb{T}.$$

We apply Hausdorff-Young inequality to the convolution in (4.2) :

$$\|\phi_{\rho,\alpha} * |\varphi|\| \leq \|\phi_{\rho,\alpha}\|_{L^1(\mathbb{T})} \|\varphi\|_{L^2(\mathbb{T})}. \tag{4.3}$$

This leads us to compute the L^1−norm of $\phi_{\rho,\alpha}$. For this, we make use of the binomial formula in (3.5) and we obtain that

$$\begin{aligned} \|\phi_{\rho,\alpha}\|_{L^1(\mathbb{T})} &= \sum_{0 \leq j,k < \infty} \frac{\Gamma(\operatorname{Re}\alpha + j)}{(\Gamma(\operatorname{Re}\alpha))^2} \frac{(1 - \rho^2)^{\operatorname{Re}\alpha}}{\Gamma(1 + j)\Gamma(1 + k)} \rho^j \rho^k \int_{\mathbb{T}} \zeta^j \overline{\zeta}^k d\sigma(\zeta) \\ &= 2\pi \left(1 - \rho^2\right)^{\operatorname{Re}\alpha} {}_2F_1\left(\operatorname{Re}\alpha, \operatorname{Re}\alpha, 1, \rho^2\right). \end{aligned} \tag{4.4}$$

Now, in view of (4.2), (4.3) and (4.4), we get

$$\int_{\mathbb{T}} |f(\rho\omega)|^2 \, d\sigma(\omega) \leq \left(2\pi \left(1 - \rho^2\right)^{\operatorname{Re}\alpha} {}_2F_1\left(\operatorname{Re}\alpha, \operatorname{Re}\alpha, 1, \rho^2\right)\right)^2 \|\varphi\|_{L^2(\mathbb{T})}^2.$$

Making use of the identity ([3], p.1042) :

$${}_2F_1(a, b, c, 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \ \operatorname{Re} c > \operatorname{Re}(a + b)$$

for the increasing function $\rho \rightarrow_2 F_1 (\operatorname{Re}\alpha, \operatorname{Re}\alpha, 1, \rho^2)$, we obtain the following inequality

$$_2F_1\left(\operatorname{Re}\alpha, \operatorname{Re}\alpha, 1, \rho^2\right) \leq \frac{\Gamma(1-2\operatorname{Re}\alpha)}{(\Gamma(1-\operatorname{Re}\alpha))^2}, \quad 0 \leq \rho < 1.$$

Therefore,

$$\left(1-\rho^2\right)^{-2\operatorname{Re}\alpha} \int_{\mathbb{T}} |f(\rho\omega)|^2 \, d\sigma(\omega) \leq \left(\frac{2\pi\Gamma(1-2\operatorname{Re}\alpha)}{(\Gamma(1-\operatorname{Re}\alpha))^2}\right)^2 \|\varphi\|^2_{L^2(\mathbb{T})} \tag{4.5}$$

and the proof of the necessary condition is completed by taking the sup with respect to $\rho \in [0,1[$ in left side of inequality (4.5).

Conversely, let $f \in \mathcal{E}_{\alpha,B}$ with $\mathcal{N}(f) < +\infty$. By proposition 2.1 there exists a family of complex numbers $(c_{B,\alpha,k})_{k\in\mathbf{Z}}$ such that

$$f\left(\rho e^{i\theta}\right) = \left(1-\rho^2\right)^{\alpha} \sum_{k\in\mathbf{Z}} c_{B,\alpha,k} \; {}_2F_1\left(\alpha+B+\frac{|k|+k}{2}, \alpha-B+\frac{|k|-k}{2}, 1+|k|, \rho^2\right) \rho^{|k|} e^{ik\theta}. \tag{4.6}$$

Setting

$$\psi(\zeta) = \sum_{k\in\mathbf{Z}} c_{B,\alpha,k} \left(\lambda_k^{\alpha,B}\right)^{-1} \zeta^k, \quad \zeta \in \mathbb{T}$$

where $\left(\lambda_k^{\alpha,B}\right)$ are the quantities defined in (3.2), then obviously ψ satisfies $P_B^{\alpha}[\psi] = f$. It remains to prove that ψ belongs to $L^2(\mathbb{T}, d\sigma)$. For this, we apply the Parseval formula in $L^2(\mathbb{T}, d\sigma)$ to the expansion given in (4.6), and we get for each fixed $\rho \in [0,1[$ the estimate :

$$\sum_{k\in\mathbf{Z}} |c_{B,\alpha,k}|^2 \rho^{2|k|} \left|{}_2F_1\left(\alpha+B+\frac{|k|+k}{2}, \alpha-B+\frac{|k|-k}{2}, 1+|k|, \rho^2\right)\right|^2 \leq \mathcal{N}(f) < +\infty. \tag{4.7}$$

From (4.7) we can write for every fixed $l \in \mathbf{Z}_+$ the following estimate

$$\sum_{|k|\leq l} |c_{B,\alpha,k}|^2 \rho^{2|k|} \left|{}_2F_1\left(\alpha+B+\frac{|k|+k}{2}, \alpha-B+\frac{|k|-k}{2}, 1+|k|, \rho^2\right)\right|^2 \leq \mathcal{N}(f) \tag{4.8}$$

Using the functional relation ([3], p.1043)

$$_2F_1(a,b,c,x) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \, {}_2F_1(a,b,a+b-c+1,1-x)$$
$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b} \; {}_2F_1(c-a,c-b,c-a-b+1,1-x)$$

we establish by computation the limit :

$$\lim_{\rho\to 1} \rho^{2|k|} \left|{}_2F_1\left(\alpha+B+\frac{|k|+k}{2}, \alpha-B+\frac{|k|-k}{2}, 1+|k|, \rho^2\right)\right|^2$$

$$= \frac{\left|\Gamma\left(1-2\alpha\right)\Gamma\left(1+|k|\right)\right|^2}{\left|\Gamma\left(\alpha+B+\frac{|k|+k}{2}\right)\Gamma\left(\alpha-B+\frac{|k|-k}{2}\right)\right|^2}$$

Now, letting ρ goes to 1 in (4.8) ,we get that

$$\sum_{|k|\leq l}\left|c_{B,\alpha,k}\right|^2\frac{\left|\Gamma\left(1-2\alpha\right)\Gamma\left(1+|k|\right)\right|^2}{\left|\Gamma\left(\alpha+B+\frac{|k|+k}{2}\right)\Gamma\left(\alpha-B+\frac{|k|-k}{2}\right)\right|^2}\leq\mathcal{N}\left(f\right)$$

and in view of the expression of $\lambda_k^{\alpha,B}$, we can also write

$$\sum_{|k|\leq l}\left|\left(\lambda_k^{\alpha,B}\right)^{-1}c_{B,\alpha,k}\right|^2\leq\frac{\left|\Gamma\left(\alpha+B\right)\Gamma\left(\alpha-B\right)\right|^2}{\left|\Gamma\left(1-2\alpha\right)\right|^2}\mathcal{N}\left(f\right),\ \text{for all } l\in\mathbf{Z}_+$$

This proves that $\psi\in L^2\left(\mathbb{T},d\sigma\right)$.

Making use of the identity ([3] , p.1043)

$${}_2F_1\left(a,b,c,x\right)=\left(1-x\right)^{c-a-b}\ {}_2F_1\left(c-a,c-b,c,x\right),$$

we treat the case when $\operatorname{Re}\alpha>\frac{1}{2}$ in a similar manner. We get

$$\left(1-\rho^2\right)^{-2(1-\operatorname{Re}\alpha)}\int_{\mathbb{T}}\left|f\left(\rho\omega\right)\right|^2d\sigma\left(\omega\right)\leq\left(\frac{2\pi\Gamma\left(2\operatorname{Re}\alpha-1\right)}{\left(\Gamma\left(\operatorname{Re}\alpha\right)\right)^2}\right)^2\left\|\varphi\right\|^2_{L^2(\mathbb{T})}$$

as analog of (4.5). And as analog of (4.6), we write

$$f\left(\rho e^{i\theta}\right)=\left(1-\rho^2\right)^{1-\alpha}$$
$$\sum_{k\in\mathbf{Z}}c_{B,\alpha,k}\ {}_2F_1\left(1-\alpha+B+\frac{|k|+k}{2},1-\alpha-B+\frac{|k|-k}{2},1+|k|,\rho^2\right)\times\rho^{|k|}e^{ik\theta}.$$

Remark 4.1. We note that for $\operatorname{Re}\alpha=\frac{1}{2}$ there are difficulties in performing a natural condition on eigenfunctions of $\mathcal{E}_{\alpha,B}$ to be in the range of P_B^α.

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