Periodic boundary value problems for functional differential equations

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Abstract

In this paper, the method of quasilinearization has been extended to periodic boundary value problems of nonlinear functional differential equations. It is shown that monotone iterations converge to the unique solution and this convergence is semi–superlinear.

1 Introduction

Put $C_0 = C(J_0, \mathbb{R}),\ C_1 = C(J \times C_0, \mathbb{R})$ with $J_0 = [-\tau, 0],\ J = [0, T]$ for some $\tau, T > 0$. Let $g \in C_0$ and $g(0) = 0$. We shall study the following periodic boundary value problems for functional differential equations

$$(1) \qquad \begin{cases} x'(t) = f(t, x_t), \quad t \in J, \\ x(s) = g(s) + x(0), \quad s \in J_0, \quad x(0) = x(T), \end{cases}$$

where $f \in C_1$, and for any $t \in J,\ x_t \in C_0$ is defined by $x_t(s) = x(t+s)$ for $s \in J_0$. Note that g is given on J_0. If we take $g(s) = 0$ on J_0, then the boundary condition in (1) has the form $x(s) = x(0) = x(T),\ s \in J_0$.

The differential equation from problem (1) is a very general type. It includes, for example, as special cases, ordinary differential equations if $\tau = 0$, differential–difference equations, and integro–differential equations too.

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It is known that the method of quasilinearization offers an approach for obtaining approximate solutions of nonlinear differential equations (for details, see, for example [5], [7]). Recently, this method has been extended so as to be applicable to a much larger class of nonlinear problems (see, for example [2], [4]–[10]). The purpose of this paper is to show that it can be applied successfully to periodic boundary value problems of functional differential equations. Under suitable assumptions it is shown that linear iterations converge to the unique solution of our problem and this convergence is semi–superlinear.

2 Assumptions

Choose $M > 0$, and rewrite the differential equation of (1) as

$$x'(t) = -Mx(t) + Mx(t) + f(t, x_t), \quad t \in J. \tag{2}$$

Then, by variation of parameters formula, equation (2) takes the form

$$x(t) = e^{-Mt}\left\{x(0) + \int_0^t e^{Ms}\left[Mx(s) + f(s, x_s)\right] ds\right\}, \quad t \in J.$$

Since $x(0) = x(T)$, it follows that

$$x(0) = \frac{1}{e^{MT} - 1}\int_0^T e^{Ms}[Mx(s) + f(s, x_s)]ds.$$

It shows that problem (1) is equivalent to the following one

$$\begin{cases} x(t) = \dfrac{e^{-Mt}}{e^{MT} - 1}\displaystyle\int_0^T G(t, s)e^{Ms}[Mx(s) + f(s, x_s)]ds, & t \in J, \\ x(s) = x(0) + g(s), \quad s \in J_0, \end{cases} \tag{3}$$

where

$$G(t, s) = \begin{cases} e^{MT} & \text{if } 0 \le s < t, \\ 1 & \text{if } t \le s \le T. \end{cases}$$

A function $v \in \bar{C} \equiv C(\bar{J}, \mathbb{R}) \cap C^1(J, \mathbb{R}),\ \bar{J} = [-\tau, T]$ is said to be a lower solution of problem (1) if

$$\begin{cases} v'(t) \le f(t, v_t), \quad t \in J, \\ v(s) = g(s) + v(0), \quad s \in J_0, \quad v(0) \le v(T), \end{cases}$$

and an upper solution of (1) if the above inequalities are reversed.

Now, we list the following assumptions for later use.

H_1 $f \in C_1,\ g \in C_0,\ g(0) = 0,$

H_2 $y_0, z_0 \in \bar{C}$ are lower and upper solutions of problem (1) and $y_0(t) \le z_0(t)$ on J,

H_3 the Frechet derivative f_Φ exists, is a continuous linear operator satisfying:

(a) $|f_\Phi(t,\phi)v_t| \le L \max\limits_{[-\tau,t]} |v(s)|, \;\; L > 0$ for $t \in J, \;\; \phi, v_t \in C_0$,

(b) $f(t,v_2) \ge f(t,v_1) + f_\Phi(t,v_2)(v_2 - v_1)$ for $t \in J, \;\; v_1, v_2 \in C_0$ such that $y_{0,t} \le v_1 \le v_2 \le z_{0,t}$,

(c) if $v_1 \le v_2, \;\; v, v_1, v_2 \in C_0$, then $f_\Phi(t,v)v_1 \le f_\Phi(t,v)v_2$ for $y_{0t} \le v \le z_{0,t}, \;\; t \in J$,

(d) if $v, \bar{v}, V \in C_0, \;\; V \ge 0$, then

$$f_\Phi(t,v)V \ge f_\Phi(t,\bar{v})V \;\text{ for }\; t \in J, \;\; y_{0,t} \le \bar{v} \le v \le z_{0,t},$$

(e) $\int_0^T f_\Phi(s,u)v_s ds > 0$ if $u, v_t \in C_0, \;\; v(s) > 0, \;\; s \in \bar{J}$,

H_4 there exist constants $L_1 > 0$ and $\alpha \in [0,1]$ such that the condition

$$|f_\Phi(t,v_1) - f_\Phi(t,v_2)| \le L_1 |v_1 - v_2|_0^\alpha$$

holds for $t \in J, \;\; v_1, v_2 \in C_0$ with $|v|_0 = \max\limits_{s \in [-\tau,0]} |v(s)|$.

3 Existence, uniqueness results

In this section we give existence/uniqueness results both for initial and boundary value problems of functional differential equations.

Theorem 1. Let Assumption $H_3(a)$ hold. Assume that $h \in C_0, \;\; b \in C(J,\mathbb{R})$. Then the problem

$$\begin{cases} x'(t) = f_\Phi(t,u)x_t + b(t), \;\; t \in J, \;\; u \in C_0, \;\; x \in \bar{C}, \\ x(s) = h(s), \;\; s \in J_0 \end{cases}$$

has a unique solution.

Proof. To show it we can use the Banach fixed point theorem with the norm

$$|v|_* = \max_{t \in J} e^{-Kt}|v(t)| \quad \text{for} \quad K \ge L.$$

We omit the details.

Lemma 1. Let Assumption $H_3(a,e)$ hold. Then the problem

$$\begin{cases} \alpha'(t) = f_\Phi(t,u)\alpha_t, \;\; t \in J, \;\; u \in C_0, \;\; \alpha \in \bar{C}, \\ \alpha(s) = \alpha(0) = \alpha(T), \;\; s \in J_0 \end{cases} \tag{4}$$

has only zero solution.

Proof. Note that $\alpha(t) = 0, \;\; t \in \bar{J}$ is a solution of (4). Suppose that problem (4) has another solution w. Let $B = \{t_k \in J : w(t_k) = 0\}$. Assume that $t_0 \in B$. If $t_0 = 0$ or $t_0 = T$, then $w(0) = 0$. Hence $w(t) = 0, \;\; t \in \bar{J}$ since the initial problem has only one solution, by Theorem 1. It is a contradiction. If $0 < t_0 < T$, then $w(t_0) = 0$ showing that $w(t) = 0$ on $[t_0, T]$. Since $w(T) = 0$ and $w(T) = w(0)$, so $w(t) = 0$ on $\bar{J}$. It is a contradiction again. If we assume that $w(t) > 0, \;\; t \in \bar{J}$, then

$$w(t) = w(0) + \int_0^t f_\Phi(s,u)w_s ds, \;\; t \in J.$$

Note that $w(T) > w(0)$ because $\int_0^T f_\Phi(s,u)w_s ds > 0$, by Assumption $H_3(e)$. It is a contradiction. Same argument holds if we assume that $w(t) < 0$ on J. It proves that problem (4) has only one solution. It ends the Proof.

The next theorem gives sufficient conditions for the uniqueness of the solution of (1) but it does not guarantee the existence of the solution.

Theorem 2. Assume that Assumptions $H_1, H_3(a,e)$ hold. Then problem (1) has at most one solution.

Proof. Assume that problem (1) has two solutions x and y. Put $p = x - y$. Then $p(s) = p(0) = p(T),\ s \in J_0$. Moreover, by a mean value theorem, we get

$$p'(t) = f(t,x_t) - f(t,y_t) = \int_0^1 f_\Phi(t, sx_t + (1-s)y_t)dsp_t,\ \ t \in J.$$

This and Lemma 1 prove that $p(t) = 0$ on $\bar{J}$ showing that problem (1) has at most one solution. It ends the Proof.

Lemma 2. Let Assumptions H_1, H_2 and H_3 hold. Then, for $t \in J, u \in \Omega$, the periodic boundary value problem

$$\text{(5)}\qquad \begin{cases} p'(t) = f(t,u) + f_\Phi(t,u)[p_t - u],\ \ t \in J,\ \ p \in \bar{C}, \\ p(0) = p(T) \ \text{ and } \ p(s) = g(s) + p(0),\ \ s \in J_0 \end{cases}$$

has a unique solution. The set Ω is defined by

$$\Omega = \{\phi \in C_0 : y_{0,t} \le \phi \le z_{0,t},\ t \in J\}.$$

Proof. Using (2) and (3), for $M > 0$, we see that problem (5) is equivalent to the following

$$\begin{cases} p(t) = \dfrac{e^{-Mt}}{e^{MT}-1}\displaystyle\int_0^T G(t,s)e^{Ms}\left[Mp(s) + f_\Phi(s,u)(p_s - u) + f(s,u)\right]ds \equiv Ap(t),\ \ t \in J, \\ p(s) = p(0) + g(s),\ \ s \in J_0. \end{cases}$$

Assumptions H_2 and $H_3(b,d)$ imply that

$$y_0'(t) \le f(t,y_{0,t}) - f(t,u) + f(t,u) \le f(t,u) + f_\Phi(t,u)[y_{0,t} - u],\ \ t \in J,$$

and

$$\begin{aligned} z_0'(t) &\ge f(t,z_{0,t}) - f(t,u) + f(t,u) \ge f(t,u) + f_\Phi(t,z_{0,t})[z_{0,t} - u] \\ &\ge f(t,u) + f_\Phi(t,u)[z_{0,t} - u],\ \ t \in J. \end{aligned}$$

Knowing that $y_0(0) \le y_0(T),\ \ z_0(0) \ge z_0(T)$, and using the above inequalities and the method of integration by substitution, we see that

$$\begin{aligned} Ay_0(t) &= \frac{e^{-Mt}}{e^{MT}-1}\int_0^T G(t,s)e^{Ms}\left\{My_0(s) + f_\Phi(s,u)[y_{0,s} - u] + f(s,u)\right\}ds \\ &\ge \frac{e^{-Mt}}{e^{MT}-1}\int_0^T G(t,s)e^{Ms}[y_0'(s) + My_0(s)]ds \\ &= \frac{e^{-Mt}}{e^{MT}-1}\left\{e^{MT}\left[e^{Mt}y_0(t) - y_0(0)\right] + e^{MT}y_0(T) - e^{Mt}y_0(t)\right\} \ge y_0(t),\ \ t \in J, \end{aligned}$$

and

$$\begin{aligned}
Az_0(t) &= \frac{e^{-Mt}}{e^{MT}-1}\int_0^T G(t,s)e^{Ms}\left\{Mz_0(s)+f_\Phi(s,u)[z_{0,s}-u]+f(s,u)\right\}ds\\
&\le \frac{e^{-Mt}}{e^{MT}-1}\int_0^T G(t,s)e^{Ms}[z_0'(s)+Mz_0(s)]ds\\
&= \frac{e^{-Mt}}{e^{MT}-1}\left\{e^{MT}\left[e^{Mt}z_0(t)-z_0(0)\right]+e^{MT}z_0(T)-e^{Mt}z_0(t)\right\}\le z_0(t),\ \ t\in J.
\end{aligned}$$

Let $v_1, v_2 \in C(\bar{J},\mathbb{R})$ and $y_0(t)\le v_1(t)\le v_2(t)\le z_0(t),\ \ t\in\bar{J},$ so $y_{0,t}\le v_{1,t}\le v_{2,t}\le z_{0,t},\ t\in J$. Then, by Assumption $H_3(c)$, we have

$$\begin{aligned}
Av_1(t) &= \frac{e^{-Mt}}{e^{MT}-1}\int_0^T G(t,s)e^{Ms}\left\{Mv_1(s)+f_\Phi(s,u)[v_{1,s}-u]+f(s,u)\right\}ds\\
&\le \frac{e^{-Mt}}{e^{MT}-1}\int_0^T G(t,s)e^{Ms}\left\{Mv_2(s)+f_\Phi(s,u)[v_{2,s}-u]+f(s,u)\right\}ds = Av_2(t)
\end{aligned}$$

showing that the operator A maps the segment $[y_0, z_0]$ into itself. Since A is a completely continuous operator on $[y_0, z_0]$, so the sequences $y_{n+1}(t)=Ay_n(t),\ z_{n+1}(t)=Az_n(t)$ converge to the fixed points $y, z\in[y_0,z_0]$ of A and $y(t)\le z(t)$ on J.

Now we are going to show that problem (5) has one solution. Assume that it has two solutions, x and y. Set $q=x-y$, so $q(s)=q(0)=q(T),\ s\in J_0$. Then

$$\begin{cases} q'(t)=f_\Phi(t,u)q_t,\ \ t\in J,\\ q(s)=q(0)=q(T),\ s\in J_0.\end{cases}$$

By Lemma 1, this problem has only zero solution. This proves that $x(t)=y(t)$ on J, so problem (5) has a unique solution.

It ends the Proof.

Lemma 3. The assertion of Lemma 2 also holds if problem (5) is replaced by the following

$$\begin{cases} p'(t)=f(t,v)+f_\Phi(t,u)[p_t-v],\ \ u,v\in\Omega,\ u\le v,\ \ p\in\bar{C},\\ p(s)=g(s)+p(0)\ \text{ and }\ p(0)=p(T).\end{cases}$$

Proof. Obviously, we see that

$$\begin{aligned}
y_0'(t) &\le f(t,y_{0,t})-f(t,v)+f(t,v)\le f(t,v)+f_\Phi(t,v)[y_{0,t}-v]\\
&\le f(t,v)+f_\Phi(t,u)[y_{0,t}-v],
\end{aligned}$$

and

$$\begin{aligned}
z_0'(t) &\ge f(t,z_{0,t})-f(t,v)+f(t,v)\ge f(t,v)+f_\Phi(t,z_{0,t})[z_{0,t}-v]\\
&\ge f(t,v)+f_\Phi(t,u)[z_{0,t}-v],\ \ t\in J.
\end{aligned}$$

The rest of this proof is similar to the proof of Lemma 2 with the operator A defined by

$$Ap(t)=\frac{e^{-Mt}}{e^{MT}-1}\int_0^T G(t,s)e^{Ms}\left[Mp(s)+f_\Phi(s,u)(p_s-v)+f(s,v)\right]ds,\ \ t\in J.$$

We omit the details.

Lemma 4. Let Assumptions H_1, H_2, H_3 hold. Let $u, v \in \bar{C}$ be lower and upper solutions of problem (1) such that $y_0(t) \le u(t) \le v(t) \le z_0(t),\ t \in J$. Then the problems

(6)
$$\begin{cases} p'(t) = f(t,u_t) + f_\Phi(t,u_t)[p_t - u_t],\ t \in J,\ \ p(0) = p(T),\ p(s) = g(s) + p(0),\ s \in J_0 \\ q'(t) = f(t,v_t) + f_\Phi(t,u_t)[q_t - v_t],\ t \in J,\ \ q(0) = q(T),\ q(s) = g(s) + q(0),\ s \in J_0 \end{cases}$$

have their unique solutions (p,q). Moreover $u(t) \le p(t) \le q(t) \le v(t),\ t \in J$.

Proof. By Lemmas 2 and 3, there exists a unique solution (p,q) of (6). We need to show that $p, q \in [u,v]$ and $p(t) \le q(t),\ t \in J$. Note that, for $M > 0$,

$$p(t) = A(t,u,p), \quad q(t) = B(t,v,q), \ \ t \in J,$$

where

$$\begin{aligned} A(t,u,p) &= \frac{e^{-Mt}}{e^{MT}-1}\int_0^T G(t,s)e^{Ms}U(s,u,p)ds, \\ B(t,u,q) &= \frac{e^{-Mt}}{e^{MT}-1}\int_0^T G(t,s)e^{Ms}U(s,v,q)ds, \\ U(t,v,p) &= Mp(t) + f_\Phi(t,u_t)[p_t - v_t] + f(t,v_t). \end{aligned}$$

Let

$$\begin{cases} p_{n+1}(t) = A(t,u,p_n),\ \ p_0(t) = u(t),\ t \in J, \\ q_{n+1}(t) = B(t,v,q_n),\ \ q_0(t) = v(t),\ t \in J. \end{cases}$$

Observe that

$$\begin{aligned} U(t,u,u) &= Mu(t) + f(t,u_t) \ge Mu(t) + u'(t), \\ U(t,v,v) &= Mv(t) + f(t,v_t) \le Mv(t) + v'(t) \end{aligned}$$

because u, v are lower and upper solutions of (1), respectively. Now, using the method of integration by substitution, we get

$$\begin{aligned} p_1(t) = A(t,u,u) &= \frac{e^{-Mt}}{e^{MT}-1}\int_0^T G(t,s)e^{Ms}U(s,u,u)ds \\ &\ge \frac{e^{-Mt}}{e^{MT}-1}\int_0^T G(t,s)e^{Ms}[Mu(s) + u'(s)]ds \\ &= \frac{e^{-Mt}}{e^{MT}-1}\Big\{(e^{MT}-1)u(t)e^{Mt} + e^{MT}[u(T) - u(0)]\Big\} \ge u(t) = p_0(t), \\ q_1(t) = B(t,v,v) &\le \frac{e^{-Mt}}{e^{MT}-1}\int_0^T G(t,s)e^{Ms}[Mv(s) + v'(s)]ds \\ &= \frac{e^{-Mt}}{e^{MT}-1}\Big\{(e^{MT}-1)v(t)e^{Mt} + e^{MT}[v(T) - v(0)]\Big\} \le v(t) = q_0(t). \end{aligned}$$

Suppose that $\alpha(t) \le \beta(t)$ on $\bar{J}$. Then,

$$\begin{aligned} U(t,u,\alpha) &= M\alpha(t) + f_\Phi(t,u_t)[\alpha_t - \beta_t + \beta_t - u_t + v_t - v_t] + f(t,u_t) \\ &\le M\beta(t) + f_\Phi(t,u_t)[\beta_t - v_t] + f_\Phi(t,u_t)[v_t - u_t] + f(t,u_t) - f(t,v_t) + f(t,v_t) \\ &\le U(t,v,\beta) \end{aligned}$$

Hence

$$\begin{aligned} p_1(t) &= \frac{e^{-Mt}}{e^{MT}-1}\int_0^T G(t,s)e^{Ms}U(s,u,u)ds \\ &\le \frac{e^{-Mt}}{e^{MT}-1}\int_0^T G(t,s)e^{Ms}U(s,v,v)ds = q_1(t), \;\; t \in J, \end{aligned}$$

so

$$p_0(t) \le p_1(t) \le q_1(t) \le q_0(t), \;\; t \in J.$$

By mathematical inductions, we are able to show that

$$p_0(t) \le p_1(t) \le \cdots \le p_n(t) \le q_n(t) \le \cdots \le q_1(t) \le q_0(t), \;\; t \in J.$$

It yields $p_n \to p,\ q_n \to q,\ p, q \in [u, v]$ and $p(t) \le q(t), t \in J$. It ends the Proof.

4 Main result

A fundamental result of this paper is the following.

Theorem 3. Assume that Assumptions from H_1 until H_4 are satisfied. Then there exist monotone sequences $\{y_n\},\ \{z_n\}$ which converge uniformly to the unique solution x of problem (1) and that convergence is semi–superlinear i.e.

$$\begin{aligned} &\max_{t\in J}|x(t) - y_{n+1}(t)| \le a_1 \max_{t\in J}|p_{n,t}|_0^{\alpha+1} + a_2 \max_{t\in J}|p_{n,t}|_0, \\ &\max_{t\in J}|x(t) - z_{n+1}(t)| \le a_3 \max_{t\in J}|p_{n,t}|_0^{\alpha}|q_{n,t}|_0 + a_4 \max_{t\in J}|q_{n,t}|_0^{\alpha+1} + a_5 \max_{t\in J}|q_{n,t}|_0 \end{aligned}$$

for some nonnegative constants c_i and $p_n = x - y_n,\ q_n = z_n - x$.

Proof. Let $y_{n+1}(s) = g(s) + y_{n+1}(0),\ z_{n+1}(s) = g(s) + z_{n+1}(0),\ s \in J_0$ and

$$\begin{aligned} y'_{n+1}(t) &= f(t, y_{n,t}) + f_\Phi(t, y_{n,t})[y_{n+1,t} - y_{n,t}], \;\; y_{n+1}(0) = y_{n+1}(T), \\ z'_{n+1}(t) &= f(t, z_{n,t}) + f_\Phi(t, y_{n,t})[z_{n+1,t} - z_{n,t}], \;\; z_{n+1}(0) = z_{n+1}(T) \end{aligned}$$

for $t \in J,\ n = 0, 1, \cdots$.

Note that the elements y_1, z_1 are well defined, by Lemmas 2 and 3. Lemma 4 asserts that

$$y_0(t) \le y_1(t) \le z_1(t) \le z_0(t), \;\; t \in J.$$

Now we prove that y_1, z_1 are lower and upper solutions of problem (1), respectively. By Assumption $H_3(b, d)$, we get

$$\begin{aligned} y'_1(t) &= f(t, y_{0,t}) + f_\Phi(t, y_{0,t})[y_{1,t} - y_{0,t}] - f(t, y_{1,t}) + f(t, y_{1,t}) \\ &\le f(t, y_{1,t}) - f_\Phi(t, y_{1,t})[y_{1,t} - y_{0,t}] + f_\Phi(t, y_{0,t})[y_{1,t} - y_{0,t}] \le f(t, y_{1,t}) \end{aligned}$$

and

$$\begin{aligned} z'_1(t) &= f(t, z_{0,t}) + f_\Phi(t, y_{0,t})[z_{1,t} - z_{0,t}] - f(t, z_{1,t}) + f(t, z_{1,t}) \\ &\ge f(t, z_{1,t}) + f_\Phi(t, z_{0,t})[z_{0,t} - z_{1,t}] + f_\Phi(t, y_{0,t})[z_{1,t} - z_{0,t}] \\ &\ge f(t, z_{1,t}). \end{aligned}$$

It proves that y_1, z_1 are lower and upper solutions of (1).

Let us assume that

$$y_0(t) \le y_1(t) \le \cdots \le y_{k-1}(t) \le y_k(t) \le z_k(t) \le z_{k-1}(t) \le \cdots \le z_1(t) \le z_0(t), \ \ t \in J,$$

and let y_k, z_k be lower and upper solutions of problem (1) for some $k \ge 1$. Then, by Lemmas 2 and 3, the elements y_{k+1}, z_{k+1} are well defined. Moreover, Lemma 4 yields

$$y_k(t) \le y_{k+1}(t) \le z_{k+1}(t) \le z_k(t), \ \ t \in J.$$

Hence, by induction, we obtain

$$y_0(t) \le y_1(t) \le \cdots \le y_n(t) \le z_n(t) \le \cdots \le z_1(t) \le z_0(t), \ \ t \in J$$

for all n. Employing standard techniques [5], it can be shown that the sequences $\{y_n\}, \{z_n\}$ converge uniformly and monotonically to the solutions y, z of (1), so $y_n \to y$, $z_n \to z$ and $y(t) \le z(t)$ on J. By Theorem 2, $y = z$. It means that the sequences $\{y_n\}, \{z_n\}$ converge to the unique solution x of problem (1).

It remains only to show that the convergence of y_n, z_n to the unique solution x of problem (1) is semi–superlinear. For this purpose, we put

$$p_{n+1}(t) = x(t) - y_{n+1}(t) \ge 0, \qquad q_{n+1}(t) = z_{n+1}(t) - x(t) \ge 0 \ \ t \in \bar{J}.$$

Note that $p_{n+1}(s) = p_{n+1}(0) = p_{n+1}(T)$, $q_{n+1}(s) = q_{n+1}(0) = q_{n+1}(T)$, $s \in J_0$. Observe that

$$\begin{aligned} p_{n+1}(t) &= x(t) - y_{n+1}(t) + y_n(t) - y_n(t) \le p_n(t) \\ q_{n+1}(t) &= z_{n+1}(t) - x(t) - z_n(t) + z_n(t) \le q_n(t). \end{aligned}$$

Choose $M > 0$. Using Assumptions H_3 and H_4, we obtain

$$\begin{aligned} p'_{n+1}(t) &= f(t, x_t) - f(t, y_{n,t}) - f_\Phi(t, y_{n,t})[y_{n+1,t} - y_{n,t}] \\ &= \int_0^1 f_\Phi(t, sx_t + (1-s)y_{n,t})p_{n,t}ds - f_\Phi(t, y_{n,t})[p_{n,t} - p_{n+1,t}] \\ &= \int_0^1 [f_\Phi(t, sx_t + (1-s)y_{n,t}) - f_\Phi(t, y_{n,t})]p_{n,t}ds + f_\Phi(t, y_{n,t})p_{n+1,t} \\ &\le L_1 \int_0^1 s^\alpha |p_{n,t}|_0^{\alpha+1} ds + f_\Phi(t, y_{n,t})p_{n+1,t} \\ &\le D + L \max_{s \le t} p_n(s) + M p_{n+1}(t) - M p_{n+1}(t) \\ &\le D + (L + M) \max_{t \in J} |p_{n,t}|_0 - M p_{n+1}(t) \equiv \bar{D} - M p_{n+1}(t) \end{aligned}$$

with $D = L_1 \max_{t \in J} |p_{n,t}|_0^{\alpha+1}$. Hence, the differential inequality yields

$$p_{n+1}(t) \le e^{-Mt} \left[p_{n+1}(0) + \frac{\bar{D}}{M} \left(e^{Mt} - 1 \right) \right], \ t \in J.$$

Since $p_{n+1}(0) = p_{n+1}(T)$, we get $p_{n+1}(0) \le \dfrac{\bar{D}}{M}$, so $p_{n+1}(t) \le \dfrac{\bar{D}}{M}$, $t \in J$. Hence, we finally obtain

$$\max_{t \in J} |p_{n+1}(t)| \le \frac{1}{M} \left[L_1 \max_{t \in J} |p_{n,t}|_0^{\alpha+1} + (L + M) \max_{t \in J} |p_{n,t}|_0 \right].$$

By a similar way, we can obtain

$$\begin{aligned}
q'_{n+1}(t) &= f(t, z_{n,t}) - f(t, x_t) + f_\Phi(t, y_{n,t})[z_{n+1,t} - x_t + x_t - z_{n,t}] \\
&= \int_0^1 f_\Phi(t, sz_{n,t} + (1-s)x_t)q_{n,t}ds + f_\Phi(t, y_{n,t})[q_{n+1,t} - q_{n,t}] \\
&= \int_0^1 [f_\Phi(t, sz_{n,t} + (1-s)x_t) - f_\Phi(t, x_t) + f_\Phi(t, x_t) - f_\Phi(t, y_{n,t})]\, q_{n,t}ds \\
&+ f_\Phi(t, y_{n,t})q_{n+1,t} \le L_1 \left[|q_{n,t}|_0^\alpha + |p_{n,t}|_0^\alpha\right] q_{n,t} + f_\Phi(t, y_{n,t})q_{n+1,t} \\
&\le P + (L+M) \max_{t\in J} |q_{n,t}|_0 - Mq_{n+1}(t), \quad t \in J,
\end{aligned}$$

where

$$P = L_1 \max_{t\in J} \left[|q_{n,t}|_0^{\alpha+1} + |p_{n,t}|_0^\alpha |q_{n,t}|_0\right].$$

Consequently

$$\max_{t\in J} |q_{n+1}(t)| \le \frac{1}{M}\left[L_1 \max_{t\in J} |p_{n,t}|_0^\alpha |q_{n,t}|_0 + L_1 \max_{t\in J} |q_{n,t}|_0^{\alpha+1} + (L+M) \max_{t\in J} |q_{n,t}|_0\right].$$

The proof is complete.

Remark 3. If $\alpha = 1$, then the convergence of sequences $\{y_n\}$ and $\{z_n\}$ to x is semi–quadratic.

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