

Minimal covers of open manifolds with half-spaces and the proper L-S category of product spaces

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Abstract

Classical results about the Lusternik-Schnirelmann category of product spaces have their analogues in the category of proper maps. By comparing the proper Lusternik-Schnirelmann category of an open manifold X with the smallest number of closed half-spaces needed to cover X , we obtain a proper analogue of Singhof's theorem on the category of $X \times S^1$.

Introduction

The Lusternik-Schnirelmann category (L-S category) $cat(X)$ of a space X is the smallest number k such that there exists an open cover $\{X_1, \dots, X_k\}$ for which each inclusion $X_j \subseteq X$ is nullhomotopic in X . The L-S category turns to be a homotopy invariant of the space X . See [9] for a survey on L-S category.

Ordinary homotopy invariants do not take care of the behaviour of spaces at infinity. So, "proper" homotopy invariants are needed for the study of non-compact spaces. Proper analogues of Lusternik-Schnirelmann numerical invariants were introduced in [2] and [3].

The crucial point in the definition of the proper L-S category is the fact that the half-line $[0, \infty)$ plays in proper homotopy part of the role played by the point in ordinary homotopy. The parallelism between both roles breaks down on the

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fibration-side of homotopy theory since product and fibration projections are not always proper maps. However, the class of proper maps still keeps the basic properties on the cofibration-side of homotopy theory which lead to a “combinatorial” homotopy in the sense of J.H.C. Whitehead [19]; see [1].

This paper continues the study of the proper L-S category of non-compact spaces. Here, we focus our interest on the behaviour of the proper L-S category on product spaces. More explicitly we work out the proper L-S category of spaces of the form $W \times S^k$ and $M \times \mathbb{R}^k$ with W and M open and closed manifolds respectively. These results can be regarded as proper analogues of a theorem due to Singhof [17] in relation with a classical question posed by Ganea; see Remark 3.9.

In this paper we follow closely the combinatorial proof of Singhof’s theorem given by Montejano [12] since proper collapses [16] and a suitable Engulfing Theorem for open manifolds [10] are available; see Appendix A for details. This way we can compare the proper L-S category of an open manifold X with the smallest number of properly embedded half-spaces needed to cover X .

Several new improvements of Singhof’s theorem have recently appeared in the literature; see [14] and [18]. However, these results depend heavily on the study of sections of certain fibrations. Since the category of proper maps provides an example of homotopy theory with good properties only on the cofibration-side, it seems to be interesting to look for alternative proofs of those results in the cofibration realm of homotopy theory.

We shall deal with the category \mathcal{P} of locally compact σ -compact Hausdorff spaces and proper maps. Recall that a *proper map* (*p-map*) is a continuous map $f : X \rightarrow Y$ such that $f^{-1}(K)$ is compact for each compact subset $K \subseteq Y$.

All maps and homotopies are assumed to be proper unless stated otherwise. We use the symbol \simeq for proper homotopy and \mathcal{P}/\simeq stands for the corresponding homotopy category. Furthermore, the symbol \mathbb{R}_+ denotes the half-line $[0, \infty)$ and more generally \mathbb{R}_+^n denotes the upper n -dimensional half-space $\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n; x_n \geq 0\}$.

1 Proper L-S category

This section contains some technical observations about the notion of proper L-S category which will be used later. Recall that, given a space X in \mathcal{P} , a *system of ∞ -neighbourhoods* of X is a decreasing sequence $\{W_j\}$ of subsets of X such that the closures $K_j = \overline{X - W_j}$ form an increasing sequence of compact subsets with $K_j \subseteq \text{int}K_{j+1}$ and $X = \cup \text{int}K_j$.

Remark 1.1. Given a locally finite sequence of pairwise disjoint compact subsets $C_i \subseteq X$ ($i \geq 1$), it is possible to choose the compact sets K_j above satisfying $C_i \cap \text{Fr} K_j = \emptyset$ for all $i, j \geq 1$. For this we consider the compact set $\hat{L}_1 = K_1 \cup (\cup\{C_i; C_i \cap K_1 \neq \emptyset\})$. By using the normality of X we find a compact set $L_1 \subseteq X$ with $\hat{L}_1 \subseteq \text{int} L_1$ and $L_1 \cap C_i = \emptyset$ whenever $C_i \cap K_1 = \emptyset$. Hence $C_i \cap \text{Fr} L_1 = \emptyset$ for all $i \geq 1$. Then we pick n_1 such that $L_1 \subseteq \text{int} K_{n_1}$ and we set $\hat{L}_2 = K_{n_1} \cup (\cup\{C_i; C_i \cap K_{n_1} \neq \emptyset\})$. Similarly we find a compact set $L_2 \subseteq X$ with $\hat{L}_2 \subseteq \text{int} L_2$ and $C_i \cap \text{Fr} L_2 = \emptyset$ for all $i \geq 1$. We proceed inductively to obtain an increasing sequence of compact sets $L_j \subseteq \text{int} L_{j+1}$ with the required properties.

In the category \mathcal{P} the constant map $X \rightarrow \{p\}$ is not defined if X is not compact. Notwithstanding, the role of the point is played partially in \mathcal{P} by the half-line \mathbb{R}_+ since for any space X in \mathcal{P} there always exists a proper map $r : X \rightarrow \mathbb{R}_+$. Moreover the map r is unique up to p-homotopy. We shall briefly describe the construction of such a map r . If $\{U_j\}_{j \geq 0}$ is a system of ∞ -neighbourhoods of X with $U_0 = X$, the Tietze Extension Theorem yields continuous maps $r_j = \overline{U_j - U_{j+1}} \rightarrow [j, j + 1]$ with $r_j(FrU_{j+1}) = j + 1, r_j(FrU_j) = j$. It is now clear that the maps r_j define a proper map $r : X \rightarrow \mathbb{R}_+$.

Definition 1.2. A proper map $r : \mathbb{R}_+ \rightarrow X$ is called a *ray* in X . Moreover a proper map $f : X \rightarrow Y$ is *properly inessential* if there exists a commutative diagram in \mathcal{P}/\simeq

$$\begin{array}{ccc}
 & & * \\
 & \nearrow \beta & \downarrow \alpha \\
 X & & Y \\
 & \searrow f &
 \end{array} \tag{1}$$

where $*$ is either \mathbb{R}_+ or the one-point space $\{p\}$. Notice that $* = \{p\}$ only if X is compact.

Given a space X in \mathcal{P} a closed subset $A \subseteq X$ is called *inessential* if the inclusion $i : A \rightarrow X$ is an inessential map. A set $A \subseteq X$ is called *properly categorical* if $A \subseteq U$ with U an open set in X and the closure \bar{U} is inessential. Moreover an open cover $\{U_\alpha\}$ of X is said to be properly categorical if each \bar{U}_α is an inessential set. The *proper Lusternik-Schnirelmann category* of X , $p - cat(X)$, is the least number n such that X admits a properly categorical open cover $\mathcal{V} = \{U_1, U_2, \dots, U_n\}$ with n elements. In case X is compact $p - cat(X) = cat(X)$ is the ordinary L-S category of X .

Remark 1.3. As in ordinary homotopy theory closed covers can also be used to define the proper L-S category of ANR-spaces. Furthermore for polyhedra in \mathcal{P} one can use covers consisting of subpolyhedra in the definition of proper L-S category; see [2] and [3] for details.

A *Freudenthal end* of a space X in \mathcal{P} is an element of the inverse limit $\mathcal{F}(X) = \varprojlim \mathcal{U}(W_j)$. Here $\{W_j\}$ is a system of ∞ -neighbourhoods of X and $\mathcal{U}(-)$ stands for the family of unbounded connected components. A subset $A \subseteq X$ is termed *unbounded* if its closure \bar{A} is non-compact. If $\mathcal{F}(X) = \{*\}$ then X is said to be *one-ended*.

Remark 1.4. Notice that a properly categorical set $A \subseteq X$ cannot contain sequences of points defining two different Freudenthal ends. Indeed, it is immediate to check that any ray $r : \mathbb{R}_+ \rightarrow X$ defines a unique Freudenthal end. As a consequence, for any system of ∞ -neighbourhoods $\{W_j\}_{j \geq 1}$ of X , there exists j_0 such that for each $j \geq j_0$ there is at most one component $U_j \in \mathcal{U}(W_j)$ with $A \cap W_j \neq \emptyset$; moreover these components form a nested sequence $U_{j_0+1} \supseteq U_{j_0+2} \supseteq \dots$ which defines a unique Freudenthal end of X .

Example 1.5. A lower bound for the proper L-S category of spaces with cylindrical ends can be easily obtained as follows. Let X be a space in \mathcal{P} with m cylindrical ends; that is, there exists a relatively compact open set $A \subseteq X$ such that $X - A$ has m unbounded components W_j ($1 \leq j \leq m$) homeomorphic to cylinders $Z_j \times [0, \infty)$ where each Z_j is compact. Then the inequality

$$p - \text{cat}(X) \geq k = \sum_{j=1}^m \text{cat}(Z_j)$$

holds. Indeed, given a properly categorical open cover $\{U_s\}_{1 \leq s \leq n}$ of X , by Remark 1.4, there is a compact subset $B \subseteq X$ such that $A \subseteq B$, and each non-empty difference $U_s - B$ is contained in exactly one component of $X - B$. Therefore, if $n \leq k - 1$ then at least one component $\Omega_{j_0} \subseteq X - B$ is covered by less than $\text{cat}(Z_{j_0})$ differences $U_{s_i} - B$ ($1 \leq i \leq q < \text{cat}(Z_{j_0})$). Moreover, we can assume without loss of generality that Ω_{j_0} is of the form $Z_{j_0} \times [t_0, \infty)$. As each U_{s_i} is properly categorical there exists $t \geq t_0$ such that the deformation of $U_{s_i} - (Z_{j_0} \times [t, \infty))$ occurs inside Ω_{j_0} . From this fact, we easily derive that the intersections $U_{s_i} \cap (Z_{j_0} \times \{t + 1\})$ ($1 \leq i \leq q < \text{cat}(Z_{j_0})$) provide an ordinary categorical cover of $Z_{j_0} \times \{t + 1\}$ which is a contradiction.

The proper homotopy class of the map β in diagram (1) is unique. However for $* = \mathbb{R}_+$ the proper homotopy class of the map α in diagram (1) depends on the set of proper homotopy classes $[\mathbb{R}_+, Y]$. Each class $[\alpha] \in [\mathbb{R}_+, Y]$ is called a *strong end* of Y . When $[\mathbb{R}_+, Y]$ consists of only one element we say that Y is *strongly one-ended*. Clearly each strong end defines a Freudenthal end. More precisely there exists an onto map $[\mathbb{R}_+, Y] \rightarrow \mathcal{F}(Y)$.

It is obvious that for strongly one-ended spaces all rays α_i can be chosen to be the same. Next proposition shows that the same holds for one-ended polyhedra.

Proposition 1.6. *Let X be a connected one-ended polyhedron in \mathcal{P} with $p - \text{cat}(X) = n$. Then there exists a properly categorical (polyhedral) cover $\{V_1, V_2, \dots, V_n\}$ of X such that in the diagrams*

$$\begin{array}{ccc}
 & & \mathbb{R}_+ \\
 & \nearrow r & \downarrow \alpha_i \\
 V_i & & X \\
 & \searrow k_i &
 \end{array} \tag{2}$$

all rays α_i ($i \leq n$) define the same strong end.

Proof. Let $\{U_1, U_2, \dots, U_n\}$ be a properly categorical cover of X consisting of subpolyhedra; see Remark 1.3. Moreover, it is well known that any ray $\alpha : \mathbb{R}_+ \rightarrow X$ is properly homotopic to a ray embedded in the 1-skeleton of X .

In case all connected components of some U_i are compact, we next show that α_i in diagram (2) can be chosen to be arbitrary. Indeed, since all components $C \subseteq U_i$ are compact, one readily checks that there is a proper deformation H which contracts each C to a point in X . Furthermore, by using the proper homotopy extension property we can replace H by a new proper deformation H' which shrinks each C to a point $x_C \in C$ relative x_C . Finally, given any ray $R \subseteq X$, we use that X is

one-ended to move each point x_C to some point $y_C \in R$ via a proper homotopy $\{x_C\} \times I \rightarrow X$.

By using the previous arguments, we can assume without loss of generality that some element of the cover $\{U_1, U_2, \dots, U_n\}$ has at least one unbounded component. Let U_1 be such an element. We assume inductively that U_i can be properly deformed to a ray $R \subseteq X$ for $i \leq k$. Now we consider U_{k+1} . By the arguments above we can assume that U_{k+1} is not compact. Moreover we can also assume that $U_1 \cap U_{k+1} \neq \emptyset$ is non-compact as well; otherwise we use the fact that X is one-ended to join U_1 to U_{k+1} with a locally finite sequence of pairwise disjoint arcs which we add to U_1 . In addition, if the intersection $U_1 \cap U_{k+1}$ contains a ray R' then the proper deformation of U_1 to R yields that both rays R and R' represent the same strong end of X , and so U_{k+1} can be properly deformed to the ray R .

It only remains to consider the case when $U_1 \cap U_{k+1}$ consists of a locally finite sequence $\{K_1, K_2, \dots\}$ of compact components. In such a case one finds two pairwise disjoint families $\{A_1, A_2, \dots\}$ and $\{B_1, B_2, \dots\}$ of compact subpolyhedra¹ of U_{k+1} with $U_{k+1} = (\cup_{i=1}^\infty A_i) \cup (\cup_{t=1}^\infty B_t)$ and $K_i \subseteq \text{int}_{U_{k+1}} A_i$ for $i \geq 1$. Also one chooses a pairwise disjoint sequence $\{L_1, L_2, \dots\}$ of compact subpolyhedra of U_1 with $K_i \subseteq \text{int}_{U_1} L_i$ for all $i \geq 1$. Finally we take a locally finite sequence of pairwise disjoint arcs $\gamma_i \subseteq U_1$ joining K_i to $Z = \overline{U_1 - \cup_{i=1}^\infty L_i}$ with $\text{int} \gamma_i \subseteq \text{int}_{U_1} L_i$. Then we replace U_1 and U_{k+1} by

$$\tilde{U}_1 = (\cup_{i=1}^\infty A_i) \cup Z \cup (\cup_{j=1}^\infty \gamma_j) \text{ and } \tilde{U}_{k+1} = (\cup_{t=1}^\infty B_t) \cup (\cup_{j=1}^\infty L_j)$$

respectively. Now by the arguments above, we can assume in addition that the disjoint union $\cup_{s=1}^\infty A_s$ is properly deformed to the discrete set $D = \cup_{j=1}^\infty (\gamma_j \cap K_j)$ relative to D . From this one easily shows that \tilde{U}_1 can be properly deformed to the ray R . The set \tilde{U}_{k+1} is a locally finite disjoint union of compact subpolyhedra, and so it can be properly deformed to the ray R as well. Since $U_1 \cup U_{k+1} = \tilde{U}_1 \cup \tilde{U}_{k+1}$ we can replace the cover $\{U_1, U_1, \dots, U_n\}$ by $\{\tilde{U}_1, U_2, \dots, U_k, \tilde{U}_{k+1}, U_{k+2}, \dots, U_n\}$. After a finite number of steps we get a properly categorical (polyhedral) cover of X such that all elements in it can be properly deformed to the ray R .

2 Proper L-S category of product spaces

For ordinary L-S category the following formula is well known for product spaces; see [5]

$$\max\{cat(X), cat(Y)\} \leq cat(X \times Y) \leq cat(X) + cat(Y) - 1 \tag{*}$$

In this section we study the proper analogue for this formula. Recall that the proper L-S category $p - cat(-)$ is a proper homotopy invariant; in fact if $f : X \rightarrow Y$ and $g : Y \rightarrow X$ are proper maps with $gf \simeq id_X$ one has $p - cat(X) \leq p - cat(Y)$. In particular, if Y is compact the projection $p_1 : X \times Y \rightarrow X$ yields

¹One finds these families as follows. Let $\{L_j\}$ be an increasing sequence of compact subpolyhedra in U_{k+1} with $K_i \cap Fr L_j = \emptyset$ for $i, j \geq 1$; see Remark 1.1. Then we pick $n_1 < n_2 < \dots$ and we choose any locally finite family of pairwise disjoint compact subpolyhedra A_i with $K_i \subseteq \text{int}_{U_{k+1}} A_i$ for $i \neq n_j$ and $K_{n_j} \cup Fr L_j \subseteq \text{int}_{U_{k+1}} A_{n_j}$ for all $j \geq 1$. It is immediate to check that the closure $\overline{U_{k+1} - \cup_{i=1}^\infty A_i} = \cup_{t=1}^\infty B_t$ is a locally finite union of pairwise disjoint compact subpolyhedra.

$$p - cat(X) \leq p - cat(X \times Y) \tag{3}$$

However, Example 3.10 below shows that inequality (3) does not hold if Y is not compact.

Concerning the right-hand side inequality in (*) we can prove the following proposition; compare [5]

Proposition 2.1. *Let X and Y be two connected polyhedra in \mathcal{P} . Assume X has at most one end in case Y is compact. Then $p-cat(X \times Y) \leq p-cat(X) + p-cat(Y) - 1$.*

Proof. Assume X and Y are not compact. Then by ([11];2.2) $X \times Y$ is strongly one-ended. Let $p - cat(X) = n$ and $p - cat(Y) = m$ and let $\{U_1, \dots, U_n\}$ and $\{W_1, \dots, W_m\}$ be families of inessential subpolyhedra whose interiors cover X and Y respectively. By using regular neighbourhoods we find new properly categorical covers $\{\tilde{U}_i\}$ and $\{\tilde{W}_j\}$ consisting of closed subpolyhedra with $U_i \subseteq int \tilde{U}_i$ and $W_j \subseteq int \tilde{W}_j$. Since $\mathbb{R}_+ \times \mathbb{R}_+$ has the proper homotopy type of \mathbb{R}_+ it is clear that each product $\tilde{U}_i \times \tilde{W}_j$ is inessential. Moreover, since $X \times Y$ is strongly one-ended we have a commutative diagram in \mathcal{P}/\simeq

$$\begin{array}{ccc} & & \mathbb{R}_+ \\ & \nearrow r & \downarrow \alpha \\ \tilde{U}_i \times \tilde{W}_j & & X \times Y \end{array} \tag{4}$$

with the same α for all i, j . We now consider the unions $A_i = \cup_{k=1}^i U_k$, $i \leq n$, and $B_j = \cup_{h=1}^j W_h$, $j \leq m$. Moreover let $C_1 \subseteq C_2 \subseteq \dots \subseteq C_{n+m-1} = X \times Y$ be the increasing sequence of closed sets

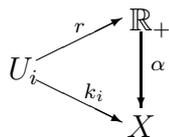
$$C_s = \cup_{i+j=s+1} A_i \times B_j, \quad 1 \leq s \leq n + m - 1.$$

From this we can write $X \times Y = \cup_{s=1}^{n+m-1} D_s$ for the sets $D_1 = C_1$ and $D_s = C_s - C_{s-1}$, $s \geq 2$. Moreover we have $D_s = \cup \{E_i \times F_j; i + j = s + 1\}$ where $E_i = A_i - A_{i-1}$ and $F_j = B_j - B_{j-1}$. We set $A_0 = B_0 = \emptyset$. Clearly the sets $E_i \times F_j \subseteq D_s$ are pairwise disjoint. Moreover they are pairwise separated; that is, we have

$$(\overline{E_i \times F_j}) \cap (E_{i'} \times F_{j'}) = (E_i \times F_j) \cap (\overline{E_{i'} \times F_{j'}}) = \emptyset$$

if $i + j = i' + j'$. Since $X \times Y$ is hereditarily normal we find for each $s \leq n + m - 1$ a pairwise disjoint family of open sets $\mathcal{V}_s = \{V_i^j\}_{i+j=s+1}$ with $E_i \times F_j \subseteq V_i^j \subseteq int \tilde{U}_i \times int \tilde{W}_j$; see ([4];2.1.7). In particular the union $\mathcal{V} = \cup_{s=1}^{n+m-1} \mathcal{V}_s$ is an open cover of $X \times Y$, and the paracompactness of $X \times Y$ provides us with a finite open refinement of \mathcal{V} , $\mathcal{Z} = \cup_{s=1}^{n+m-1} \{Z_i^j\}_{i+j=s+1}$ with $\overline{Z_i^j} \subseteq V_i^j$; see ([4]; 5.1.7). Moreover for each s the closure of the open set $\Omega_s = \cup_{i+j=s+1} Z_i^j$ is the disjoint finite union of closed sets $\overline{\Omega}_s = \cup_{i+j=s+1} \overline{Z_i^j}$. Finally diagram (4) shows that $\{\Omega_s\}_{1 \leq s \leq n+m-1}$ is a properly categorical open cover of $X \times Y$ and hence $p-cat(X \times Y) \leq p-cat(X) + p-cat(Y) - 1$. Here we use the crucial fact that the ray α is the same for all i, j .

In case Y is compact the subpolyhedra W_j are contractible to a point in Y . Moreover, since X is supposed to be at most one-ended, we can use Proposition 1.6 to assume that for the subpolyhedra U_i one has a commutative diagram



in \mathcal{P}/\simeq for all $i \geq 1$. Next we argue as above to obtain the open sets Z_i^j . Then by using the previous diagram and the fact that each W_j is contractible in Y one easily shows that the disjoint union $\cup\{Z_i^j; i + j = s + 1\}$ is an inessential set in $X \times Y$ and the result follows.

Remark 2.2. It is clear that Proposition 2.1 does not hold if X is not one-ended in case Y is compact. Indeed, it is clear that $p - cat(S^n \times \mathbb{R}) \leq 4$, and so from Example 1.5 it follows $p - cat(S^n \times \mathbb{R}) = 4 > 3 = cat(S^n) + p - cat(\mathbb{R}) - 1$.

3 Minimal covers with half-spaces and proper L-S category

If M is an open n -manifold one can consider covers consisting of closed subspaces homeomorphic to the half-space \mathbb{R}_+^n , and it is natural to ask for the comparison of $p - cat(M)$ with the smallest number, $h(M)$, of half-spaces needed to cover M . By using the proper Engulfing Theorem (A.6) we have the following result; compare ([20]; Ch. VII).

Theorem 3.1. *Let M be a one-ended open PL n -manifold. Then $p - cat(M) \leq h(M) \leq n + 1$.*

Proof. Let K be a triangulation of M . If $b(\sigma)$ denotes the barycentre of $\sigma \in K$ we consider the discrete sets $\Gamma_i = \{b(\sigma); dim\sigma = i\}$. Since M is one-ended it is easily checked that all sets $\Gamma_i, 0 \leq i \leq n$, are inessential in M and so by (A.6) there exist half-spaces H_i with $\Gamma_i \subseteq H_i$. Next we consider the regular neighbourhoods N_i of Γ_i in the second barycentric subdivision of K . Then $M = \cup_{i=0}^n N_i$ and moreover by the uniqueness of regular neighbourhoods (A.3) there exist ambient isotopies ϕ_i in M carrying N_i inside H_i . Hence $M = \cup_{i=0}^n \phi_i^{-1}(H_i)$ and the result follows.

We now proceed to prove a proper analogue of a theorem due to Singhof [17] which provides sufficient conditions for the equality $p - cat(M) = h(M)$. Recall that a space X in \mathcal{P} is said to be *properly k -connected* if for any $q \leq k$ any proper map $f : K \rightarrow X$ from a q -dimensional polyhedron K in \mathcal{P} is properly inessential. Notice that X is properly 0-connected if and only if X is one-ended.

Lemma 3.2. *Let P be a properly k -connected polyhedron in \mathcal{P} and let (K, L) be a polyhedral pair in \mathcal{P} with $dim(K - L) \leq k + 1$. Then any proper map $f : L \rightarrow P$ admits a proper extension $\tilde{f} : K \rightarrow P$. In particular, if $R \subset Q \subset P, dim(Q - R) \leq k$, and R is properly inessential then so is Q .*

Proof. Assume that $\tilde{f} : K^r \cup L \rightarrow P$ exists. Then the restriction of \tilde{f} to the union $\Delta = \cup\{\partial\sigma; \sigma^{r+1} \in K\}$ is properly inessential and hence $\tilde{f}|_{\Delta}$ admits a proper extension to K^{r+1} which yields a proper extension of \tilde{f} to $K^{r+1} \cup L$. For the second part, let $H : R \times I \rightarrow P$ be a proper deformation of R with H_1 a composite $H_1 : R \xrightarrow{r} \mathbb{R}_+ \xrightarrow{\alpha} P$. Then we apply the first part of the lemma to $K = Q \times I, L = R \times I \cup Q \times \{1\}$ and $f = H \cup \tilde{r} : L \rightarrow P$ where $\tilde{r} : Q \rightarrow \mathbb{R}_+$ is any proper extension of r .

Theorem 3.3. *Let M be a properly c -connected open PL n -manifold ($c \geq 0, n \geq 4$). If $p - \text{cat}(M) \geq \frac{n+c+4}{2(c+1)}$ then $p - \text{cat}(M) = h(M)$.*

The proof of Theorem 3.3 follows closely the proof due to Montejano [12] for ordinary L-S category. More precisely we first prove the following proper analogue of ([13], Thm. 1). See Appendix A for the definition of a proper collapse $Y \searrow_p X$.

Proposition 3.4. *Let P be a properly c -connected n -dimensional polyhedron in \mathcal{P} , and let $\{P_1, \dots, P_m\}$ be a properly categorical (polyhedral) cover of P . Then, for each $0 \leq q \leq c$, there is a properly categorical cover $\{R_1, \dots, R_m\}$ of P such that, for each $1 \leq i \leq m, R_i \searrow_p N_i$ where $\dim(N_i) \leq \max\{n - (m - 1)(q + 1), q\}$.*

Proof. Let T be a triangulation of P such that T_1, \dots, T_m are subcomplexes which triangulate P_1, \dots, P_m . Let L_1 be the $(n - (m - 1)(q + 1))$ -skeleton of T_1 and let L'_1 be its dual skeleton. By Lemma A.2, there exists a polyhedral cover $\{R_2^1, \dots, R_m^1\}$ of $|L'_1| \cup (\bigcup_{i=2}^m P_i)$ such that $R_i^1 \searrow_p P_i \cup N_i$ and $\dim(N_i) \leq q, 2 \leq i \leq m$. Let $R_1^1 = |J|$, where J is a second derived neighbourhood of L_1 in T such that $P = |J| \cup (\bigcup_{i=2}^m R_i^1)$. Notice that each $R_i^1 (i \geq 2)$ is properly categorical by Lemma 3.2. Moreover, $R_1^1 \searrow_p |L_1| \subseteq P_1$ with $\dim(L_1) \leq n - (m - 1)(q + 1)$, and hence R_1^1 is also properly categorical. Next, let us suppose we have constructed a polyhedral cover $\{R_1^k, \dots, R_m^k\}$ satisfying

- (a) R_i^k is properly categorical, $1 \leq i \leq m$.
- (b) $_k$ $R_i^k \searrow_p N_i$, with $\dim(N_i) \leq \max\{n - (m - 1)(q + 1), q\}, 1 \leq i \leq k < m$.

By replacing R_1^1 with R_{k+1}^k and using the same argument as above one constructs a polyhedral cover $\{R_1^{k+1}, \dots, R_m^{k+1}\}$ such that $R_i^{k+1} \searrow_p R_i^k \cup N'_i, \dim(N'_i) \leq q (i \neq k+1)$ and $R_{k+1}^{k+1} \searrow_p |L_{k+1}| \subseteq R_{k+1}^k$ with $\dim(L_{k+1}) \leq n - (m - 1)(q + 1)$. Moreover, for each $1 \leq i \leq k, R_i^{k+1} \searrow_p R_i^k \cup N'_i$ and $R_i^k \searrow_p N_i$. Thus, by Lemma A.1, there exist polyhedra M_i with $\dim(M_i) \leq \dim(N'_i) \leq q$ and such that $R_i^k \cup N'_i \searrow_p N_i \cup M_i = N''_i$, whence $R_i^{k+1} \searrow_p N''_i$ and $\dim(N''_i) \leq \max\{n - (m - 1)(q + 1), q\}$. By Lemma 3.2, each $R_i^{k+1} (i \neq k + 1)$ is properly categorical. Moreover, $R_{k+1}^{k+1} \searrow_p |L_{k+1}| \subseteq R_{k+1}^k$ and hence R_{k+1}^{k+1} is also properly categorical. Therefore, the polyhedral cover $\{R_1^{k+1}, \dots, R_m^{k+1}\}$ satisfies properties (a) and (b) $_{k+1}$.

Proof of 3.3. Let $q = \min\{c, n - 3\}$. According to Proposition 3.4 there exists a properly categorical cover $M = R_1 \cup \dots \cup R_m (m = p - \text{cat}(M))$ such that $R_i \searrow_p N_i$ with $\dim N_i \leq \max\{n - (m - 1)(q + 1), q\}$. If $c \leq n - 3$ then $q = c$ and $\dim N_i \leq \frac{n+c-2}{2} \leq n - 3$. Otherwise $c \geq n - 2$ and $q = n - 3$ yield $\dim N_i \leq n - 3 \leq \frac{n+c-2}{2}$. Hence by the proper Engulfing Theorem (A.6) we can find m half-spaces H_1, \dots, H_m with $N_i \subseteq H_i$. As $R_i \searrow_p N_i$ any regular neighbourhood Ω_i of R_i is a regular neighbourhood of N_i and by the uniqueness of regular neighbourhoods there exists an isotopy ϕ_i carrying Ω_i into H_i . See (A.3). Hence $M = \bigcup_{i=1}^m \phi_i^{-1}(H_i)$ and the proof is finished.

According to 2.1 and (2) above, for any one-ended open manifold M we see that $p - \text{cat}(M \times S^k)$ is either $p - \text{cat}(M)$ or $p - \text{cat}(M) + 1$. As a consequence of Theorem 3.3 we can determine the proper L-S category of $M \times S^k$ in some cases. More explicitly,

Theorem 3.5. *Let M be a one-ended open PL n -manifold, $n \geq 3$. If $p - cat(M) \geq \frac{n+k}{2} + 2$ then $p - cat(M \times S^k) = p - cat(M) + 1$. In particular if $p - cat(M) \geq \frac{n+5}{2}$ we have $p - cat(M \times S^1) = p - cat(M) + 1$.*

Example 3.6. It is clear that Theorem 3.5 does not hold if M is not one-ended. Indeed, for $M = S^2 \times \mathbb{R}$ we have $p - cat(M \times S^1) \leq 6$ since $cat(S^2 \times S^1) = 3$. Hence from Example 1.5 we get $p - cat(M \times S^1) = 6 > 5 = p - cat(M) + 1$.

In the proof of Theorem 3.5 we need the following

Lemma 3.7. *Let X and Y be path connected spaces in \mathcal{P} . If Y is compact then the projection $p_1 : X \times Y \rightarrow X$ induces a bijection $p_{1*} : [\mathbb{R}_+, X \times Y] \cong [\mathbb{R}_+, X]$ between strong ends.*

Proof. Given $y_0 \in Y$ let $j : X \rightarrow X \times Y$ be the inclusion $j(x) = (x, y_0)$. It is clear that $pj = id_X$ and hence p_{1*} is onto and j_* is injective. Moreover, given any ray $r : \mathbb{R}_+ \rightarrow X \times Y$ let $H : p_2r \simeq c_{y_0}$ be a homotopy where c_{y_0} is the constant map $c_{y_0}(t) = y_0$. Although H is not a proper map the map $\tilde{H}(t, s) = (p_1r(t), H(t, s))$ is a proper homotopy such that $\tilde{H}(t, 0) = r(t)$ and $\tilde{H}(t, 1)$ is a ray in $X \times \{y_0\}$. We have shown that j_* is onto and hence j_* as well as p_{1*} are bijections.

Proof of 3.5 We have $p - cat(M) \leq p - cat(M \times S^k) \leq p - cat(M) + 1$ by Proposition 2.1. Assume for a moment $s = p - cat(M \times S^k) = p - cat(M) \geq \frac{n+k}{2} + 2$. By Theorem 3.3 with $c = 0$ applied to $M \times S^k$ there are closed half-spaces H_1, H_2, \dots, H_s with $M \times S^k = \cup_{i=1}^s H_i$. Let $R \subseteq H_1$ be an embedded ray with H_1 collapsing properly to R and hence H_1 is a regular neighbourhood of R ; see (A.3). Let $x_0 \in S^k$ be any point. By Lemma 3.7 we can find an embedded ray $R' \subseteq M \times \{x_0\}$ such that both R and R' define the same strong end of $M \times S^k$. Hence there exists a proper homotopy $G : \mathbb{R}_+ \times I \rightarrow M \times S^k$ with $G(\mathbb{R}_+ \times \{0\}) = R$ and $G(\mathbb{R}_+ \times \{1\}) = R'$. As $dim M \times S^k \geq 4$ there exists an ambient isotopy of $M \times S^k$ which carries R to R' ; see (A.5). Now we use the uniqueness of regular neighbourhoods (A.3) to find an isotopy carrying H_1 to a small regular neighbourhood N of R' with $N \cap (M \times \{x\}) = \emptyset$ for some $x \neq x_0$. Hence $M \times \{x\} \subseteq Z = H_2 \cup \dots \cup H_s$ and so the restriction $r = p|_Z : Z \rightarrow M \times \{x\}$ of the obvious projection is a proper retraction. Hence $p - cat(M \times \{x\}) \leq p - cat(Z) \leq s - 1$ which is a contradiction.

Recently Rudyak ([14]; 3.8) has proved that Singhof's Theorem implies the stronger result $cat(M \times S^k) = cat(M) + 1$. Rudyak's arguments can be repeated here to derive from Theorem 3.5 the following

Theorem 3.8. *Let M be a one-ended open PL n -manifold, $n \geq 3$. If $p - cat(M) \geq \frac{n+5}{2}$ we have $p - cat(M \times S^k) = p - cat(M) + 1$ and $p - cat(M \times S^{m_1} \times \dots \times S^{m_k}) = p - cat(M) + k$ for all $k \geq 1$.*

Remark 3.9. It has been a long standing conjecture due to Ganea that the equality $cat(X \times S^k) = cat(X) + 1$ always holds for any finite CW-complex X . In 1998, Iwase [7] gave counterexamples to this conjecture. In addition, Iwase [8] has obtained recently a closed manifold M for which $cat(M \times S^k) = cat(M)$. At present the authors do not know whether the corresponding version of Ganea's conjecture is true for the proper L-S category of one-ended open manifolds.

It is natural to ask for the behaviour of the invariant $p - \text{cat}(-)$ with respect to the product with \mathbb{R}^n . Next example shows the anomalous behaviour of the proper L-S category on the product of two open manifolds.

Example 3.10. Let $W^4 = \text{int}M^4$ be the interior of a contractible 4-manifold $M \neq B^4$. We have $p - \text{cat}(W) = \text{cat}(\partial M) = 4$; see [2] and [6]. But $W \times \mathbb{R}^k$ is homeomorphic to \mathbb{R}^{4+k} ([20]; Ch. VII) and hence $p - \text{cat}(W \times \mathbb{R}^k) = 2 < 5 = p - \text{cat}(W) + 1$.

However, if M is a closed manifold we have a formula similar to Theorem 3.5. Namely,

Theorem 3.11. *Let M be a closed n -manifold with $\text{cat}(M) \geq \frac{n+5}{2}$. Then $p - \text{cat}(M \times \mathbb{R}^k) = \text{cat}(M) + 1$, $k \geq 2$.*

Proof. Let $p - \text{cat}(M \times \mathbb{R}^k) = m$ and let $\{U_1, \dots, U_m\}$ be an open cover of $M \times \mathbb{R}^k$ for which each closure \overline{U}_j is properly categorical. Moreover, we can assume that the proper deformations $H^j : \overline{U}_j \times I \rightarrow M \times \mathbb{R}^k$ satisfy $H_1^j(\overline{U}_j) \subseteq \{x_0\} \times \{0\} \times \mathbb{R}_+$ for some $x_0 \in M$. Here we use that $M \times \mathbb{R}^k$ is strongly one-ended. Then let $s \geq 1$ such that H^j carries $\Omega_j = [\overline{U}_j \cap (M \times S^{k-1} \times [s, \infty))] \times I$ into $M \times S^{k-1} \times [1, \infty)$ for $j = 1, \dots, m$. The obvious proper retraction of $[1, \infty)$ onto $[s, \infty)$ shows that $\{\Omega_1, \dots, \Omega_m\}$ is a properly categorical cover of $M \times S^{k-1} \times [s, \infty)$. Hence

$$\begin{aligned} \text{cat}(M) + 1 = \text{cat}(M \times S^{k-1}) &= p - \text{cat}(M \times S^{k-1} \times [s, \infty)) \\ &\leq p - \text{cat}(M \times \mathbb{R}^k) \leq \text{cat}(M) + 1. \end{aligned}$$

Here we use Rudyak's theorem, see Theorem 3.8 for the first equality and Proposition 2.1 for the last inequality.

Appendix: Some results on the PL topology of open manifolds

This appendix is a collection of the results on open PL manifolds which are used in this paper. We start recalling the notion of *proper collapse*. Given two polyhedra X and Y in \mathcal{P} it is said that there is an elementary proper collapse from Y onto X and we write $Y \searrow_{ep} X$ if $Y = X \cup C_1 \cup C_2 \cdots \cup C_n \dots$ where $\{C_i\}$ is a sequence of compact polyhedra satisfying $(C_i - X) \cap (C_j - X) = \emptyset$ if $i \neq j$ and $C_i \searrow C_i \cap X$. Then a proper collapse $Y \searrow_p X$ is a finite sequence of elementary proper collapses. A polyhedron X in \mathcal{P} is said to be *properly collapsible* if $X \searrow_p \mathbb{R}_+$.

Many results concerning ordinary collapses can be readily generalized to proper collapses. In this paper we will use the following proper analogue of ([13], Lemma 2.1)

Lemma A.1. *Let R, P and L be polyhedra in \mathcal{P} , and suppose $R \searrow_p P$. Then, there exists a polyhedron L' in \mathcal{P} such that $R \cup L \searrow_p P \cup L'$ and $\dim(L') \leq \dim(L)$.*

Proof. It suffices to show the result in the case $R \searrow_{ep} P$, i.e., $R = P \cup C_1 \cup C_2 \cup \dots$, where the C_i 's satisfy the conditions mentioned above for an elementary proper collapse. Thus, $P \cup C_i \searrow P$, for each i . The proof of ([15], Lemma 1.6.4) shows that there are polyhedra L'_i such that $P \cup C_i \cup L \searrow P \cup L'_i$, for each i , and

$\dim(L'_i) \leq \dim(L)$. Let $L' = \bigcup_i L'_i$. Then, $R \cup L \searrow_p P \cup L'$ and $\dim(L') \leq \dim(L)$.

It is worth noting that if $K \subseteq T$ are locally finite simplicial complexes and N is a second derived neighbourhood of K in T , then $|N| \searrow_p |K|$ ([16], Lemmas 6-7). This leads to the proper analogue of ([13], Lemma 2.2); namely,

Lemma A.2. *Let P be a polyhedron in \mathcal{P} , and $X, Y \subseteq P$ be subpolyhedra. Let $\{P_1, \dots, P_k\}$ be a polyhedral cover of X in P , and suppose n_1, \dots, n_k are non-negative integers with $\dim(Y) < \sum_{i=1}^k (n_i + 1)$. Then, there is a polyhedral cover $\{R_1, \dots, R_k\}$ of $X \cup Y$ in P such that $\bigcup_{i=1}^k R_i$ is a neighbourhood of Y in P and for each $1 \leq i \leq k$, $R_i \searrow_p P_i \cup N_i$ with $\dim(N_i) \leq n_i$.*

Proof. Let T be a triangulation of P such that K and L are subcomplexes of T which triangulate X and Y respectively. Note that if $k = 1$ we can just take $N_1 = Y$ and $R_1 = |J|$, where J is a second derived neighbourhood of $K \cup L$ in T . The proof is by induction on k . Let T_1, \dots, T_k be subcomplexes of T which triangulate P_1, \dots, P_k . Let L_1 be the n_1 -skeleton of L and let L_2 be its dual skeleton. Then, $\dim(L_2) \leq \dim(Y) - (n_1 + 1)$. Let $X' = X \cap (\bigcup_{i=2}^k P_i)$. By induction, there exists a polyhedral cover $\{R_2, \dots, R_k\}$ of $X' \cup |L_2|$ satisfying the required conditions with respect to the subpolyhedra $X', |L_2| \subseteq P$. Let $R_1 = |J|$, where J is a second derived neighbourhood of $T_1 \cup L_1$ such that $X \cup Y \subset |J| \cup (\bigcup_{i=2}^k R_i)$. Then, $\{R_1, \dots, R_k\}$ is the desired polyhedral cover of $X \cup Y$ in P .

For open manifolds we will use the following results involving the notion of proper collapse.

Theorem A.3. ([16]) *Let M be an open PL manifold and $X \subseteq M$ a closed subpolyhedron. Then a subpolyhedron $N \subseteq M$ is a regular neighbourhood of X in M if N is a topological neighbourhood of X in M and N properly collapses onto X . Moreover any two regular neighbourhoods of X are ambient isotopic, keeping X fixed.*

Theorem A.4. ([10]) *Suppose M is an open PL n -manifold and $X \subseteq M$ is a non-compact properly collapsible subpolyhedron. Then a regular neighbourhood of X in M is PL homeomorphic to \mathbb{R}_+^n .*

We also use two further results due to Maxwell [10]. The first one is easily derived from the proper unknotting theorem in ([10];5.2)

Theorem A.5. *Let M^n and Q^q be two open PL manifolds with $q \geq 2n + 2$. Suppose $f, g : M \rightarrow Q$ are two proper embeddings such that $f \simeq g$ relative ∂M . Then f is ambient isotopic to g .*

The second is the following proper engulfing theorem ([10];3.11)

Theorem A.6. *Let M be an open PL n -manifold. Suppose $Q \subseteq M$ is an inessential subpolyhedron of dimension $q \leq n - 3$. If M is properly c -connected with $c \geq 2q - n + 2$ there exists a properly collapsible subpolyhedron $P \subseteq M$ with $Q \subseteq P$ and $\dim(P - Q) \leq q + 1$. In particular Q is included in a half-space $H \subseteq M$.*

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