Existence of labeled resolvable block designs *

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Abstract

It is proved in this paper that there exists a labeled almost resolvable 3–fold block design of order v and block size 4 if and only if $v \equiv 1 \pmod 4$, and that there exists a labeled resolvable 3–fold block design of order v and block size 4 if and only if $v \equiv 0 \pmod 4$ with 22 possible exceptions.

1 Introduction

A λ–fold balanced incomplete block design of order v and block size k, denoted by $B(k, \lambda; v)$, is a pair $(X, \mathbf{B})$ where X is a v–set and $\mathbf{B}$ is a collection of k–subsets (called blocks) of X such that each pair of distinct elements of X is contained in exactly λ blocks.

For brevity, a balanced incomplete block design is also called a block design or BIB design.

Let $(X, \mathbf{B})$ be a $B(k, \lambda; v)$. A subset $\mathbf{P}$ of $\mathbf{B}$ is called a parallel class if $\mathbf{P}$ partitions X. A $B(k, \lambda; v)$ is called resolvable and denoted by $RB(k, \lambda; v)$ if all the blocks can be partitioned into parallel classes.

The existence of resolvable block designs has been studied extensively, the interested reader may refer to [3]. In this paper, we study a special class of resolvable block designs — labeled resolvable block designs.

Let $(X, \mathbf{B})$ be a $B(k, \lambda; v)$ where $X = \{a_1, a_2, \cdots, a_v\}$ is a totally ordered v–set with ordering $a_1 < a_2 < \cdots < a_v$. For each block $B = \{x_1, x_2, \cdots, x_k\}$, we may suppose that

$$x_1 < x_2 < \cdots < x_k.$$

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Let

$$\phi : \mathbf{B} \to \mathbb{Z}_\lambda^{\binom{k}{2}}$$

be a mapping where for each $B = \{x_1, x_2, \cdots, x_k\} \in \mathbf{B}$,

$$\phi(B) = (\phi(x_1, x_2), \cdots, \phi(x_1, x_k), \phi(x_2, x_3), \cdots, \phi(x_{k-1}, x_k)),$$
$$\phi(x_i, x_j) \in \mathbb{Z}_\lambda, \quad \forall 1 \le i < j \le k.$$

If there exists a mapping ϕ satisfying the following two conditions:

(i) For each pair $\{x, y\} \subset X$ with $x < y$, let $B_1, B_2, \cdots, B_\lambda$ be the λ blocks containing $\{x, y\}$ and let $\phi(x, y)_i$ be the value of $\phi(x, y)$ corresponding to B_i, $1 \le i \le \lambda$. Then for $1 \le i, j \le \lambda$,

$$\phi(x, y)_i \equiv \phi(x, y)_j \pmod{\lambda}$$

if and only if $i = j$.

(ii) For each block $B = \{x_1, x_2, \cdots, x_k\}$, we have

$$\phi(x_r, x_s) + \phi(x_s, x_t) \equiv \phi(x_r, x_t) \pmod{\lambda}, \quad \forall 1 \le r < s < t \le k.$$

Then $B(k, \lambda; v)$ is called a labeled block design and denoted by $LB(k, \lambda; v)$; its blocks will be denoted in the following form:

$$(x_1, x_2, \cdots, x_k; \phi(x_1, x_2), \cdots, \phi(x_1, x_k), \phi(x_2, x_3), \cdots, \phi(x_{k-1}, x_k)).$$

A labeled $RB(k, \lambda; v)$ is denoted by $LRB(k, \lambda; v)$. Here is for example an $LRB(4, 3; 8)$:

$$\begin{array}{rcl} X & = & \{0, 1, 2, \cdots, 7\}, \\ \mathbf{B} & : & (0,1,3,6;0,0,1,0,1,1), (2,4,5,7;1,2,0,1,2,1); \\ & & (0,1,2,4;2,2,2,0,0,0), (3,5,6,7;1,2,0,1,2,1); \\ & & (1,2,3,5;2,2,2,0,0,0), (0,4,6,7;1,2,1,1,0,2); \\ & & (2,3,4,6;2,2,2,0,0,0), (0,1,5,7;1,2,2,1,1,0); \\ & & (0,3,4,5;1,0,0,2,2,0), (1,2,6,7;1,2,2,1,1,0); \\ & & (1,4,5,6;1,0,0,2,2,0), (0,2,3,7;1,2,0,1,2,1); \\ & & (0,2,5,6;0,1,0,1,0,2), (1,3,4,7;1,2,0,1,2,1). \end{array}$$

The concept of labeled resolvable block design was introduced in [6] and further studied in [8, 9]. It provides a powerful technique in the construction of resolvable group divisible designs.

Let v and λ be two given positive integers and K and M two sets of positive integers. A group divisible design $GD(K, \lambda, M; v)$ is a triple $(X, \mathbf{G}, \mathbf{A})$ where X is a v–set, $\mathbf{G}$ is a set of subsets of X (called groups) forming a partition of X, and $\mathbf{A}$ is a collection of subsets of X (called blocks) such that

(i) $|B| \in K$, $\forall B \in \mathbf{A}$,

(ii) $|G| \in M$, $\forall G \in \mathbf{G}$,

(iii) $|B \cap G| \leq 1$, $\forall B \in \mathbf{A}$, $G \in \mathbf{G}$,

(iv) Each pair of elements of X from distinct groups is contained in precisely λ blocks.

If $K = \{k\}$ and $M = \{m\}$, then a $GD(\{k\}, \lambda, \{m\}; v)$ is called uniform and simply denoted by $GD(k, \lambda, m; v)$.

A $GD(K, \lambda, M; v)$ is called resolvable and denoted by $RGD(K, \lambda, M; v)$ if the set of blocks can be partitioned into parallel classes.

For the application of labeled resolvable block designs in the construction of resolvable group divisible designs, we have the following theorem:

Theorem 1 ([8])
If there exists an $LRB(k, \lambda; v)$ with $\lambda = m$, then there exists an $RGD(k, 1, m; mv)$.

Example 1
By Theorem 1, since an $LRB(4, 3; 8)$ is constructed in the example above, then there exists an $RGD(4, 1, 3; 24)$.

In fact, labeled resolvable block designs played an important role in the construction of $RGD(4, 1, 3; v)$s [9]. We also note that, for quite a long time, not a single example of an $RGD(4, 1, 2; v)$ was known [7]. The first example, an $RGD(4, 1, 2; 32)$, was constructed from an $LRB(4, 2; 16)$ [10].

In the rest of this paper, we will give several direct and recursive constructions for $LRB(4, 3; v)$s. It can be easily seen that if an $LRB(4, 3; v)$ exists, then

$$v \equiv 0 \pmod 4, \quad v \geq 8. \tag{1}$$

Our main purpose is to prove that (1) is also sufficient for the existence of an $LRB(4, 3; v)$, with at most 22 possible exceptions.

2 Labeled almost resolvable block designs

Let $(X, \mathbf{B})$ be a $B(k, \lambda; v)$. A subset $\mathbf{P}$ of $\mathbf{B}$ is called an almost parallel class if $\mathbf{P}$ forms a partition of $X \setminus \{x\}$ for some $x \in X$. A $B(k, \lambda; v)$ is called almost resolvable and denoted by $ARB(k, \lambda; v)$ if $\mathbf{B}$ can be partitioned into almost parallel classes.

A labeled $ARB(k, \lambda; v)$ is denoted by $LARB(k, \lambda; v)$.

Labeled almost resolvable block designs will be used later in the construction of labeled resolvable block designs. In this section we will completely determine the existence of $LARB(4, 3; v)$.

Lemma 1 ([8])
If there exists an $LARB(4, 3; v)$, then

$$v \equiv 1 \pmod 4. \tag{2}$$

To prove that (2) is also sufficient for the existence of an $LARB(4,3;v)$, we need the concept of pairwise balanced design (briefly PBD).

Let v and λ be given positive integers and K be a set of positive integers, a pairwise balanced design $B(K,\lambda;v)$ is a pair $(X,\mathbf{B})$ where X is a v–set and $\mathbf{B}$ is a collection of subsets (called blocks) of X such that $|B| \in K$ for each $B \in \mathbf{B}$ and each pair of distinct elements of X is contained in precisely λ blocks.

In this paper, we only need PBD designs with $\lambda = 1$.

For a given set K of positive integers, let

$$B(K) = \{v \mid \exists \text{ a } B(K,1;v)\}.$$

K is called PBD–closed if $B(K) = K$.

Let

$$LAB^*(4,3) = \{v \mid \exists \text{ an } LARB(4,3;v)\}.$$

Lemma 2 ([8])
$LAB^*(4,3)$ is a PBD–closed set.

Lemma 3 ([5])
Let

$$H^4 = \{n \mid n \equiv 1 \pmod 4\},$$

then H^4 is a PBD–closed set and

$$H^4 = B(\{5,9,13,17,29,33\}).$$

Lemma 4 ([8])
If $q \equiv 1 \pmod 4$ and q is a prime power, then $q \in LAB^*(4,3)$.

Theorem 2
There exists an $LARB(4,3;v)$ if and only if

$$v \equiv 1 \pmod 4.$$

Proof. By Lemma 1, It is equivalent to prove that

$$LAB^*(4,3) = H^4. \tag{3}$$

Since $LAB^*(4,3)$ and H^4 are PBD–closed sets and $H^4 = B(\{5,9,13,17,29,33\})$, then it is sufficient to prove that

$$\{5,9,13,17,29,33\} \subset LAB^*(4,3).$$

By Lemma 4, there exists an $LARB(4,3;v)$ for $v = 5,\ 9\ 13,\ 17$ and 29. We form an $LARB(4,3;33)$ below. Let $X = \mathbb{Z}_{33}$ with ordering $0 < 1 < \cdots < 32$.

$$\mathbf{B}:\ \left.\begin{array}{l}
(i+1, i+8, i+18, i+19; 1,0,0,2,2,0),\\
(i+2, i+3, i+24, i+31; 1,2,2,1,1,0),\\
(i+4, i+13, i+21, i+23; 2,1,0,2,1,2),\\
(i+5, i+11, i+17, i+20; 1,0,2,2,1,2),\\
(i+6, i+10, i+26, i+29; 2,0,0,1,1,0),\\
(i+7, i+12, i+15, i+16; 0,1,0,1,0,2),\\
(i+9, i+22, i+28, i+30; 2,2,2,0,0,0),\\
(i+14, i+25, i+27, i+32; 0,1,2,1,2,1).
\end{array}\right\}\ i \in \mathbb{Z}_{33}$$

This completes the proof. ∎

3 Labeled resolvable transversal designs

A $GD(k, \lambda, m; v)$ with $v = km$ is called a transversal design and denoted by $TD(k, \lambda, m)$. A resolvable $TD(k, \lambda, m)$ is denoted by $RTD(k, \lambda, m)$. It is well known that the existence of a $TD(k, 1, m)$ is equivalent to the existence of an $RTD(k-1, 1, m)$ and equivalent to the existence of $k-2$ mutually orthogonal latin squares of order m.

As for labeled resolvable block designs, we may also give labelings to the blocks of a resolvable transversal design and define the concept of labeled transversal design.

Let $(X, \mathbf{G}, \mathbf{B})$ be an $RTD(k, \lambda, m)$ with $\mathbf{G} = \{G_i \mid 1 \le i \le k\}$ and $X = \bigcup_{i=1}^{k} G_i$ be a partially ordered set such that for any $x, y \in X$, $x < y$ if and only if $x \in G_i$ and $y \in G_j$, $1 \le i < j \le k$. If there exists a mapping

$$\phi : \mathbf{B} \to \mathbb{Z}_{\lambda}^{\binom{k}{2}}$$

satisfying the two conditions in the definition of labeled block designs, then the $RTD(k, \lambda, m)$ is called a labeled resolvable transversal design and denoted by $LRTD(k, \lambda, m)$.

What we need in this paper are labeled resolvable transversal designs with block size 4 and $\lambda = 3$. First we form an $LRTD(4, 3, m)$ for each $m \in \{3, 5, 8\}$.

Lemma 5
There exists an $LRTD(4, 3, 3)$.

Proof. Let $X = \bigcup_{1 \le i \le 4} G_i$, $\mathbf{G} = \{G_1, G_2, G_3, G_4\}$, where

$$G_i = \{(i, j) \mid j \in \mathbb{Z}_3\}, \quad i = 1, 2, 3, 4.$$

Let $\mathbf{B}$ be the union of the following 9 parallel classes:

$$\begin{aligned}
&\{((1,j),(2,j),(3,j),(4,j);1,2,0,1,2,1) \mid j \in \mathbb{Z}_3\};\\
&\{((1,j),(2,j),(3,j+1),(4,j+1);2,2,2,0,0,0) \mid j \in \mathbb{Z}_3\};\\
&\{((1,j),(2,j),(3,j+2),(4,j+2);0,2,1,2,1,2) \mid j \in \mathbb{Z}_3\};\\
&\{((1,j),(2,j+1),(3,j),(4,j+1);0,0,0,0,0,0) \mid j \in \mathbb{Z}_3\};\\
&\{((1,j),(2,j+1),(3,j+1),(4,j+2);1,0,2,2,1,2) \mid j \in \mathbb{Z}_3\};\\
&\{((1,j),(2,j+1),(3,j+2),(4,j);2,0,1,1,2,1) \mid j \in \mathbb{Z}_3\};\\
&\{((1,j),(2,j+2),(3,j),(4,j+2);2,1,0,2,1,2) \mid j \in \mathbb{Z}_3\};\\
&\{((1,j),(2,j+2),(3,j+1),(4,j);0,1,2,1,2,1) \mid j \in \mathbb{Z}_3\};\\
&\{((1,j),(2,j+2),(3,j+2),(4,j+1);1,1,1,0,0,0) \mid j \in \mathbb{Z}_3\}.
\end{aligned}$$

Then $(X, \mathbf{G}, \mathbf{B})$ is an $LRTD(4, 3, 3)$. ∎

Lemma 6
There exists an $LRTD(4, 3, 5)$.

Proof. Let $X = \bigcup_{1 \le i \le 4} G_i$, $\mathbf{G} = \{G_1, G_2, G_3, G_4\}$, where

$$G_i = \{(i, j) \mid j \in \mathbb{Z}_5\}, \quad i = 1, 2, 3, 4.$$

Let $\mathbf{B}$ be the union of the following 15 parallel classes:

$$\begin{array}{l}
\{((1,j),(2,j),(3,j),(4,j);2,0,0,1,1,0) \mid j\in\mathbb{Z}_5\};\\
\{((1,j),(2,j+1),(3,j+2),(4,j+3);0,2,1,2,1,2) \mid j\in\mathbb{Z}_5\};\\
\{((1,j),(2,j+2),(3,j+4),(4,j+1);0,0,0,0,0,0) \mid j\in\mathbb{Z}_5\};\\
\{((1,j),(2,j+3),(3,j+1),(4,j+4);2,0,2,1,0,2) \mid j\in\mathbb{Z}_5\};\\
\{((1,j),(2,j+4),(3,j+3),(4,j+2);1,1,1,0,0,0) \mid j\in\mathbb{Z}_5\};\\
\{((1,j),(2,j),(3,j),(4,j+2);1,1,0,0,2,2) \mid j\in\mathbb{Z}_5\};\\
\{((1,j),(2,j+1),(3,j+2),(4,j);1,1,2,0,1,1) \mid j\in\mathbb{Z}_5\};\\
\{((1,j),(2,j+2),(3,j+4),(4,j+3);2,1,0,2,1,2) \mid j\in\mathbb{Z}_5\};\\
\{((1,j),(2,j+4),(3,j+3),(4,j+4);0,2,0,2,0,1) \mid j\in\mathbb{Z}_5\};\\
\{((1,j),(2,j+3),(3,j+1),(4,j+1);1,1,2,0,1,1) \mid j\in\mathbb{Z}_5\};\\
\{((1,j),(2,j),(3,j),(4,j+3);0,2,2,2,2,0) \mid j\in\mathbb{Z}_5\};\\
\{((1,j),(2,j+1),(3,j+2),(4,j+1);2,0,1,1,2,1) \mid j\in\mathbb{Z}_5\};\\
\{((1,j),(2,j+2),(3,j+4),(4,j+4);1,2,1,1,0,2) \mid j\in\mathbb{Z}_5\};\\
\{((1,j),(2,j+3),(3,j+1),(4,j+2);0,2,2,2,2,0) \mid j\in\mathbb{Z}_5\};\\
\{((1,j),(2,j+4),(3,j+3),(4,j);2,0,1,1,2,1) \mid j\in\mathbb{Z}_5\}.
\end{array}$$

Then $(X,\mathbf{G},\mathbf{B})$ is an $LRTD(4,3,5)$. ■

Lemma 7
There exists an LRTD(4, 3, 8).

Proof. Let $X=\bigcup_{1\le i\le 4} G_i$, $\mathbf{G}=\{G_1,G_2,G_3,G_4\}$, where

$$G_i=\{(i,j)\mid j\in\mathbb{Z}_8\},\quad i=1,2,3,4.$$

For $0\le t\le 7$, let π_t be the following permutations on $\mathbb{Z}_8$:

i	0	1	2	3	4	5	6	7
$\pi_0(i)$	0	1	2	3	4	5	6	7
$\pi_1(i)$	1	0	4	7	2	6	5	3
$\pi_2(i)$	2	4	0	5	1	3	7	6
$\pi_3(i)$	3	7	5	0	6	2	4	1
$\pi_4(i)$	4	2	1	6	0	7	3	5
$\pi_5(i)$	5	6	3	2	7	0	1	4
$\pi_6(i)$	6	5	7	4	3	1	0	2
$\pi_7(i)$	7	3	6	1	5	4	2	0

Let $\mathbf{B}$ be the union of the following 24 parallel classes:

$$\{((1,j),(2,\pi_0(j)),(3,\pi_0(j)),(4,\pi_0(j));2,1,0,2,1,2)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_1(j)),(3,\pi_2(j)),(4,\pi_3(j));2,1,0,2,1,2)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_2(j)),(3,\pi_3(j)),(4,\pi_4(j));2,0,1,1,2,1)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_3(j)),(3,\pi_4(j)),(4,\pi_5(j));2,2,1,0,2,2)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_0(j)),(3,\pi_0(j)),(4,\pi_1(j));0,0,1,0,1,1)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_1(j)),(3,\pi_2(j)),(4,\pi_7(j));0,0,2,0,2,2)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_2(j)),(3,\pi_3(j)),(4,\pi_2(j));0,2,0,2,0,1)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_3(j)),(3,\pi_4(j)),(4,\pi_6(j));0,1,0,1,0,2)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_0(j)),(3,\pi_0(j)),(4,\pi_5(j));1,2,2,1,1,0)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_1(j)),(3,\pi_2(j)),(4,\pi_2(j));1,2,2,1,1,0)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_2(j)),(3,\pi_3(j)),(4,\pi_7(j));1,1,0,0,2,2)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_3(j)),(3,\pi_4(j)),(4,\pi_0(j));1,0,2,2,1,2)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_4(j)),(3,\pi_5(j)),(4,\pi_6(j));1,1,1,0,0,0)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_5(j)),(3,\pi_6(j)),(4,\pi_7(j));2,2,1,0,2,2)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_6(j)),(3,\pi_7(j)),(4,\pi_1(j));2,2,2,0,0,0)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_7(j)),(3,\pi_1(j)),(4,\pi_2(j));0,1,1,1,1,0)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_4(j)),(3,\pi_5(j)),(4,\pi_5(j));0,2,0,2,0,1)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_5(j)),(3,\pi_6(j)),(4,\pi_3(j));1,0,1,2,0,1)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_6(j)),(3,\pi_7(j)),(4,\pi_0(j));1,0,1,2,0,1)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_7(j)),(3,\pi_1(j)),(4,\pi_4(j));1,0,0,2,2,0)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_4(j)),(3,\pi_5(j)),(4,\pi_1(j));2,0,0,1,1,0)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_5(j)),(3,\pi_6(j)),(4,\pi_4(j));0,1,2,1,2,1)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_6(j)),(3,\pi_7(j)),(4,\pi_6(j));0,1,2,1,2,1)\mid j\in\mathbb{Z}_8\};$$
$$\{((1,j),(2,\pi_7(j)),(3,\pi_1(j)),(4,\pi_3(j));2,2,2,0,0,0)\mid j\in\mathbb{Z}_8\}.$$

Then $(X,\mathbf{G},\mathbf{B})$ is an $LRTD(4,3,8)$. ∎

For the application of labeled resolvable transversal designs in the construction of $LRB(4,3;v)$s, we have the following theorem:

Theorem 3
If there exists an $RGD(4,1,m;v)$, an $LRTD(4,3,t)$ and an $LRB(4,3;tm)$, then there exists an $LRB(4,3;tv)$.

Proof. Let $(X,\mathbf{G},\mathbf{A})$ be an $RGD(4,1,m;v)$. Assign to each point $x\in X$ weight t, i.e., x may be considered as a t–set $x=\{x_1,x_2,\cdots,x_t\}$. Let $\mathbf{P}$ be an arbitrary parallel class of the $RGD(4,1,m;v)$. Let $B=\{x,y,z,w\}$ be a block of $\mathbf{P}$. Form an $LRTD(4,3,t)$ with $\{x_1,\cdots,x_t\}$, $\{y_1,\cdots,y_t\}$, $\{z_1,\cdots,z_t\}$ and $\{w_1,\cdots,w_t\}$ as groups, and let $Q_1(B),Q_2(B),\cdots,Q_{3t}(B)$ be the parallel classes of the $LRTD(4,3,t)$. Let

$$Q_i(\mathbf{P})=\bigcup_{B\in\mathbf{P}}\mathbf{Q}_i(B),\quad 1\le i\le 3t.$$

Then each $Q_i(\mathbf{P})$ is a parallel class of the desired $LRB(4,3;tv)$. Let $\mathbf{B}_1$ be the union of all such parallel classes.

For each group $G\in\mathbf{G}$, form an $LRB(4,3;tm)$ on the set

$$\bigcup_{x\in G}\{x_1,x_2,\cdots,x_t\}$$

Let $\mathbf{R}_1(G), \mathbf{R}_2(G), \cdots, \mathbf{R}_{tm-1}(G)$ be the parallel classes of the $LRB(4,3;tm)$. Let

$$\mathbf{R}_i = \bigcup_{G \in \mathbf{G}} \mathbf{R}_i(G), \quad 1 \leq i \leq tm-1$$

Then each $\mathbf{R}_i$ is also a parallel class of the desired $LRB(4,3;tm)$. Let

$$\mathbf{B}_2 = \bigcup_{i=1}^{tm-1} \mathbf{R}_i, \quad \mathbf{B} = \mathbf{B}_1 \cup \mathbf{B}_2,$$

and let

$$Y = \bigcup_{x \in X} \{x_1, x_2, \cdots, x_t\},$$

then $(Y, \mathbf{B})$ is an $LRB(4,3;tv)$. This completes the proof. ∎

Theorem 4
If there exist an $RGD(4,1,m;v)$, an $LRTD(4,3,t)$ and an $LARB(4,3;tm)$, then there exists an $LRB(4,3;tv)$.

Proof. Let $(X, \mathbf{G}, \mathbf{A})$ be an $RGD(4,1,m;v)$. Assign to each point weight t. We may form the set $\mathbf{B}_1$, which can be partitioned into parallel classes, as in Theorem 3.

For each group $G \in \mathbf{G}$, let $(S(G), \mathbf{B}(G))$ be an $LARB(4,3;tm)$ where

$$S(G) = \bigcup_{x \in G} \{x_1, x_2, \cdots, x_t\}.$$

For each $a \in S(G)$, let $R_a(G)$ be the almost parallel class missing a.

Let $\mathbf{Q}_0$ be a fixed parallel class of $\mathbf{B}_1$. For an arbitrary block $B = \{a,b,c,d\} \in \mathbf{Q}_0$, if $a \in S(G_1)$, $b \in S(G_2)$, $c \in S(G_3)$, $d \in S(G_4)$, let

$$R(B) = R_a(G_1) \cup R_b(G_2) \cup R_c(G_3) \cup R_d(G_4) \cup B.$$

Let

$$\mathbf{B}_2 = \mathbf{Q}_0 \cup \{ \bigcup_{G \in \mathbf{G}} \mathbf{B}(G),$$

then $\mathbf{B}_2$ can also be partitioned into parallel classes. Now let

$$\begin{aligned} Y &= \bigcup_{x \in X} \{x_1, x_2, \cdots, x_t\}, \\ \mathbf{B} &= \mathbf{B}_2 \cup \{\mathbf{B}_1 \setminus \mathbf{Q}_0\}, \end{aligned}$$

then $(Y, \mathbf{B})$ is an $LRB(4,3;tv)$ as required. This completes the proof. ∎

4 Further recursive constructions

To prove our main theorem, we also need the following constructions for labeled resolvable designs.

Theorem 5
If there is an $RGD(k,1,m;v)$ such that there exist an $LRB(4,3;k)$ and an $LRB(4,3;m)$, then there exists an $LRB(4,3;v)$.

Proof. Let $(X,\mathbf{G},\mathbf{A})$ be an $RGD(k,1,m;v)$ and let $\mathbf{P}$ be an arbitrary parallel class. For each block $B \in \mathbf{P}$, form an $LRB(4,3;k)$ on the k–set B and let $\mathbf{Q}_1(B),\cdots,\mathbf{Q}_{k-1}(B)$ be the parallel classes. Let

$$\mathbf{Q}_i(\mathbf{P}) = \bigcup_{B\in\mathbf{P}} \mathbf{Q}_i(B), \quad 1 \le i \le k-1.$$

Then each $\mathbf{Q}_i(\mathbf{P})$ is a parallel class of the desired $LRB(4,3;v)$.

For each group $G \in \mathbf{G}$, form an $LRB(4,3;m)$ on the m–set G and let $\mathbf{R}_1(G),\cdots,\mathbf{R}_{m-1}(G)$ be the parallel classes. Let

$$\mathbf{R}_i = \bigcup_{G\in\mathbf{G}} \mathbf{R}_i(G), \quad 1 \le i \le m-1.$$

Then each $\mathbf{R}_i$ is also a parallel class. Now let $\mathbf{B}$ be the union of all the parallel classes $\mathbf{R}_i$, $1 \le i \le m-1$, and all the parallel classes $\mathbf{Q}_i(\mathbf{P})$, $1 \le i \le k-1$, for all $\mathbf{P}$. It can be easily verified that $(X,\mathbf{B})$ is an $LRB(4,3;v)$. This completes the proof. ■

Similarly, using the technique in the proof of Theorem 4, we have the following constructions.

Theorem 6
If there is an $RGD(k,1,m;v)$ such that there exist an $LRB(4,3;k)$ and an $LARB(4,3;m)$, then there exists an $LRB(4,3;v)$.

Theorem 7
If there is an $RTD(5,1,m)$ such that there exists an $LRB(4,3;m)$, then there exists an $LRB(4,3;5m)$.

Theorem 8 ([9])
If there is a $GD(K,1,M;v)$ such that there exist an $LARB(4,3;k)$ for each $k \in K$, and an $LRB(4,3;m+v_0)$ containing an $LRB(4,3;v_0)$ as a subdesign for each $m \in M$, then there exists an $LRB(4,3;v+v_0)$.

5 Main results

The following lemmas will be used in proving our main theorem.

Lemma 8
There exists an $LRB(4,3;v)$ for each $v \in \{8,12,16,20,24,28,32\}$.

Proof. The existence of an $LRB(4,3;v)$ for $v \in \{8,12,24\}$ was proved in [9]. An $LRB(4,3;28)$ was constructed in [10]. Form an $LRB(4,3;8)$ on each group of the $LRTD(4,3;8)$ in Lemma 7, this gives an $LRB(4,3;32)$. Form an $LARB(4,3;5)$ on each group of the $LRTD(4,3,5)$ given in Lemma 6 and use the technique in the proof of Theorem 4, we get an $LRB(4,3;20)$. Finally, we form an $LRB(4,3;16)$ on $X = Z_{15} \cup \{\infty\}$ as follows:

$$\left.\begin{array}{r}\{i, 2+i, 3+i, 8+i; 2,2,2,0,0,0\}\\ \{7+i, 11+i, 12+i, 14+i; 1,2,2,1,1,0\}\\ \{4+i, 6+i, 10+i, 13+i; 1,1,1,0,0,0\}\\ \{1+i, 5+i, 6+i, \infty; 2,1,0,2,1,2\}\end{array}\right\} i \in Z_{15}.$$

■

Lemma 9
There exists an $LRB(4,3;36t+12)$ for each $t \geq 1$.

Proof. It is well known [4] that there is an $RGD(4,1,4;12t+4)$ for each $t \geq 1$. Since there exist an $LRTD(4,3,3)$ by Lemma 5 and an $LRB(4,3;12)$ by Lemma 8, then the conclusion follows from Theorem 3. ■

Similarly, the following two results follow from the existence of an $LRTD(4,3,5)$ and an $LRB(4,3;20)$, or an $LRTD(4,3,8)$ and an $LRB(4,3;32)$.

Lemma 10
There exists an $LRB(4,3;60t+20)$ for each $t \geq 1$.

Lemma 11
There exists an $LRB(4,3;96t+32)$ for each $t \geq 1$.

Lemma 12
If there exists an $LRB(4,3;v)$, then there exists an $LRB(4,3;9v)$.

Proof. By Theorem 1, if there is an $LRB(4,3;v)$, then there exists an $RGD(4,1,3;3v)$. Since there exist an $LRTD(4,3,3)$ and an $LARB(4,3;9)$, the conclusion then follows from Theorem 4. ■

Lemma 13
If there is an $RTD(k,1,m)$ such that there exist an $LRB(4,3;k)$ and an $LRB(4,3;m)$ (or $LARB(4,3;m)$), then there exists an $LRB(4,3;km)$.

Proof. These are special cases of Theorem 5 and Theorem 6. ■

Lemma 14
If there is a $TD(k,1,m)$, an $LARB(4,3;k)$ and an $LRB(4,3;m+1)$, then there exists an $LRB(4,3;km+1)$.

Proof. This is a special case of Theorem 8. ■

Let E be the following set of 22 integers:

44,	52,	68,	88,	92,	112,	124,	132,	152,	164,
184,	188,	208,	212,	220,	268,	284,	292,	304,	308,
312,	788.								

Lemma 15
There exists an $LRB(4,3;v)$ *if* $v \equiv 0 \pmod 4$, $8 \le v \le 320$, $v \notin E$, *or* $v \in \{772, 780, 784, 792, 796, 804\}$.

Proof. Let
$$LRB(4,3) = \{v \mid \exists \text{ an } LRB(4,3;v)\}.$$
By Lemmas 9–11, we have
$$\{48, 84, 120, 156, 192, 228, 264, 300, 804, 80, 140, 200, 260, 320, 128, 224\} \subset LRB(4,3).$$
By Lemma 12, we have
$$\{72, 108, 144, 180, 216, 252, 288\} \subset LRB(4,3).$$
In Lemma 13, let $k = 8$, then we have
$$\{64, 104, 136, 232, 256, 296\} \subset LRB(4,3).$$
Let $k = 12$ or 16, we have
$$\{204, 272, 784\} \subset LRB(4,3).$$
In lemma 14, let $k = 5$, we have
$$\{36, 56, 76, 96, 116, 176, 196, 236, 276, 316, 796\} \subset LRB(4,3).$$
Let $k = 9$ or 13, we have
$$\{172, 244, 248\} \subset LRB(4,3).$$
In Theorem 7, let $k = 5$, then we have
$$\{40, 60, 100, 160, 240, 280, 780\} \subset LRB(4,3).$$

From the $LRTD(4,3,8)$, we may form an $LRB(4,3;32)$ containing an $LRB(4,3;8)$ as a subsystem. From a $TD(5,1,7)$ we may form an $LRB(4,3,36)$ containing an $LRB(4,3;8)$. From an $RTD(5,1,8)$ we may form an $LRB(4,3;40)$ containing an $LRB(4,3;8)$. Since $148 = 5 \times 28 + 8$, $168 = 5 \times 32 + 8$, then by Theorem 8, we may form an $LRB(4,3;148)$ from a $TD(5,1,28)$, an $LRB(4,3;168)$ from a $TD(5,1;32)$.

We may form an $LRB(4,3;156)$ containing an $LRB(4,3;32)$ from a $TD(5,1,31)$, and so there is an $LRB(4,3;156)$ containing an $LRB(4,3;8)$. As there is a $GD(5,1,\{148,24\};764)$ [2], then by Theorem 8, there exists an $LRB(4,3;772)$. Finally, for $v = 792$, since there is an $RGD(8,1,8;792)$, then there exists an $LRB(4,3;792)$, by Theorem 5. This completes the proof. ■

Lemma 16 ([1])
If there is a $TD(17,1,n)$, *then there is a* $GD(K,1,M;v)$ *where* $K = \{5,17\}$, $M = \{n, n+4m_1, n+4m_2\}$, $v = 17n+4(m_1+m_2)$, *for any* m_1, m_2 *satisfying* $0 \le m_1,\ m_2 \le n$.

Now we are ready to prove our main theorem.

Theorem 9
If $v \equiv 0 \pmod 4$, $v \geq 8$ *and* $v \notin E$, *then there exists an* $LRB(4,3;v)$.

Proof. For $v \leq 320$, or $772 \leq v \leq 804$, see Lemma 15. Suppose $v \geq 324$. First, let $n = 19$ and $m_1,\ m_2 \in \{0,1,2,3,4,5,7,9,10,11,13,14,15,16,19\}$ in Lemma 16. Then there is a $GD(K,1,M;v)$ where $K = \{5,17\}$, $M = \{19, 19+4m_1, 19+4m_2\}$, $v = 323 + 4(m_1+m_2)$ such that there is an $LARB(4,3;k)$ for each $k \in K$, and there is an $LARB(4,3;m+1)$ for each $m \in M$. By Theorem 8, there exists an $LRB(4,3;v+1)$. This proves that there exists an $LRB(4,3;v)$ for each $v \equiv 0 \pmod 4$, $324 \leq v \leq 464$.

Now let

$$n = 23, 27, 31, 47, 59, 71, 83, 107, 139, 171, 203, 243, 331$$

and

$$3^{2s-1}\cdot 17, 3^{2s-1}\cdot 25, 3^{2s-1}\cdot 37, 3^{2s-1}\cdot 49, 3^{2s}\cdot 23, 3^{2s}\cdot 31, 3^{2s}\cdot 43, \quad s = 2,3,\cdots.$$

Choose m_1 and m_2 appropriately and apply Theorem 8 recursively. This gives an $LRB(4,3;v)$ for each $v \equiv 0 \pmod 4$, $v \geq 324$, $v \notin \{772, 780, 784, 788, 792, 796, 804\}$. Combining this with Lemma 15 completes the proof. ∎

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