Linear Geometries of Baer subspaces

Harm Pralle Johannes Ueberberg

Abstract

In $PG(2,q^2)$, q a prime power, we study the set $\mathcal{T}$ of Baer subplanes that contain a fixed triangle PQR. To construct a linear rank 2–geometry over $\mathcal{T}$, we determine the dihedral groups, their orders and possible extensions that are generated by the involutions of two Baer subplanes of $\mathcal{T}$. If $q+1$ is an odd prime, the $(q+1)^2$ Baer subplanes through the triangle PQR are the points of an affine plane $\mathcal{A}_{q+1}$ of order $q+1$. Since it is necessary for this construction that $q+1$ is an odd prime, the change of the order from q to $q+1$ occurs only for $q = 2^{2^r}$, $r \in \mathbb{N}$, i.e. for the Fermat primes.

Coordinatizing the affine plane $\mathcal{A}_{q+1}$, we show that $\mathcal{A}_{q+1}$ is desarguesian. Finally, we generalize the construction of $AG(d, q+1)$ out of $PG(d,q^2)$ to dimensions $d \geq 2$ constructing the corresponding vector space.

1 Introduction

A *Baer subspace* of a finite desarguesian projective space $PG(d,q^2)$ of square order is a subspace of the same dimension d and order q. Each Baer subspace consists of the fixed elements of $PG(d,q^2)$ of a unique so called *Baer involution* (for properties of Baer subspaces and their intersection configurations see [3], [6], [1], [10]). Using the description of Baer subspaces by involutions, we induce a geometric structure on the set of Baer subspaces. For, we consider the subgroup in the automorphism group $P\Gamma L(d+1,q^2)$ of $PG(d,q^2)$ that is generated by the Baer involutions corresponding

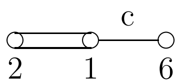
Received by the editors October 1997- In revised form in May 1998.
Communicated by J. Thas.
1991 *Mathematics Subject Classification :* 51A15, 51E15, 51E20, 51E26.
Key words and phrases : Baer subspaces, dihedral groups, affine planes, affine spaces, Fermat primes.

to the set of Baer subspaces under consideration, and we investigate the structure in this group.

In his programmatic paper [2], Cameron outlines the possible consequences of Aschbacher's classification theorem for future research in geometry. Aschbacher's theorem gives rise to eight families $\mathcal{C}_1, ..., \mathcal{C}_8$ of 'large' subgroups of $PGL(d+1, q)$. Cameron indicates how to obtain a geometric interpretation of these groups using the concepts of buildings and diagram geometry. A description of the state of the art is included.

In [7], it is shown that, given two disjoint Baer subplanes B_1 and B_2 of $PG(2, q^2)$, there exists a unique Singer cycle for which the corresponding partition of $PG(2, q^2)$ into Baer subplanes contains B_1 and B_2. Since Singer groups belong to the class $\mathcal{C}_3$, this theorem is a possible geometric interpretation of a group which belongs to $\mathcal{C}_3$ and is a subgroup of a projective linear group of square order.

The Frobenius map $GF(q^3) \to GF(q^3)$, $x \mapsto x^q$, induces a projective collineation φ of order 3 on $PG(2, q) \simeq GF(q^3) \pmod{GF(q)}$. The conjugates of φ are called Frobenius collineations. In [8], the Frobenius collineations of $PG(2, q^2)$ with $q \equiv 2 \bmod 3$ are used to construct a rank 3–geometry with a group operating transitively on its maximal flags. The geometries are called Frobenius spaces of order q. The Frobenius space of order 2 (see [9]) belongs to the diagram



.

As the Frobenius collineations are induced by a field automorphism fixing a subfield, the Frobenius spaces can be seen as a geometric interpretation of a group belonging to $\mathcal{C}_5$ and again being a subgroup of a projective linear group of square order.

The aim of [5] is the geometric interpretation of the dihedral groups generated by any two Baer involutions. It investigates a certain linear structure on sets of Baer subplanes of $PG(2, q^2)$. More precisely, let B and B' be two Baer subplanes with involutions τ, τ', resp. Let $\delta := \tau\tau'$ be the product of the Baer involutions τ and τ' and let $s := ord(\delta)$. For the geometric interpretation of the dihedral group $\mathcal{D} := \langle \tau, \tau' \rangle$, we define the sets

$$\begin{aligned} \mathcal{B}_0(B, B') &:= \{ \delta^i(B) \mid i = 0, 1, ..., s-1 \}, \\ \mathcal{B}_1(B, B') &:= \{ \delta^i(B') \mid i = 0, 1, ..., s-1 \} \text{ and} \\ \mathcal{B}(B, B') &:= \mathcal{B}_0 \cup \mathcal{B}_1 . \end{aligned}$$

Considering the Baer subplanes as points and the sets $\mathcal{B}(B, B')$ as lines, we obtain a structure in the set of Baer subplanes that is induced by any two Baer subplanes. In general, the sets $\mathcal{B}(\cdot, \cdot)$ differ for two different pairs of Baer subplanes (B, B') and (C, C'). On the one hand, this depends on the intersection configuration $B \cap B'$, on the other hand, it depends on the particular site of the two Baer subplanes B and B' in $\mathcal{P} = PG(2, q^2)$.

So far, the definition of $\mathcal{B}(B, B')$ may be used for arbitrary pairs (B, B') of Baer subplanes of $\mathcal{P}$. Furthermore, there is no condition on the dimension d of

$\mathcal{P}$. In this paper, we present a rank 2–geometry in the set $\mathcal{T}$ of Baer subplanes through a common triangle $\Delta = PQR$ of $PG(2, q^2)$ studying the corresponding dihedral groups. In general, it is not linear, but achieves linearity for certain orders q. Theorem 1.1 determines the dihedral group $\mathcal{D}$ and the set $\mathcal{B}(B, B')$ for two Baer subplanes $B, B' \in \mathcal{T}$.

Theorem 1.1. *Let B, B' be two Baer subplanes of $\mathcal{P} = PG(2, q^2)$ with involutions τ, τ', resp., that have a triangle PQR in common. Let $\delta := \tau\tau'$, $s := ord(\delta)$, $\mathcal{D} := \langle \tau, \tau' \rangle$ and let $\mathcal{B}(B, B')$ be defined as above.*

(a) s is a divisor of $q + 1$.

(b) $\mathcal{D}$ is a dihedral group with $|\mathcal{D}| = 2s$. The reflections of $\mathcal{D}$ are the involutions of the Baer subplanes of $\mathcal{B}(B, B')$.

(c) There exists a Baer subplane $\tilde{B} \in \mathcal{T}$ such that $\mathcal{B}(B, B') \subseteq \mathcal{B}(B, \tilde{B})$ and $|\mathcal{B}(B, \tilde{B})| = q + 1$.

Definition 1.2. *Let P, Q and R be three non collinear points of $\mathcal{P} = PG(2, q^2)$ and let $\mathcal{T}$ be the set of Baer subplanes through P, Q and R. The geometry $\mathcal{A}_{q+1}$ is defined as follows:*

- *The points of $\mathcal{A}_{q+1}$ are the Baer subplanes of $\mathcal{T}$.*
- *The lines are the sets $\mathcal{B}(B, B')$ with $|\mathcal{B}(B, B')| = q+1$ for two Baer subplanes $B, B' \in \mathcal{T}$.*
- *Incidence is inclusion.*

Our main result reads as follows:

Theorem 1.3. *Let the geometry $\mathcal{A}_{q+1}$ be defined as in Definition 1.2. If $q + 1$ is a prime number, then $\mathcal{A}_{q+1}$ is a desarguesian affine plane of order $q + 1$.*

A nice example for the presented method is the construction of an affine plane of order 3 out of $PG(2, 4)$. Let PQR be a triangle in $PG(2, 4)$. Each Baer subplane of $PG(2, 4)$ through PQR has exactly one point off the lines PQ, QR and PR. So, there is a bijective map between the points of the geometry $\mathcal{A}_3$ and the points of $PG(2, 4)$ off the triangle lines. We have just to determine the lines of $\mathcal{A}_3$. Nine of the 13 lines of $\mathcal{A}_3$ are induced by the lines of $PG(2, 4)$ through the points P, Q and R different from the triangle lines through these points. The remaining four lines of $\mathcal{A}_3$ cannot be seen in $PG(2, 4)$ without calculating them by means of the dihedral groups.

Finally, the results generalize in $d \geq 2$ dimensions. Considering the set $\mathcal{Q}$ of Baer subspaces of $PG(d, q^2)$ that contain a given basis $\{P_0, P_1, ..., P_d\}$, $\mathcal{Q}$ is the point set of a desarguesian affine space $AG(d, q + 1)$, when $q + 1$ is a prime number.

The paper is organized as follows. In Section 2, we give the necessary information on Baer subspaces and Baer involutions. In Section 3, we prove Theorems 1.1 and

1.3. In Section 4, we generalize our results in $d \geq 2$ dimensions. To do this, we define the geometry Ω_{q+1} that corresponds to the 2–dimensional geometry $\mathcal{A}_{q+1}$, we formulate the results corresponding to the plane case in Theorem 4.2 and we sketch a proof developing the vector space that coordinatizes the affine space Ω_{q+1} if $q+1$ is a prime number.

2 Baer subspaces and dihedral groups

Proposition 2.1. *Let τ, τ' be the involutions of two Baer subspaces B, B' of a finite desarguesian projective space $\mathcal{P}$. There exists a collineation of $\mathcal{P}$ mapping B onto B'. If α is such a collineation, it follows that $\tau' = \alpha\tau\alpha^{-1}$. Furthermore, the product $\delta := \tau\tau'$ is a projectivity.*

Proof. Ueberberg [7, Propositions 2.1 and 2.2] ∎

Proposition 2.2. *Let τ, τ' be the involutions of two Baer subspaces B, B' of a finite desarguesian projective space and let $\delta := \tau\tau'$ and $s := ord(\delta)$. Let τ_k and τ'_k denote the involutions of the Baer subspaces $B_k := \delta^k(B)$ and $B'_k := \delta^k(B')$, resp.*

(a) $\tau_k = \delta^{2k}\tau$ *and* $\tau'_k = \delta^{2k}\tau'$.

(b) For s odd, $\tau'_k = \tau_{k+\frac{s-1}{2}}$.

(c) For s even, there is no $k \in \{0, 1, ..., s-1\}$ such that $\tau' = \tau_k$.

Proof. (a) By Proposition 2.1, the involutions of Baer subspaces of $\mathcal{B}_0(B, B')$ are the conjugates under $\langle\delta\rangle$. It follows

$$\tau_k = \delta^k\tau\delta^{-k} = \delta^k\tau(\tau\tau')^{-k} = \delta^k\tau(\tau'\tau)^k = \delta^k(\tau\tau')^k\tau = \delta^{2k}\tau$$

and similarly $\tau'_k = \delta^{2k}\tau'$ for the involutions of the Baer subspaces of $\mathcal{B}_1(B, B')$.

(b) By *(a)*, we have $\tau'_k = \delta^{2k}\tau' = \delta^{2k-1}(\tau\tau')\tau' = \delta^{2k-1}\tau = \delta^{2(k+\frac{s-1}{2})}\tau = \tau_{k+\frac{s-1}{2}}$, since s is odd.

(c) By *(a)*, it follows $\tau'_k = \delta^{2k}\tau' = \delta^{2k-1}(\tau\tau')\tau' = \delta^{2k-1}\tau$. Since s is even, there is no $k \in \{0, 1, ..., s-1\}$ such that $2k-1 \equiv 0 \bmod s$. Hence there is no k such that $\tau' = \tau_k$. ∎

A *basis* $\mathcal{M}$ of a d–dimensional projective space $\mathcal{P}$ is a set of $d+1$ points generating $\mathcal{P}$. A *frame* of a d–dimensional projective space $\mathcal{P}$ is a set of $d+2$ points of which any $d+1$ points are a basis. It is well known that a Baer subspace of $\mathcal{P} = PG(d, q^2)$ is uniquely determined by a frame.

Proposition 2.3. *In $\mathcal{P} = PG(d, q^2)$, there are $(q+1)^d$ Baer subspaces through a basis.*

Proof. Let the basis be $\mathcal{M} = \{P_0, P_1, ..., P_d\}$. Since a Baer subspace is uniquely determined by a frame, we count the frames that contain the basis $\mathcal{M}$. First, we count the points off the union of the hyperplanes

$$H_i := \langle P_0, ..., P_{i-1}, P_{i+1}, ..., P_d \rangle \ , \ i = 0, ..., d \, .$$

Step 1. There are $(q^2-1)^d$ points in $\mathcal{P} - \bigcup_{i=0}^{d} H_i$.
We coordinatize $\mathcal{P}$ such that the points P_i get the coordinates

$$P_i = (p_0, p_1, ..., p_d) \text{ with } p_i = 1 \text{ and } p_j = 0 \text{ for all } j \neq i, \ i = 0, 1, ..., d \, .$$

Hence the points in $\mathcal{P} - \bigcup_{i=0}^{d} H_i$ are exactly those with nonzero coordinates. There are $(q^2-1)^d$ such points in $\mathcal{P}$.

Similarly, every Baer subspace B through $\mathcal{M}$ contains $(q-1)^d$ points off the hyperplanes H_i, $i = 0, 1, ..., d$.

Step 2. There are $(q+1)^d$ Baer subspaces containing $\mathcal{M}$.
Since a Baer subspace is uniquely determined by a frame, the $(q^2-1)^d$ points of $\mathcal{P}$ off the hyperplanes H_i are partitioned by the Baer subspaces. Since every Baer subspace contains $(q-1)^d$ points off the hyperplanes H_i, $i = 0, 1, ..., d$, there are

$$\frac{(q^2-1)^d}{(q-1)^d} = (q+1)^d$$

Baer subspaces through the basis $\mathcal{M}$. ■

We need the following proposition about prime numbers.

Proposition 2.4. *If q is a power of a prime number and $q+1$ is a prime, there is an $r \in \mathbb{N}$ such that $q = 2^{2^r}$.*

Proof. Since q is a power of a prime and $q+1$ is a prime, it is a power of 2, i.e. $q = 2^h$. Assume that h has an odd factor m, hence $h = mn$ for an $n \in \mathbb{N}$. Since

$$2^h + 1 = 2^{mn} + 1 = (2^n + 1)(2^{n(m-1)} - 2^{n(m-2)} + \cdots + 2^{n2} - 2^n + 1) \ ,$$

h cannot have an odd factor. ■

Remark. Prime numbers of the form $2^{2^r} + 1$, $r \in \mathbb{N}$, are called *Fermat primes*. So far, only five Fermat primes are known, namely for $r = 0, 1, 2, 3, 4$.

3 Proofs of the main results

Notation. For diagonal (3×3)–matrices $A = (a_{i,j})_{1 \leq i,j \leq 3}$ we write $diag(a_{11}, a_{22}, a_{33})$.

Proof of Theorem 1.1. We coordinatize $PG(2, q^2)$ such that $P = (1, 0, 0)$, $Q = (0, 1, 0)$ and $R = (0, 0, 1)$. Since a Baer subplane is uniquely determined by a quadrangle, every point off the triangle lines PQ, QR and PR is in exactly one

Baer subplane of $\mathcal{T}$. Let B be defined by $S = (1, 1, 1)$, and let B' be defined by $S' = (1, a, b)$ with $(1, a, b) \notin (GF(q))^3$.

(a) The Baer involution τ corresponding to B is induced by the Frobenius automorphism

$$\varphi : GF(q^2) \to GF(q^2),\ \varphi(x) = x^q \ .$$

By Proposition 2.1, B' has the involution $\tau' = \alpha\tau\alpha^{-1}$ where α is a projectivity that maps B onto B'. For instance, let α be induced by $diag(1, a, b)$. It follows, that the product $\delta = \tau\tau' = \tau\alpha\tau\alpha^{-1}$ is induced by the linear mapping $D := diag(1, a^{q-1}, b^{q-1})$. Hence we get

$$s = ord(\delta) = ord(D) = lcm(ord(a^{q-1}), ord(b^{q-1})) \ .$$

So, s divides $q + 1$.

(b) One easily checks that $\mathcal{D}$ is a dihedral group of order $2s$. The reflections of $\mathcal{D}$ are the conjugates of τ and τ' under the rotation group $\langle \delta = \tau\tau' \rangle$.

(c) Let t be a primitive element of $GF(q^2)$. There are $e, f \in \mathbb{N}$, such that $a = t^e$ and $b = t^f$. It follows

$$s = ord(\delta) = (q + 1)(\ q + 1\ ,\ (e, f))^{-1} \ .$$

Let $h := (e, f)$ and let Z be the point with coordinates $Z = (1, t^{e/h}, t^{f/h})$. Let $\tilde{B}$ be the Baer subplane in $\mathcal{T}$ that contains Z. Since $\tau(Z) \neq Z$, it is $\tilde{B} \neq B$. The projectivity γ induced by $C := diag(1, t^{e/h}, t^{f/h})$ maps B onto $\tilde{B}$. By Proposition 2.1, the involution $\tilde{\tau}$ of $\tilde{B}$ is $\tilde{\tau} = \gamma\tau\gamma^{-1}$. So, the product $\tilde{\delta} = \tau\tilde{\tau}$ is induced by the mapping $\tilde{D} := diag(1, t^{e/h(q-1)}, t^{f/h(q-1)})$ and it follows

$$ord(\tilde{\delta}) = ord(\tilde{D}) = ord(t^{e/h(q-1)}), ord(t^{f/h(q-1)})) = q + 1 \ .$$

Furthermore, we have $\delta = \tilde{\delta}^h$. So, it follows $\mathcal{D} = \langle \tau, \tau' \rangle = \langle \delta, \tau \rangle \subseteq \langle \tilde{\delta}, \tau \rangle = \langle \tau, \tilde{\tau} \rangle$ and

$$\mathcal{B}(B, B') \subseteq \mathcal{B}(B, \tilde{B}) \ .$$

The cardinality $|\mathcal{B}(B, \tilde{B})|$ is the number of different Baer subplanes in $\mathcal{B}(B, \tilde{B})$. By Proposition 2.2, for $ord(\tilde{\delta})$ odd, it is $\mathcal{B}_0(B, \tilde{B}) = \mathcal{B}_1(B, \tilde{B})$ and $|\mathcal{B}(B, \tilde{B})| = ord(\tilde{\delta}) = q + 1$, for $ord(\tilde{\delta})$ even, we have $\mathcal{B}_0(B, \tilde{B}) \cap \mathcal{B}_1(B, \tilde{B}) = \emptyset$ and $|\mathcal{B}(B, \tilde{B})| = ord(\tilde{\delta}) = q + 1$. ∎

Proof of Theorem 1.3. We prove the Theorem in four steps.

Step 1. $\mathcal{A}_{q+1}$ is an affine plane if $q+1$ is a prime number. The existence of a line through two points B, B' of $\mathcal{A}_{q+1}$ with Baer involutions τ, τ', resp., is given by Theorem 1.1 *(c)*. The uniqueness of the line follows by the condition on $q + 1$ to be a prime number: For, let $\mathcal{B}(C, C')$ be a line containing B and B'. The line $\mathcal{B}(B, B')$ consists of all the Baer subplanes of $\mathcal{T}$ which have involutions in the dihedral group $\mathcal{D} = \langle \tau, \tau' \rangle$. The line $\mathcal{B}(C, C')$ consists of all the Baer subplanes of $\mathcal{T}$ that have involutions in the dihedral group $\mathcal{C} = \langle \gamma, \gamma' \rangle$, where γ and γ' are the involutions of C and C', resp. So, $B, B' \in \mathcal{B}(C, C')$ means $\tau, \tau' \in \mathcal{C}$, i.e. $\mathcal{D} \subseteq \mathcal{C}$. Since $q + 1$ is a prime number, we have $|\mathcal{D}| = |\mathcal{C}| = 2(q+1)$ by Theorem 1.1 *(a)*, i.e. $\mathcal{D} = \mathcal{C}$ and finally $\mathcal{B}(B, B') = \mathcal{B}(C, C')$.

Since by Proposition 2.3 there are $(q+1)^2$ points in $\mathcal{A}_{q+1}$ and since by Theorem 1.1 there are $q+1$ points on every line, the geometry $\mathcal{A}_{q+1}$ is an affine plane of order $q+1$ by [4, Theorem 3.2.4 (b)].

It remains to show that $\mathcal{A}_{q+1}$ is desarguesian. To do so, we shall make use of the so called inversive planes. For definitions, the reader is referred to [4].

For the following, we remind the reader that $q+1$ is assumed to be a prime number.

Step 2. We introduce coordinates for the points of $\mathcal{A}_{q+1}$. Let $g = PQ$ and $h = PR$ be two of the triangle lines in $\mathcal{P} = PG(2, q^2)$. Again, we coordinatize $\mathcal{P}$ such that the three non collinear points P, Q and R get the coordinates $P = (1, 0, 0)$, $Q = (0, 1, 0)$ and $R = (0, 0, 1)$.

A Baer subplane $B \in \mathcal{T}$ is uniquely determined by the two point sets $B \cap g$ and $B \cap h$, which we call *Baer subline of* g and *Baer subline of* h, resp. Since we consider only Baer subplanes that contain the points P, Q and R, the intersections of the point sets of the Baer subplanes of $\mathcal{T}$ with the points of g are elements of the bundle with carriers P and Q, when we consider g as an inversive plane with point set $PG(1, q^2)$ and blocks the sublines $PG(1, q)$. We denote this bundle by

$$\mathcal{X} = \{\ B \cap g \mid B \in \mathcal{T}\ \}\ .$$

Similarly, considering the line h as an inversive plane, the intersections of the point sets of the Baer subplanes of $\mathcal{T}$ with h are elements of the bundle with carriers P and R. We denote it by

$$\mathcal{Y} = \{\ B \cap h \mid B \in \mathcal{T}\ \}\ .$$

So, any pair (X, Y) with $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ determines uniquely a Baer subplane of $\mathcal{T}$ and vice versa. Hence there is a bijective map from $\mathcal{X} \times \mathcal{Y}$ onto $\mathcal{T}$. By means of this map, we introduce coordinates for the point set $\mathcal{T}$ of the geometry $\mathcal{A}_{q+1}$.

According to our coordinatization of $\mathcal{P} = PG(2, q^2)$ in setting $P = (1, 0, 0)$ and $Q = (0, 1, 0)$, the points of $g \backslash \{P, Q\}$ have coordinates $X = (1, x, 0)$ with $x \in GF(q^2)^*$. Hence we may denote them by the elements of $GF(q^2)^*$. So, the set $\mathcal{X}$ can be identified with the factor group $\mathcal{F} = GF(q^2)^*/GF(q)^*$. Similarly, $\mathcal{Y}$ can be identified with the factor group $\mathcal{F}$ denoting the points of $h \backslash \{Q, R\}$ by the elements of $GF(q^2)^*$. Let t be a primitive element of $GF(q^2)$. The set

$$\mathcal{R} = \left\{\ (t^{q-1})^i \mid i = 0, 1, ..., q\ \right\}$$

is a group of representatives of the factor group $\mathcal{F}$ in $GF(q^2)^*$, i.e.

$$\mathcal{F} = \left\{\ (t^{q-1})^i\ GF(q)^* \mid i = 0, 1, ..., q\ \right\}\ .$$

We denote the element $X \in \mathcal{X}$ by the number i, if $(t^{q-1})^i$ is the representative of X in $\mathcal{R}$. So, the unique Baer subplane through X and Y with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ gets the coordinates (i, j), if $(t^{q-1})^i$ is a representative of X and $(t^{q-1})^j$ is a representative of Y.

Step 3. Let (i,j) and (l,k) be two points of $\mathcal{A}_{q+1}$. The line $\langle(i,j),(l,k)\rangle$ is the following set of points:

$$\langle(i,j),(k,l)\rangle = \\ \{\ ((i+n2(i-k)) \bmod (q+1), (j+n2(j-l)) \bmod (q+1)) \mid n = 0,1,...,q\ \}\ .$$

Let B and B' be the two Baer subplanes of $\mathcal{T}$ that have the coordinates (i,j) and (k,l) in $\mathcal{A}_{q+1}$, resp. Furthermore, let τ and τ' be the corresponding Baer involutions. By definition of the coordinates in $\mathcal{A}_{q+1}$, B is the Baer subplane defined by P,Q,R and $(1,(t^{q-1})^i,(t^{q-1})^j)$. Similarly, B' is determined by P,Q,R and $(1,(t^{q-1})^k,(t^{q-1})^l)$. Hence we have

$$\tau = \theta\varphi\ ,$$

where φ denotes the Frobenius automorphism $(z,x,y) \overset{\varphi}{\mapsto} (z^q,x^q,y^q)$ and θ denotes the projectivity induced by $T = diag(1,(t^{q-1})^{2i},(t^{q-1})^{2j})$. Similarly, we have $\tau' = \theta'\varphi$, where θ' is the projectivity induced by $T' = diag(1,(t^{q-1})^{2k},(t^{q-1})^{2l})$.

By definition, the line through the two points $B=(i,j)$ and $B'=(k,l)$ is the set of Baer subplanes with involutions in the dihedral group $\langle\tau,\tau'\rangle$, i.e. the conjugates of τ under the product $\delta = \tau\tau'$ of the two Baer involutions (see Section 1.). The projectivity $\delta = \tau\tau' = \theta\varphi\theta'\varphi$ is induced by

$$D = T(T')^{(q)} = diag(1,(t^{q-1})^{2(i-k)},(t^{q-1})^{2(j-l)})\ .$$

So, the image of B under δ^n, $n = 0,1,...,q$, is determined by the points P,Q,R and

$$\delta^n(1,(t^{q-1})^i,(t^{q-1})^j) = (1,(t^{q-1})^{(i+2n(i-k))},(t^{q-1})^{(j+2n(j-l))})\ .$$

Hence in the coordinates of the affine plane $\mathcal{A}_{q+1}$, $\delta^n(B)$ gets the coordinates

$$((i+2n(i-k)) \bmod (q+1)\ ,\ (j+2n(j-l)) \bmod (q+1))\ .$$

Step 4. We are now able to conclude the proof of Theorem 1.3. By Step 3, a line l of $\mathcal{A}_{q+1}$ can be described as

$$l = \{((x+n\xi) \bmod (q+1),\ (y+nv) \bmod (q+1)) \mid n = 0,1,...,q\ \}$$

with $x,y,\xi,v \in \{0,1,...,q\}$ and $(\xi,v) \neq (0,0)$. Hence $\mathcal{A}_{q+1}$ is desarguesian. ∎

4 Generalization for $d \geq 2$ dimensions

To prove Theorem 4.2 corresponding to Theorem 1.3 in d dimensions with $d \geq 2$, we coordinatize the geometry Ω_{q+1} of Definition 4.1 in a similar way to the previous section. Let us start with the definition of the geometry Ω_{q+1}.

Definition 4.1. *Let $\mathcal{P} = PG(d,q^2)$, $d \geq 2$, and let $\mathcal{M}$ be a basis of $\mathcal{P}$. Let $\mathcal{Q}$ be the set of Baer subspaces of $\mathcal{P}$ that contain $\mathcal{M}$. The geometry Ω_{q+1} is defined as follows:*

- *The points of* Ω_{q+1} *are the Baer subspaces of* $\mathcal{Q}$.
- *The lines of* Ω_{q+1} *are the sets* $\mathcal{B}(B, B')$ *with* $|\mathcal{B}(B, B')| = q + 1$.
- *The incidence is inclusion.*

Theorem 4.2. *Let* B, B' *be two Baer subspaces of* $\mathcal{P} = PG(d, q^2)$, $d \geq 2$, *with involutions* τ, τ', *resp., that have a basis* $\mathcal{M} = \{P_0, P_1, ..., P_d\}$ *in common. Let* $\delta := \tau\tau'$, $s := ord(\delta)$, $\mathcal{D} := \langle \tau, \tau' \rangle$ *and let* $\mathcal{B}(B, B')$ *be defined as above.*

(a) s *is a divisor of* $q + 1$.

(b) $\mathcal{D}$ *is a dihedral group with* $|\mathcal{D}| = 2s$. *The reflections of* $\mathcal{D}$ *are the involutions of the Baer subspaces of* $\mathcal{B}(B, B')$.

(c) There exists a Baer subspace $\tilde{B} \in \mathcal{Q}$ *such that* $\mathcal{B}(B, B') \subseteq \mathcal{B}(B, \tilde{B})$ *and* $|\mathcal{B}(B, \tilde{B})| = q + 1$.

(d) Let Ω_{q+1} *be defined as in Definition 4.1. If* $q + 1$ *is a prime number,* Ω_{q+1} *is a* d*–dimensional desarguesian affine space of order* $q + 1$.

Sketch of a proof. We coordinatize $\mathcal{P}$ such that the points of $\mathcal{M}$ get the coordinates

$$P_i = (p_0, p_1, ..., p_d) \ , \ p_i = 1, p_j = 0 \text{ for all } j \neq i, \ i = 0, 1, ..., d \ .$$

Since all Baer subspaces of $\mathcal{Q}$ contain the points of $\mathcal{M}$, projectivities mapping one Baer subspace on another of $\mathcal{Q}$ may be induced by diagonal matrices. So, the proof of Theorem 4.2 *(a)-(c)* is similar to that of Theorem 1.1, the matrices are diagonal $((d+1) \times (d+1))$–matrices instead of diagonal (3×3)–matrices.

(d) To show that Ω_{q+1} is an $AG(d, q+1)$, when $q + 1$ is prime, we construct the corresponding vector space. Similarly to the case $d = 2$ discussed in Section 3, a Baer subspace through $\mathcal{M}$ is a unique element of the d–dimensional crossproduct of the factor group $\mathcal{F} = GF(q^2)^*/GF(q)^*$. For, consider the d connection lines of the points of the basis $\mathcal{M}$ through one of them, say P_0. Any Baer subspace through $\mathcal{M}$ is uniquely determined by the corresponding Baer sublines on these d lines P_0P_i , $i = 1, 2, ..., d$, hence by a d–tuple of elements of $\mathcal{F}$. So, there is a bijective map of the d–dimensional crossproduct of $\mathcal{F}$ onto the point set of Ω_{q+1}.

In the previous section, we have defined the set

$$\mathcal{R} = \left\{ (t^{q-1})^i \mid i = 0, 1, ..., q \right\} ,$$

where t is a primitive element of $GF(q^2)$. $\mathcal{R}$ is a group of representatives of the factor group $\mathcal{F}$ in $GF(q^2)$. Denoting an element $(t^{q-1})^i GF(q)^*$ of $\mathcal{F}$ by the exponent i of its representative $(t^{q-1})^i \in \mathcal{R}$ and calculating mod$(q+1)$ as we have done it in the previous section, we can consider $\mathcal{F}$ as $\mathbf{Z}_{q+1}$. Hence the d–dimensional crossproduct of $\mathcal{F}$ is a d–dimensional vector space $\mathcal{V}$.

As mentioned before, there is a bijection of the points of the geometry Ω_{q+1} onto the elements of $\mathcal{V}$. We have to show that the lines of Ω_{q+1} are the cosets of the one dimensional subspaces of $\mathcal{V}$. Recall that the points of a line l of Ω_{q+1} are all Baer subspaces of $\mathcal{Q}$ with involutions in one dihedral group $\mathcal{D} = \langle \tau, \tau' \rangle = \langle \delta, \tau \rangle$ where τ, τ' are two Baer involutions and $\delta = \tau\tau'$. Since the Baer subspaces of $\mathcal{Q}$ contain the basis $\mathcal{M}$, by our coordinatization, the projectivity δ is induced by a diagonal $((d+1) \times (d+1))$–matrix. The calculations for a line through two points B and B' of Ω_{q+1} are similar to that of the proof of Proposition 3. Instead of (3×3)–matrices there are $((d+1) \times (d+1))$–matrices. So, a line l of Ω_{q+1} has the form

$$l = \{(x + ny) \bmod (q+1) \mid n = 0, 1, ..., q\} \ , \ x, y \in \mathcal{V} \text{ with } y \neq 0 \ .$$

■

5 Concluding Remark

The uniqueness of a line through two points of the affine plane $\mathcal{A}_{q+1}$ of Theorem 1.3 or in other words, that $\mathcal{F}$ is a field, follows from the condition that $q+1$ is a prime number. By Proposition 2.4, to satisfy this condition q has to be of the form $q = 2^{2^r}$, $r \in \mathbb{N}$. As mentioned there, the prime numbers $2^{2^r} + 1$ are the *Fermat primes*. So far, only five Fermat primes are known, namely those for $r = 0, 1, 2, 3, 4$.

Acknowledgement. Harm Pralle wishes to thank Chris Fisher for a very stimulating time for his Diplomarbeit [5] at the University of Western Ontario out of which this work has been grown. We thank Jörg Eisfeld for some valuable comments.

References

[1] R. C. Bose, J. W. Freeman und D. G. Glynn. On intersections of two Baer Subplanes in a finite projective plane. *Utilitas Mathematica* **17** (1980), 65-77.

[2] P. J. Cameron. Finite geometry after Aschbacher's Theorem: $PGL(n, q)$ from a Kleinian point of view. In *Geometry, Combinatorial Designs and Related Structures: Proceedings of the First Pythagorean Conference* (ed. J.W.P. Hirschfeld, S.S. Magliveras, M.J. de Resmini), Cambridge University Press, 1997, 43-61.

[3] J. Cofman. Baer Subplanes in finite projective and affine planes. *Can. J. Math.* **24** (1972), 90-97.

[4] P. Dembowski. *Finite Geometries.* Springer–Verlag Berlin–Heidelberg–New York, 1997. Reprint of the 1968 Edition.

[5] H. Pralle. *Die Geometrie der Baerunterebenen von* $PG(2, q^2)$. Diplomarbeit, Justus–Liebig–Universität Giessen, 1997.

[6] M. Sved. Baer Subspaces in the n–dimensional Projective Space. *Lecture Notes, Springer* **1036** (1982), 375-391.

[7] J. Ueberberg. Projective Planes and Dihedral Groups. *Discrete Math.* **174** (1997), 337-345.

[8] J. Ueberberg. Frobenius Collineations in Finite Projective Spaces. *Bull. Belg. Math. Soc.* **4** (1997), 473-492.

[9] J. Ueberberg. A Class of Partial Linear Spaces Related to $PGL_3(q^2)$. *Europ. J. Combinatorics* **18** (1997), 103-115.

[10] K. Vedder. A note on the intersection of two Baer Subplanes. *Arch. Math.* **37** (1981), 287-288.

Harm Pralle
Math. Institut, Arndtstr. 2, D–35392 Giessen, Germany
email: harm.pralle@math.uni-giessen.de

Johannes Ueberberg
debis IT Security Services, Rabinstr. 8, D–53111 Bonn, Germany
email: j-ueberberg@itsec-debis.de