Harmonic-Killing vector fields*

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Abstract

In this paper we consider the harmonicity of the 1-parameter group of local infinitesimal transformations associated to a vector field on a (pseudo-) Riemannian manifold, to study this class of vector fields, which we call harmonic-Killing vector fields.

1 Introduction

Different properties have been considered for the integral flows corresponding to vector fields. For instance, when the corresponding 1-parameter group of local transformations consists of isometric, affine or conformal maps, a vector field is called respectively Killing, affine-Killing or conformal. However, harmonicity has only been used to study other aspects of vector fields. In [13] harmonic vector fields are defined as those having harmonic associated 1-form. Several authors ([6], [10]) use the harmonicity of the section induced on the tangent bundle with different lift metrics: Sasaki, complete, ...

We introduced the term 1-*harmonic-Killing* vector field for the case when the transformations have zero linear part of their tension field, which [9] had referred to as harmonic infinitesimal transformations. The approach emphasizes the importance of the complete lift metric for tangent bundles in the study of harmonicity. We point out that a vector field is a Jacobi field along the identity map if and only if it is a 1-harmonic-Killing vector field.

Given a 1-harmonic-Killing vector field, X, on a (semi-)Riemannian manifold (M, g) and a parallel $(1,1)$-tensor field, T, we use the definition of harmonic $(1,1)$-tensor field ([6]), as a harmonic map from (TM, g^C) to itself, to show that TX is

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1-harmonic-Killing if and only if T commutes with the Ricci operator of (M, g). This result gives examples of 1-harmonic-Killing vector fields on (co-)Kähler manifolds.

We study also the notion of *harmonic-Killing* vector fields, those for which the 1-parameter group of local transformations consist of harmonic maps. Such vector fields are characterized as Jacobi vector fields with harmonic flows. We show the relationship among Killing, affine-Killing, conformal and harmonic-Killing vector fields and provide the characterization of these kinds of vector fields with respect to the sections that they define.

2 Harmonic maps

Let (M, g) and (N, h) be Riemannian (or pseudo-Riemannian) manifolds with dim$M$ $= m$ and dim$N = n$; denote by ∇^M and ∇^N the Levi-Civita connections on M and N, respectively. A smooth map $\phi : (M, g) \to (N, h)$ defines a vector bundle $\phi^*TN \hookrightarrow TM$, with projection $\pi_1 : \phi^*TN \to M$. The set $\Gamma(\phi^*TN)$, of sections of ϕ^*TN, are called vector fields along ϕ.

There exists a unique linear connection, $\phi^*\nabla^N$, induced by ϕ on $\phi^*(TN)$, defined for all $x \in M$, $X \in \Gamma(TM)$ and $Y' \in \Gamma(TN)$ by

$$(\phi^*\nabla^N)_X(Y' \circ \phi)(x) = (\nabla^N_{d\phi(X)}Y') \circ \phi(x) = \nabla^N_{(d\phi)_x(X(x))}(Y'(\phi(x))).$$

Let us denote by ∇' the naturally induced connection on the tensor product $T^*M \otimes \phi^*(TM)$ by the connection ∇^M on T^*M, and the connection $\phi^*\nabla^N$ on $\phi^*(TN)$. Then

$$\Big(\nabla'(d\phi)\Big)_X(Y) = (\phi^*\nabla^N)_X\Big(d\phi(Y)\Big) - (d\phi)(\nabla^M_X Y),$$

is the second fundamental form of ϕ and the section of $\phi^*(TN)$, $\tau(\phi) = trace_g\Big(\nabla'(d\phi)\Big)$, is called the tension field of ϕ. ϕ is said to be *harmonic* if $\tau(\phi) = 0$, and *totally geodesic* if $\Big(\nabla'(d\phi)\Big) = 0$. (See eg. [3], [4].)

Now, let $U \subset M$ and $V \subset N$ be domains with coordinates $(x^1, \ldots, x^m)$ and $(y^1, \ldots, y^n)$ respectively, such that $\phi(U) \subset V$. Locally, the map ϕ has the representation: $y^a = \phi^a(x^1, \ldots, x^m)$. Then the second fundamental form of ϕ at $x \in U$ can be expressed locally by:

$$(\nabla'(d\phi)) = (\nabla'(d\phi))^a_{ij} dx^i \otimes dx^j \otimes \partial_a$$

for $i, j, k = 1, \ldots, m$; $a, b, c = 1, \ldots, n$, where

$$(\nabla'(d\phi))^a_{ij}(x) = \frac{\partial^2\phi^a}{\partial x^i \partial x^j}(x) - {}^g\Gamma^k_{ij}(x)\frac{\partial \phi^a}{\partial x^k}(x) + {}^h\Gamma^a_{bc}(\phi(x))\left(\frac{\partial \phi^b}{\partial x^i}(x)\frac{\partial \phi^c}{\partial x^j}(x)\right).$$

With respect to the usual basis $dx^i \otimes \partial_a$ of the fibre of $T^*M \otimes \phi^*TN$ at $x \in M$, we have the expression $(i, j = 1, \ldots, m;\ a = 1, \ldots, n)$

$$\tau(\phi) = \tau(\phi)^a\partial_a = g^{ij}(\nabla'(d\phi))^a_{ij}\partial_a \in (\phi^*TN)_x.$$

3 Harmonic-Killing and 1-harmonic-Killing vector fields

It is well known that any vector field $X \in \Gamma(TM)$ gives rise to a local 1-parameter group of local diffeomorphisms $I \ni t \mapsto \varphi_t \in \mathrm{Diff}(M)$, where I is some neighborhood of $0 \in \mathbb{R}$, by solving the autonomous system of ordinary differential equations,

$$X \circ \varphi_t = \dot{\varphi}_t, \quad \varphi_0 = 1.$$

Harmonic-Killing vector fields will be characterized by the property that they have integral flows that act on the manifold as (local) harmonic diffeomorphisms. In general, harmonic maps are not preserved under composition, so we might not have a group formed by harmonic diffeomorphisms. However, [9] used the term harmonic infinitesimal transformation in this context to mean that the linear part of the tension field vanishes; we shall refer to this property as 1-harmonicity. Corresponding definitions are given in [14] for isometric, totally geodesic (motions and affine motions in his terminology) and conformal infinitesimal transformations.

Definition 3.1. *A vector field X on a pseudo-Riemannian m-manifold (M, g) is called* 1*-harmonic-Killing (*1*-h-K) if the local 1-parameter group of infinitesimal transformations associated to X,$\{\varphi_t\}$ $t\in I$, verifies:*

$$\frac{d\tau(\varphi_t)}{dt}|_{t=0} = 0,$$

where $\tau(\varphi_t)$ is the tension field of φ_t.

Rewriting some known results ([3], [5], [9], [10]) we have the following characterization theorem.

Theorem 3.1. *On a pseudo-Riemannian manifold (M, g) the following statements are equivalent:*

(i) *X is a* 1*-harmonic-Killing vector field.*

(ii) *$g^{ij}(\mathcal{L}_X\Gamma^k_{ij}) = 0$, $i, j, k = 1, \ldots, m$, where $\mathcal{L}$ denotes the Lie derivative and g^{ij} are the components of the inverse matrix of the metric g and Γ^k_{ij} are the Christoffel symbols of the Levi-Civita connection of g.*

(iii) *$X : (M, g) \longrightarrow (TM, g^C)$ is a harmonic section, where g^C denotes the complete lift of g.*

(iv) *$\triangle X = Ric(X, .)$, where $\triangle = d\delta + \delta d$, ($d$ = differential, δ = codifferential) and Ric denotes the Ricci tensor of (M, g).*

(v) *X is a Jacobi vector field along the identity.* ■

In [13] is used the following notation, $\Box X = \triangle X - 2QX$ where $\triangle X =: \triangle\xi = (\delta d + d\delta)\xi$, ($\xi$ is the 1-form associated to X) and QX is the $(1, 1)$-tensor field called the *Ricci operator* and defined by $g(QA, B) = Ric(A, B)$; A, B vector fields on M. In our context

$$\Box X(x) = -g^{ij}(x)(\mathcal{L}_X\Gamma^k_{ij}(x))\frac{\partial}{\partial x^k} = -\tau(\varphi_t)^{k+m}(x)\frac{\partial}{\partial x^k}.$$

Having in mind the definition of a harmonic $(1,1)$-tensor field ([6]), as a harmonic map from (TM, g^C) to itself, the characterization (iii) of 1-h-K vector fields and the fact that the composition of harmonic maps is not, in general, a harmonic map, we obtain the following results.

Lemma 3.1. *Let (M,g) be a Riemannian manifold, X a 1-h-K vector field, and T a $(1,1)$-tensor field on M, then TX is a 1-h-K vector field if and only if:*

$$g^{ij}X^l[\,\partial^2_{ij}T^k_l - \Gamma^t_{ij}(\partial_t T^k_l) - (\partial_l\Gamma^t_{ij})T^k_t + (\partial_t\Gamma^k_{ij})T^t_l + \Gamma^k_{it}(\partial_j T^t_l) \\ + \Gamma^k_{jt}(\partial_i T^t_l) + (\nabla_i T)^k_j + (\nabla_j T)^k_i\,] = 0,$$

where $i,j,k,l,t = 1,\ldots,m$, X^i and T^i_j are the components of the vector field X and the $(1,1)$-tensor field, respectively, and $(\nabla_i T)^k_j$ denotes the components of the covariant derivative of T with respect to $\frac{\partial}{\partial x^i}$.

Proof: The tension field of TX can be considered as the tension field of the composition map $T \circ X : (M,g) \longrightarrow (TM, g^C) \longrightarrow (TM, g^C)$. Using the expression of the tension field of a composite map and the results obtained in [6] and in [10], we have

$$\tau(T\circ X) = \tau(T\circ X)^k \frac{\partial}{\partial x^k} + \tau(T\circ X)^{k+m}\frac{\partial}{\partial x^{k+m}},$$

where $\tau(T\circ X)^k = 0$, and

$$\begin{aligned}\tau(T\circ X)^{k+m} \;=\;& \tau(X)^{l+m}T^k_l + \\ & g^{ij}X^l[\,\partial^2_{ij}T^k_l - \Gamma^t_{ij}(\partial_t T^k_l) - (\partial_l\Gamma^t_{ij})T^k_t + (\partial_t\Gamma^k_{ij})T^t_l + \\ & \Gamma^k_{it}(\partial_j T^t_l) + \Gamma^k_{jt}(\partial_i T^t_l) + (\nabla_i T)^k_j + (\nabla_j T)^k_i\,],\end{aligned}$$

$i,j,k,l,t = 1,\ldots,m$.

If X is a 1-h-K vector field on M then $\tau(X) = 0$, and we obtain the result. ■

Theorem 3.2. *If X is a 1-h-K vector field on M and T is a parallel $(1,1)$-tensor field, then TX is a 1-h-K vector field on M if and only if $T(Q(X)) = Q(T(X))$, where Q is the Ricci operator.*

Proof: If T is a parallel $(1,1)$-tensor field then

$$\partial^2_{ij}T^k_l = \Gamma^t_{lj}(\partial_i T^k_t) + (\partial_i\Gamma^t_{lj})T^k_t - \Gamma^k_{tj}(\partial_i T^t_l) - (\partial_i\Gamma^k_{tj})T^t_l,$$

therefore

$$\begin{aligned}\tau(T\circ X)^{k+m} \;=\;& g^{ij}X^l[\,T^k_t(\partial_i\Gamma^t_{lj} - \partial_l\Gamma^t_{ij} + \Gamma^t_{in}\Gamma^n_{lj} - \Gamma^t_{nl}\Gamma^n_{ij}) + \\ & T^n_l(\partial_n\Gamma^k_{ij} - \partial_i\Gamma^k_{nj} + \Gamma^k_{nt}\Gamma^t_{ij} - \Gamma^k_{it}\Gamma^t_{nj})],\end{aligned}$$

$i,j,k,l,n,t = 1,\ldots,m$.

Considering the local components of the curvature

$$R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^l} = R^k_{ijl}\frac{\partial}{\partial x^k} = [\partial_i\Gamma^k_{lj} - \partial_j\Gamma^k_{il} + \Gamma^k_{it}\Gamma^t_{lj} - \Gamma^k_{tj}\Gamma^t_{il}]\frac{\partial}{\partial x^k},$$

we obtain

$$\begin{aligned}\tau(T\circ X)^{k+m} &= g^{ij}X^l[\,T^k_t R^t_{ilj} + T^t_l R^k_{tij}\,] = g^{ij}X^l[\,T^k_t R^t_{ilj} - T^t_l R^k_{itj}\,]\\ &= T^k_t R^t_l X^l - R^k_t T^t_l X^l,\end{aligned}$$

$i,j,k,l,t = 1,\ldots,m$. So globally, $T\circ X$ harmonic if and only if $T(Q(X)) = Q(T(X))$.

■

Example 3.1. On a Kähler manifold (M,g,J), the complex structure J satisfies $\nabla J = 0$ and $QJ = JQ$, therefore for all 1-h-K vector field on (M,g,J), JX is also 1-h-K.

Example 3.2. A co-Kähler $2m+1$-dimensional manifold (M,g,φ,ξ,η) is an almost-contact manifold where the $(1,1)$-tensor φ satisfies $\nabla\varphi = 0$, with respect to the Levi-Civita connection associated to g. A co-Kähler manifold (M,g,φ,ξ,η) is locally the product $(N\oplus\mathbb{R}, h\oplus dt^2, J\oplus 0, \frac{\partial}{\partial t}, dt)$, where (N,h,J) a $2m$-dimensional Kähler manifold, moreover $Q\varphi = \varphi Q$. So for all 1-h-K vector fields on (M,g,φ,ξ,η), φX is also 1-h-K.

Example 3.3. Let (M,g) be a (semi-)Riemannian manifold with parallel Ricci operator, Q. As Q is a selfcommuting $(1,1)$-tensor, then, for all 1-h-K vector fields X, the vector field QX is 1-h-K.

A vector field satisfying the condition (iv) in *Theorem* 3.1 was called a geodesic vector field in [16], where it was conjectured that the flow of such a vector field would preserve geodesics 'on average'. More precisely, does the flow of a Jacobi field consist of harmonic maps?. *Theorem* 3.1 says that the Jacobi vector fields along the identity are the 1-h-K vector fields.

It is clear that harmonicity implies 1-harmonicity, but the converse is not true in general (see *Example* 4.1).

Definition 3.2. *A vector field X on a pseudo-Riemannian manifold (M,g) is called harmonic-Killing (h-K) if the local 1-parameter group of infinitesimal transformations associated to X,$\{\varphi_t\}$ $t\in I$, is a group of harmonic maps.*

Remark 3.1. Let (M,g) be a compact Riemannian flat manifold. If X is a 1-h-K vector field then (using Bochner techniques) X is a parallel vector field and therefore Killing. As the isometries are harmonic maps, we obtain that X is a h-K vector field. In conclusion, for a compact Riemannian flat manifold 1-h-K and h-K vector fields coincide.

Remark 3.2. It is known that all holomorphic maps between Kähler manifolds are harmonic. We know also (using the Lichnerowicz Rigidity Theorem [3, p.38]), that if the infinitesimal transformations are harmonic variations of the identity, which is holomorphic, then they are holomorphic variations. Therefore, X is h-K on a compact *Kähler* manifold if and only if X is holomorphic (see [2] for more applications on Kähler manifolds.)

Rewritten in our terminology, the following result in relation with the Ricci curvature is known.

Proposition 3.1. [16] *If in a compact Riemannian manifold* (M, g)*, the Ricci tensor Ric is negative semi-definite, (i.e. for all vector fields* V *on* M*,* $Ric(V, V) \leq 0$*), then a vector field,* X*, is* 1*-h-K if and only if* X *is parallel. Moreover, if Ric is negative definite, (i.e.* $Ric(V, V) = 0$ *iff* $V = 0$*), then the zero sections are the only* 1*-h-K vector fields.* ■

The proof of this proposition is based on the classical Bochner technique for compact Riemannian manifolds (as in *Remark* 3.1), which does not work so well in the pseudo-Riemannian case. A study of *h-K* vector fields on pseudo-Riemannian manifolds (in particular timelike *h-K* vector fields in the Lorentzian case) can be found in [1], where a similar approach to that of [11] is followed.

4 Relations among Killing, affine-Killing, conformal and harmonic-Killing vector fields

We recall the following classical terminology:

(i) X is called a *Killing* vector field if the 1-parameter group of infinitesimal transformations generated by X is a group of isometries. Equivalently, $\mathcal{L}_X g = 0$.

(ii) X is called an *affine-Killing* vector field if the 1-parameter group of infinitesimal transformations generated by X is a group of totally geodesic maps. Equivalently, $\mathcal{L}_X \nabla = 0$.

(iii) X is called a *conformal* vector field if the 1-parameter group of infinitesimal transformations generated by X are conformal maps. Equivalently, $\mathcal{L}_X g = 2\rho g$, for some function ρ.

It is important to point out that for these types of vector fields it does not make sense to talk about 1-Killing, 1-affine-Killing and 1-conformal as separated from Killing, affine-Killing and conformal. Indeed, in the early fifties (see [14]) the corresponding concepts to 1-isometry, 1-affine and 1-conformal map for the local 1-parameter group of infinitesimal transformations associated to a vector field X, $\{\varphi_t\}$ $t \in I$, were called *infinitesimal motion, infinitesimal affine collineation and infinitesimal conformal transformation.* It is well known that the modern characterization of these kind of vector fields includes the proof of the equivalence between infinitesimal motion and isometry, infinitesimal affine collineation and affine (or totally geodesic) map, and finally, infinitesimal conformal and conformal for the map φ_t (see [7]). However, the case of 1-harmonic and harmonic is completely different as we will show in the following example.

Example 4.1. Consider $(\mathbb{R}^2, \delta_{ij})$, where $g_{ij} = \delta_{ij}$ is the euclidean metric of $\mathbb{R}^2$. Let $X = (e^x \sin(y), 0)$ be a vector field on $\mathbb{R}^2$. The 1-parameter group of transformations, φ_t, associated to X is given by the expression:

$$\varphi_t(x, y) \;=\; (x, y) + t(e^x \sin(y), 0) + \frac{1}{2}t^2(e^{2x}(\sin(y))^2, 0) + O(t^3).$$

and its tension field is the following:

$$\tau(\varphi_t)(x,y) = t^2(e^{2x},0) + O(t^3).$$

Therefore, the linear part of the tension field vanishes and X is a 1-h-K vector field, but the 1-parameter group of transformations, φ_t, associated to X it is not formed by harmonic maps. This example suggests introducing the definition of Jacobi fields with harmonic flows, that is, harmonic-Killing vector fields.

Let (M,g) be a semi-Riemannian manifold and X a vector field on M. The pull-back of the metric g^C by the section $X : (M,g)\rightarrow(TM,g^C)$, (X^*g^C), has the expression:

$$\begin{aligned}(X^*g^C)(A,B) &= g((\nabla X)(A),B) + g(A,(\nabla X)(B)) \\ &= (\mathcal{L}_X g)(A,B)\end{aligned} \tag{4.1}$$

for all vector fields A, B on M. Then, as in $Theorem$ 3.1, the kind of vector fields defined before can be characterized by properties of $X : (M,g) \rightarrow (TM,g^C)$.

The equation (4.1) proves that X is Killing if and only if $Null(g^C) = dX(TM)$. Moreover, it is an easy calculation that the second fundamental form of the section X is equal to $\mathcal{L}_X\nabla$. Therefore, X is affine-Killing if and only if X is a totally geodesic section. Finally it is well known [13] that X is conformal if and only if $\mathcal{L}_X g = -\frac{2\,divX}{n} g$, which implies that $(X^*g^C)(A,B) = -\frac{2\,divX}{n} g(A,B)$, for all vector fields A, B on M. This shows that X is a conformal vector field if and only if X defines a conformal section of (TM,g^C).

Example 4.2. Consider the real plane $\mathbb{R}^2$ with the Euclidean metric $g_{ij} = \delta_{ij}, i,j = 1,2$, and the vector field

$$X(x^1,x^2) = X^1(x^1,x^2)\frac{\partial}{\partial x^1} + X^2(x^1,x^2)\frac{\partial}{\partial x^2}$$

satisfying

$$\frac{\partial^2 X^i}{\partial x^1 \partial x^1} = -\frac{\partial^2 X^i}{\partial x^2 \partial x^2} \neq 0, \quad \text{with } i = 1,2.$$

That is, X^i, $i = 1,2$, are harmonic non null functions from $\mathbb{R}^2$ to $\mathbb{R}$.

The totally geodesic condition (affine-Killing) for this type of example is equivalent to:

$$\frac{\partial^2 X^k}{\partial x^i \partial x^j} = 0, \quad \text{for all } i,j,k = 1,2.$$

In other words, the vector field X is 1-h-K if and only if

$$div(grad(X^i)) = 0, \quad \text{for } i = 1,2,$$

and totally geodesic if and only if the Jacobian matrix of $(grad(X^i))$ vanishes.

The condition for Killing is

$$\frac{\partial X^k}{\partial x^i} + \frac{\partial X^i}{\partial x^k} = 0, \quad \text{for all } i,k = 1,2.$$

Finally the condition for conformal vector field is

$$\frac{\partial X^k}{\partial x^i} + \frac{\partial X^i}{\partial x^k} = 2\rho(x^1, x^2)\delta_{ik}, \quad \text{for } \rho \text{ a differentiable function and all } i, k = 1, 2.$$

Then $X^1 = \frac{1}{2}(x^1)^2 - \frac{1}{2}(x^2)^2$ and $X^2 = 0$ provide an X that is 1-*h*-*K* but not affine-Killing because $\dfrac{\partial^2 X^1}{\partial x^1 \partial x^1} \neq 0$, not Killing because $2\dfrac{\partial X^1}{\partial x^1} \neq 0$, not conformal because $\dfrac{\partial X^1}{\partial x^2} \neq 0$ and not *h*-*K* because $\tau(\varphi_t)(x^1, x^2) = t^2(x^1, 0) + O(t^3)$. This example generalizes to $\mathbb{R}^n$, by taking n harmonic functions from $\mathbb{R}^n$ to $\mathbb{R}$.

A vector field X on a compact orientable Riemannian manifold without boundary is Killing if and only if $\square X = 0$ and $\delta X = 0$, also if X is affine-Killing then it is Killing [13].

Clearly Killing and affine-Killing vector fields are *h*-*K*. In the case of 1-*h*-*K* vector fields the next result tells us the condition for the converse.

Proposition 4.1. *Let X be a 1-h-K vector field on a compact orientable Riemannian manifold without boundary. Then it is Killing if and only if $\delta X = 0$.*

Proof: We have that X is 1-*h*-*K* if and only if $\square X = 0$ from *Theorem* 3.1 and this proves the result. ∎

For conformal vector fields we have the following result.

Proposition 4.2. [13] *A necessary and sufficient condition for a vector field X, on an m-dimensional compact orientable Riemannian manifold without boundary, to be a conformal vector field is that:*

$$\square X + \frac{m-2}{m} D\delta X = 0,$$

where $D\delta X$ is the vector field associated to the 1-form $d\delta X$. ∎

Therefore.

Proposition 4.3. *Conformal and 1-h-K vector fields on a compact orientable Riemannian manifold without boundary are related by the following:*

- *For $m = 2$ there is a $1:1$ correspondence between conformal and 1-h-K vector fields.*
- *For $m > 2$ if X is a conformal vector field then it is 1-h-K if and only if $d\delta X = 0$.* ∎

Remark 4.1. The *h*-*K* vector fields on (M, g) give rise to a special kind of harmonic variation of the identity id_M. Smith [12] studied the variations of $id_{\mathbb{S}^n}$, obtaining that in the case of $n \geq 3$ all harmonic variations of the identity of $\mathbb{S}^n$ are given by infinitesimal isometries. So, the *h*-*K* vector fields on $\mathbb{S}^n$, $n \geq 3$, are the Killing ones and therefore 1-*h*-*K*.

Smith [12] proves that the Jacobi fields along the identity on $\mathbb{S}^2$ are generated by 6 vector fields; 3 of these are given by infinitesimal isometries and the others by conformal transformations from $\mathbb{S}^2$ to $\mathbb{S}^2$, which are also harmonic maps. We can say that there are 6 generators of 1-h-K vector fields on $\mathbb{S}^2$ and all are h-K. Note that Lee and Tóth [8] have proved that there are no harmonic 'geodesic'-variations on $\mathbb{S}^2$.

[8] consider when a vector field X generates 'geodesic'-variations of $id_{\mathbb{S}^n}$, given by $f_t = \exp \circ (tX)$. They denote by $V(id_{\mathbb{S}^n})$ the set of all harmonic 'geodesic'-variations of the $id_{\mathbb{S}^n}$. The authors prove that $V(id_{\mathbb{S}^n}) = 0$ when n is even, and $V(id_{\mathbb{S}^{2n-1}})$ is a double cone over $SO(2n)/U(n)$. This result points out the difference with the variations obtained from the infinitesimal transformations associated to X.

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