Formula balancing and continuously valuated models

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Abstract

Uniform spaces can be Cauchy-completed; and if the base space was a first-order structure, this structure can be naturally extended to the completion. While common in algebra, this construction has been more recently used to produce new models of special set theories. We investigate here a natural way to "twist" the semantics of any structure according to a uniformity on its universe. We use it to relate the (classical first-order) theories of structures and dense substructures and apply it to the case of Cauchy-completions.

1 Introduction

Given a first-order structure endowed with a structure of uniform (or metric) space, the purpose of the current article is to study properties of formulas which remain invariant under "sufficiently small" movements: we want to be able to perform tests on the structure "with a precision of at most V", for any entourage V, and recover information on the original structure from the result of these tests for all V. More precisely, we try to approximate the classical first-order theory of the structure with uniformly continuous valuations.

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As a first application, this gives explicit links between the theory of a first-order structure with that of its Cauchy-completion under some uniformity[1]. This follows from the fact that the properties we are seeking for are invariant under the operation of restriction to a dense substructure.

The naive approach is to associate to a first-order structure X a family X_ϵ of first-order structures, with the same universe, but in which the equality and the other relations are interpreted "up to small movements": if, say, d is a metric on X and $\epsilon > 0$, we say that $X_\epsilon \vDash R[\vec{x}]$ if and only if there is a $\vec{y}$ which is componentwise ϵ-close to $\vec{x}$ and for which $X \vDash R[\vec{y}]$. However, unless we introduce serious syntactical restrictions on the formula φ, there is in general no relationship between $X \vDash \varphi$ and $\{\epsilon \mid X_\epsilon \vDash \varphi\}$.

In the present paper we suggest a variant of this approach for which explicit relationships can be given, and formalize it inside of a variant of continuous many-valued model theory. Section 2 introduces the framework; sections 3 and 4 introduce and study a particular valuation; section 5 gives related results and examples. In 5.4 links with large-cardinal-based set-theoretical constructions are given. For a less formal introduction to the underlying notion of formula balancing and the way it links the theories of first-order structures to that of their Cauchy-completions, sections 3 and 5 (in particular 5.2) can be read independently.

2 Uniform model theory

We first introduce a basic formalism for uniformly continuous model theory. The topological background can be found e.g. in [10]. Here are the basic definitions:

Let $(X, \mathcal{E})$ be a uniform space[2], where $\mathcal{E} \subseteq \mathcal{P}(X \times X)$ is its filter of entourages. Its **Hausdorff hyperspace**, denoted by $\mathcal{H}(X)$, is the set of all closed subsets[3] of X endowed with the uniformity generated by $\{V^\star \mid V \in \mathcal{E}\}$, where for any two closed subsets F and G of X we say that $(F, G) \in V^\star$ if, and only if, for all $x \in F$ there is a $y \in G$ such that $(x, y) \in V$ and conversely for all $y \in G$ there is a $x \in F$ such that $(x, y) \in V$.

Definition 1. *We call* **uniform logic** *a uniform space* Ω *endowed with a family of uniformly continuous maps, called* **uniform connectives***, as follows:*

$$\wedge_\Omega \colon \Omega^2 \to \Omega \qquad \vee_\Omega \colon \Omega^2 \to \Omega \qquad \sim_\Omega \colon \Omega \to \Omega$$

$$\forall_\Omega : \mathcal{H}(\Omega) \to \Omega \qquad \exists_\Omega : \mathcal{H}(\Omega) \to \Omega$$

plus a constant $\top_\Omega$*, seen as a* 0*-ary map. Naturally,* Ω *is also called the* **space of truth-values***.*

Note that we do not expect any particular relationship between the connectives. Of course, the current definition immediately generalizes to logics with an arbitrary set of connectives and quantifiers.

[1]This is of course related to the ultraproduct construction, although the correspondence is less straightforward in our case.

[2]We always assume that uniform spaces are Hausdorff (T_2).

[3]We take the empty set as well.

The notion of first-order language is as usual; it may include constant, relation and function symbols. Given such a language $\mathcal{L}$ and a uniform logic Ω, a **uniform first-order structure** is a non-empty uniform space $(M, \mathcal{E})$ with a valuation $[\![\ \ ]\!]$ that interprets each symbol of $\mathcal{L}$ as follows:

- a constant symbol c is interpreted by a point $[\![c]\!]$ of M;
- an n-ary function symbol f is interpreted by a uniformly continuous map $[\![f]\!] : M^n \to M$;
- an m-ary relation symbol R is interpreted by a uniformly continuous map $[\![R]\!] : M^m \to \Omega$.

In the present paper we will only study the elementary model-theoretical results before we apply it to a particular kind of models. To this end note that the notions of term and formula are as usual. The interpretation of a term $t(x_1, \ldots, x_j)$ in M gives by induction a uniformly continuous map $[\![t]\!] : M^j \to M$, starting from $[\![x]\!] = \mathrm{id}_M$ for terms that are single variables.[4] We then define by induction[5]

$$\begin{array}{rcl}
[\![R(t_1, \ldots, t_m)]\!](\vec{x}) & = & [\![R]\!]([\![t_1]\!](\vec{x}), \ldots, [\![t_m]\!](\vec{x})) \\
[\![\psi \wedge \psi']\!](\vec{x}) & = & [\![\psi]\!](\vec{x}) \wedge_\Omega [\![\psi']\!](\vec{x}) \\
[\![\psi \vee \psi']\!](\vec{x}) & = & [\![\psi]\!](\vec{x}) \vee_\Omega [\![\psi']\!](\vec{x}) \\
[\![\neg\psi]\!](\vec{x}) & = & \sim_\Omega [\![\psi]\!](\vec{x}) \\
[\![(\forall v)\psi]\!](\vec{x}) & = & \forall_\Omega \left(\mathrm{Cl}\left\{[\![\psi]\!](\vec{x}, v) \,\middle|\, v \in M\right\}\right) \\
[\![(\exists v)\psi]\!](\vec{x}) & = & \exists_\Omega \left(\mathrm{Cl}\left\{[\![\psi]\!](\vec{x}, v) \,\middle|\, v \in M\right\}\right)
\end{array}$$

where Cl denotes the topological closure in Ω. We can check that $[\![\varphi]\!]$ is uniformly continuous for all φ. From this it follows that both occurrences of Cl can be removed from the above definitions if M is compact.

Definition 2. *We call* CL**-reduction filter** *a subset $\mathcal{F}$ of Ω which is stable under logical connectives in the following sense: for any $a, b \in \Omega$ and for any compact subset $\mathcal{K}$ of Ω,*

(a) $\top_\Omega \in \mathcal{F}$;

(b) if $a \in \mathcal{F}$ and $b \in \mathcal{F}$, then $a \wedge_\Omega b \in \mathcal{F}$;

(c) if $a \in \mathcal{F}$ or $b \in \mathcal{F}$, then $a \vee_\Omega b \in \mathcal{F}$;

(d) if $a \notin \mathcal{F}$, then $\sim_\Omega a \in \mathcal{F}$;

(e) if $\mathcal{K} \subseteq \mathcal{F}$, then $\forall_\Omega(\mathcal{K}) \in \mathcal{F}$;

(f) if $\mathcal{K} \cap \mathcal{F} \neq \emptyset$, then $\exists_\Omega(\mathcal{K}) \in \mathcal{F}$.

[4] This is the equivalent in uniform spaces of the Kripke-Joyal semantics in topoi.

[5] $\vec{x}$ abbreviates $x_1, \ldots, x_j$, where we assume as usual that we have enumerated all free variables of the left-hand side formulas.

The elements of a CL-reduction filter are of course the *designated* ("true enough") truth-values. This will let us easily relate the uniform semantics to the classical semantics. Of course, "CL" stands for "classical logic", and (a)-(f) are all instances of the general rule that when any given connective takes the value "true" in classical logic, then so should the corresponding connective from the uniform logic. The same mechanism is easily generalized to let any uniform logic with any set of connectives be reduced to any logic which implements the same set of connectives; CL could be replaced by any one of many common many-valued logics.

Example 1. The restriction on compact subsets in the previous definition is important. For example, suppose that Ω has a structure of complete boolean algebra in which $\wedge$, $\vee$, $\top$ and $\neg$ have their usual meaning, and $\forall$ and $\exists$ are the greatest lower bound and least upper bound of sets of values, respectively. The uniformity on Ω is arbitrary as long as it makes these connectives uniformly continuous. Then all CL-reduction filters are ultrafilters, and conversely all ultrafilters which are closed subsets of Ω are CL-reduction filters. Indeed, if $\mathcal{K}$ is a compact subset of Ω, then $\forall(\mathcal{K})$ is easily checked to adhere to $\{a_1 \wedge \ldots \wedge a_n \mid n \in \omega,\ a_1, \ldots, a_n \in \mathcal{K}\}$.

As previously mentioned, given a CL-reduction filter $\mathcal{F}$, any uniform first-order structure M can be turned into a classical structure $M_{\mathcal{F}}$ with the same universe and in the same language: for any m-ary relation symbol R and $x_1, \ldots, x_m \in M$ we define

$$M_{\mathcal{F}} \vDash R[x_1, \ldots, x_m] \iff [\![R]\!](x_1, \ldots, x_m) \in \mathcal{F}$$

In the sequel we will need to assume that formulas can be put in negation normal form (i.e. no negation sign appears but immediately before an atomic sub-formula [2]) without changing their valuation. A uniform logic Ω is called **negation-regular** if, and only if, for any $a, b \in \Omega$ and $\mathcal{A} \subseteq \Omega$,

$$\sim_\Omega (a \wedge_\Omega b) = \sim_\Omega a \ \vee_\Omega \sim_\Omega b \qquad \sim_\Omega \forall_\Omega(\mathrm{Cl}\,\mathcal{A}) = \exists_\Omega(\mathrm{Cl}\,\{\sim_\Omega c \mid c \in \mathcal{A}\})$$

$$\sim_\Omega (a \vee_\Omega b) = \sim_\Omega a \ \wedge_\Omega \sim_\Omega b \qquad \sim_\Omega \exists_\Omega(\mathrm{Cl}\,\mathcal{A}) = \forall_\Omega(\mathrm{Cl}\,\{\sim_\Omega c \mid c \in \mathcal{A}\})$$

$$\text{and} \qquad \sim_\Omega \sim_\Omega (a) = a.$$

Lemma 1. *If $\mathcal{F}$ is a CL-reduction filter in a negation-regular uniform logic and M is a compact uniform first-order structure, then for any first-order formula $\varphi(x_1, \ldots, x_m)$, we have*

for any $x_1, \ldots, x_m \in M$, if $M_{\mathcal{F}} \vDash \varphi[x_1, \ldots, x_m]$, then $[\![\varphi]\!](x_1, \ldots, x_m) \in \mathcal{F}$.

The proof is a straightforward induction on φ in negation-normal form.

3 Formula balancing

We now introduce a natural way to turn any classical first-order structure into a uniform structure.

In this paper we will consider (in a language $\mathcal{L}$) first-order structures $(M, \mathcal{E})$ endowed with a uniformity which is Hausdorff (T_2) and **compatible** in the following sense: any m-ary relation symbol of $\mathcal{L}$ is interpreted by a closed subset of M^m (for the box product topology), and any function symbol of $\mathcal{L}$ is interpreted by a uniformly continuous function.

Let R be any m-ary relation symbol of $\mathcal{L}$. Let $V \in \mathcal{E}$ be an entourage. For each such R and V we introduce a new m-ary relation symbol R_V; let $\mathcal{L}^\sim$ be the language obtained by replacing in $\mathcal{L}$ every R by the corresponding family of new symbols R_V. The constant and function symbols are left unchanged. Note that equality deserves no special treatment here; in $\mathcal{L}^\sim$ it will be replaced by a family of symbols $=_V$. So (in any case) $\mathcal{L}^\sim$ is a language without equality – actually we will not need to assume that $\mathcal{L}$ contained the equality in the first place.

Definition 3. *We let $\boldsymbol{M^\sim}$ be the $\mathcal{L}^\sim$-structure defined on the same universe as M by interpreting the new symbols R_V as "R holds up to V in M"; formally, for $\vec{x} \in M^m$,*

$$\boldsymbol{M^\sim} \vDash \boldsymbol{R_V[\vec{x}]} \iff \exists \vec{y} \text{ with } M \vDash R[\vec{y}] \text{ and } (x_1, y_1) \in V, \dots, (x_m, y_m) \in V.$$

Let φ be a formula of $\mathcal{L}$. An **approximation** of φ is a formula of $\mathcal{L}^\sim$ whose image under the obvious reduction map ($R_V \mapsto R$) is exactly φ. An approximation $\varphi_2^\sim$ of φ is **finer** than an approximation $\varphi_1^\sim$ of φ if each time that a R_V symbol appears in $\varphi_1^\sim$, the corresponding R_W symbol that appears at the same place in $\varphi_2^\sim$ satisfies $W \subseteq V$.

Definition 4. *A formula $\varphi(\vec{x})$ in $\mathcal{L}$ with n free variables is said* **balanced in** M **at the point** $\vec{x} \in M^n$ *if, and only if, for any approximation of φ there is a finer approximation $\varphi^\sim$ such that $M^\sim \vDash \varphi^\sim[\vec{x}]$.*

The core of the paper is dedicated to the study of the relationships between formula balancing and satisfaction in M.

Lemma 2. *Let $\varphi_1^\sim$ be a formula in $\mathcal{L}^\sim$. Let $\varphi_2^\sim$ be the formula obtained from $\varphi_1^\sim$ by replacing any occurrence of a R_V symbol by R_W, for some $W \subseteq V$. Then:*

- *if the occurrence we replaced is enclosed by an* even *number of negation signs, then $M^\sim \vDash \varphi_2^\sim[\vec{x}] \ \Rightarrow\ M^\sim \vDash \varphi_1^\sim[\vec{x}]$;*
- *if the occurrence we replaced is enclosed by an* odd *number of negation signs, then $M^\sim \vDash \varphi_2^\sim[\vec{x}] \ \Leftarrow\ M^\sim \vDash \varphi_1^\sim[\vec{x}]$.*

Proof. Write $\varphi_1^\sim$ and $\varphi_2^\sim$ in negation normal form. Clearly, as $W \subseteq V$, we have $R_W^{M^\sim} \subseteq R_V^{M^\sim}$, and subsequently the extension of $\neg R_W$ in $M^\sim$ contains the extension of $\neg R_V$. The rest follows by induction. ∎

Remark 1. We can define for any $V, W \in \mathcal{E}$ the **$(\frac{W}{V})$-approximation** of a formula φ to be its approximation obtained by replacing each occurrence of all relation symbols R by either R_V or R_W, depending on whether the occurrence was enclosed by an even or odd number of negation signs. It follows from lemma 2 that φ is balanced at the point $\vec{x}$ if, and only if, for any $V \in \mathcal{E}$ there is a $W \in \mathcal{E}$ (optionally with $W \subseteq V$) such that $M^\sim$ satisfies the $(\frac{W}{V})$-approximation of φ at $\vec{x}$.

4 Approximation logic

We now proceed to build a uniform logic appropriate to formula balancing. We want to show:

Theorem 3. *Let $(M, \mathcal{E})$ be a uniform space. There exists a negation-regular uniform logic Ω and a CL-reduction filter $\mathcal{F}_{Pd}$ such that for any first-order structure on M in any first-order language $\mathcal{L}$, compatible with the uniformity, there is a valuation $[\![\ ]\!]$ providing a uniform structure on M that captures the notion of formula balancing in the following sense: for any formula φ of $\mathcal{L}$ with n free variables and any $\overrightarrow{x} \in M^n$, φ is balanced at $\overrightarrow{x}$ if and only if $[\![\varphi]\!](\overrightarrow{x}) \in \mathcal{F}_{Pd}$.*

Remark 2. The uniform logic Ω can actually be built in a way that only depends on "structural" properties of $\mathcal{E}$ and not on the universe M; for example, there is a single Ω that captures balancing for all metric spaces.

In view of the above theorem, the following is a corollary of lemma 1:

Theorem 4. *Assume that M is a classical first-order structure with a compact compatible uniformity. Then for any sentence φ of $\mathcal{L}$, if $M \vDash \varphi$, then φ is balanced in M.*

We will also see that with syntactical restrictions, similar relationships can be derived in the other direction, as well as in the non-compact case.

In the sequel of the section we prove theorem 3.

Let $(M, \mathcal{E})$ be a uniform space. We put on $\mathcal{E}$ the order defined by: for any $U, V \in \mathcal{E}$, we say that $U \prec V$ if, and only if, there is a $W \in \mathcal{E}$ such that[6] $W \circ U \circ W \subseteq V$. We extend the order to $\mathcal{E} \times \mathcal{E}$ as follows: for any $U_1, U_2, V_1, V_2 \in \mathcal{E}$, we say that $(U_1, U_2) \prec (V_1, V_2)$ if, and only if, $U_1 \succ V_1$ and $U_2 \prec V_2$. Let

$$\begin{aligned}
\Omega &= \{A \subseteq \mathcal{E} \times \mathcal{E} \mid \forall (V, W) \in A \ \ \forall V' \supseteq V \ \ \forall W' \subseteq W, \ \ (V', W') \in A\} \\
j &: \Omega \longrightarrow \Omega \\
& \quad A \longmapsto \{T \in \mathcal{E} \times \mathcal{E} \mid \forall S \prec T \ \ S \in A\} \\
\Omega_j &= \text{image of } j \text{ in } \Omega
\end{aligned}$$

j is a "closure operation" on Ω; a set $A \in \Omega_j$ is called a *closed set* of Ω. The uniform logic we are looking for is Ω_j, which plays the role of "cuts" of $\mathcal{E} \times \mathcal{E}$; we endow it with the "cut uniformity" generated by the basis $(W^*)_{W \in \mathcal{E}}$, where for $A, B \in \Omega_j$, we say that $(A, B) \in W^*$ if, and only if, the following two conditions hold for any $V_1, V_2 \in \mathcal{E}$:

1. if $(V_1, W \circ V_2 \circ W) \in A$ then $(W \circ V_1 \circ W, V_2) \in B$;
2. if $(V_1, W \circ V_2 \circ W) \in B$ then $(W \circ V_1 \circ W, V_2) \in A$.

[6] $V \circ W$ denotes the composition of binary relations, that is, $\{(x, z) \mid \exists y \ (x, y) \in V, (y, z) \in W\}$

Remark 3. As a uniform space, Ω_j is not related to $\mathcal{H}(\mathcal{E}\times\mathcal{E})$. For example, in the latter the empty set is an isolated point, whereas Ω_j has no isolated point in general. Moreover, in hyperspaces, $(A,B)\mapsto A\cup B$ is continuous but $(A,B)\mapsto A\cap B$ is not [13]; both are uniformly continuous in Ω_j.

The construction $W\circ V_1\circ W$ can be seen as a way to make V_1 a little bit larger, in the sense that if you see V_1 as much bigger than W, then what you do is add a "W-little bit" to both sides of V_1. We add it to both sides for symmetry reasons.

The uniform connectives are given by (for any $A,B\in\Omega_j$ and $\mathcal{A}$ closed subset of Ω_j):

$$\begin{array}{rcl}\top_j & = & \mathcal{E}\times\mathcal{E}\\ A\wedge_j B & = & A\cap B\\ A\vee_j B & = & A\cup B\\ \sim_j A & = & j(\{(V,W)\mid (W,V)\notin A\})\quad (\text{“transposed complement”})\\ \forall_j\mathcal{A} & = & \bigcap\mathcal{A}\\ \exists_j\mathcal{A} & = & j(\bigcup\mathcal{A})\end{array}$$

Remark 4. Ω and Ω_j are complete Heyting algebras: the former is the internal locale of the topos of presheaves on the poset $(\mathcal{E}\times\mathcal{E},(\supseteq,\subseteq))$; the latter is generated from the former by the (sheaf) topology j. For more about Grothendieck topoi see e.g. [11]. Note that $\sim_j$ is not an intuitionistic negation, but a paraconsistent one: $A\wedge_j(\sim_j A)$ might take non-$\perp$ values.[7]

We split the proof of the theorem into the following lemmas:

Lemma 5. *Ω_j with the above-defined uniform connectives is a uniform logic.*

Lemma 6. *The following defines a CL-reduction filter on Ω_j:*

$$\mathcal{F}_{Pd}=\{A\in\Omega_j\mid \pi_2[A]=\mathcal{E}\}$$

where $\pi_2:\mathcal{E}\times\mathcal{E}\to\mathcal{E}$ is the projection on the second factor and the notation $f[X]$ means $\{f(x)\mid x\in X\}$.

Note that $\mathcal{F}_{Pd}$ is neither open nor closed in Ω_j; in fact, both $\mathcal{F}_{Pd}$ and $\Omega_j\setminus\mathcal{F}_{Pd}$ are dense in Ω_j.

Lemma 7. *Suppose that $(M,\mathcal{E})$ is endowed with a first-order structure in a language $\mathcal{L}$. For any m-ary relation symbol R of $\mathcal{L}$ define*

$$\begin{array}{rcl}[\![R]\!]:M^n & \longrightarrow & \Omega_j\\ \vec{x} & \longmapsto & j\left(\left\{(V,W)\in\mathcal{E}\times\mathcal{E}\;\middle|\; M^{\sim}\vDash R_V[\vec{x}]\right\}\right)\end{array}$$

[7] It can even take designated values, although it cannot be equal to $\top$.

The valuation of constant and function symbols is given by their (unmodified) interpretation in M. Then $[\![\ ]\!]$ provides a uniform structure for the logic Ω_j and the valuation of any formula φ of $\mathcal{L}$ is equal to

$$\begin{aligned} [\![\varphi]\!] : M^n &\longrightarrow \Omega_j \\ \vec{x} &\longmapsto j\left(\left\{(V,W) \in \mathcal{E}\times\mathcal{E} \;\middle|\; M^\sim \vDash ((\tfrac{W}{V})\text{-}approx\ of\ \varphi)[\vec{x}]\right\}\right) \end{aligned}$$

Lemma 8. *The uniform logic Ω_j is negation-regular.*

Lemma 9. *For any formula φ of $\mathcal{L}$ with n free variables and any $\vec{x} \in M^n$, φ is balanced at $\vec{x}$ if and only if $[\![\varphi]\!](\vec{x}) \in \mathcal{F}_{Pd}$.*

Proof of lemma 5. We easily check that Ω_j as defined is a T_2 uniform space.

- $\wedge_j$ is clearly uniformly continuous.
- $\vee_j$ is well defined, i.e. the union of two closed sets is a closed set: indeed, let $A = j(A)$ and $B = j(B)$ and assume that $T \in j(A \cup B)$ but $T \notin A$. There is a $(U_1, U_2) \prec T$ such that $(U_1, U_2) \notin A$. We must show that $T \in B = j(B)$. This is true because for any $(V_1, V_2) \prec T$ we can show that $S = (V_1 \cap U_1, V_2 \cup U_2)$ satisfies $S \prec T$ but $S \notin A$, hence $S \in B$ and $(V_1, V_2) \in B$.

  The uniform continuity of $\vee_j$ is clear.
- $\sim_j$ is uniformly continuous: although it might be false that $\forall (A, A') \in W^*$ $(\sim_j A, \sim_j A') \in W^*$, it is the case that $\forall (A, A') \in W^*$ $(\sim_j A, \sim_j A') \in (W \circ W)^*$.
- $\forall_j$ is clear.
- $\exists_j$ is similar to $\sim_j$. ∎

Proof of lemma 6. $\mathcal{F}_{Pd}$ clearly satisfies conditions (a)-(d) and (f) of definition 2. The proof of (e) is more involved. First note that with the same definition $\mathcal{F}_{Pd}$ can be extended to the whole of Ω; then for any $A \in \Omega$ we have $A \in \mathcal{F}_{Pd}$ if, and only if, $j(A) \in \mathcal{F}_{Pd}$.

We must prove that if $\mathcal{K}$ is a compact subset of $\mathcal{F}_{Pd}$, then $\bigcap \mathcal{K} \in \mathcal{F}_{Pd}$. For $T \in \mathcal{E}$ we let $H_T = \{(V,W) \in \mathcal{E}\times\mathcal{E} \mid V \not\prec T\}$. Clearly, $H_T \in \Omega_j$. We claim that for any $T \in \mathcal{E}$, if $A \not\subseteq H_T$ for all $A \in \mathcal{K}$, then there exists $T' \in \mathcal{E}$ such that $(T, T') \in \bigcap \mathcal{K}$.[8] The lemma follows from this claim by checking that $A \in \mathcal{F}_{Pd}$ if and only if $(\forall T \in \mathcal{E})$ $A \not\subseteq H_T$.

To prove the claim, let $T \in \mathcal{E}$.

Let $A \in \mathcal{K}$. By hypothesis, there exists a $(V_0, W_0) \in A \setminus H_T$. By definition of H_T there exists a V' such that $V' \circ V_0 \circ V' \subseteq T$. By definition of uniform spaces we can find a $W \in \mathcal{E}$ such that $W \circ W \circ W \subseteq W_0 \cap V'$. We have $(A \cup H_T, H_T) \notin W^*$ because $(V_0, W \circ W \circ W) \in A \cup H_T$ but $(W \circ V_0 \circ W, W) \notin H_T$. There exists an

[8] The stronger result $\bigcap \mathcal{K} \not\subseteq H_T$ cannot be deduced here.

$S \in \mathcal{E}$ such that $S^* \circ S^* \subseteq W^*$, so that we have $S^*[A \cup H_T] \cap S^*[H_T] = \emptyset$.[9] For each $A \in \mathcal{K}$ we get such an S; let us call it S_A.

The function $A \mapsto A \cup H_T$ is uniformly continuous, so that the image $\mathcal{K}'$ of the compact $\mathcal{K}$ is itself compact. When A ranges over $\mathcal{K}$, the neighbourhoods $S_A^*[A \cup H_T]$ built above completely cover this compact $\mathcal{K}'$; there exists a finite family $\mathcal{V}$ of points of Ω_j such that the $S_A^*[A \cup H_T]$ for $A \in \mathcal{V}$ already cover it. On the other hand, $\bigcap_{A \in \mathcal{V}} S_A^*[H_T]$ is a finite intersection of neighbourhoods of H_T, so it is still a neighbourhood of H_T. It means that there exists a $U \in \mathcal{E}$ such that $U^*[H_T]$ is disjoint from $\mathcal{K}'$.

To conclude we show that $(T,\ U \circ U) \in \bigcap \mathcal{K}$. Let $A \in \mathcal{K}$. As we have shown, $(A \cup H_T, H_T) \notin U^*$. By definition of U^*, and because $H_T \subseteq A \cup H_T$, we find $V_1, V_2 \in \mathcal{E}$ such that

$$\begin{cases} (V_1, U \circ V_2 \circ U) \in A \cup H_T \qquad \text{but} \\ (U \circ V_1 \circ U, V_2) \notin H_T \end{cases}$$

so that $U \circ V_1 \circ U \prec T$, and in particular $V_1 \prec T$, hence $(V_1, U \circ V_2 \circ U) \in A$ and $(T,\ U \circ U) \in A$. ∎

Proof of lemma 7. Let R be an m-ary relation symbol of $\mathcal{L}$. By an argument similar to that of lemma 5 we can check that $[\![R]\!]$ is uniformly continuous, so that $[\![\ ]\!]$ provides a uniform structure.

Let φ be a first-order formula in the language $\mathcal{L}$. We prove the second part of the result by induction on φ. If we ignore all occurrences of j, the result is clear by construction of the uniform connectives. We have to check that the connectives are stable under j, i.e. (for any $A, B \in \Omega$ and $\mathcal{A} \subseteq \Omega$):

- $j(A \cap B) = j(j(A) \cap j(B))$ (which is $j(A) \cap j(B)$)
- $j(A \cup B) = j(j(A) \cup j(B))$ (which is $j(A) \cup j(B)$)
- $j(\sim_\Omega A) = j(\sim_j j(A))$ (which is $\sim_j j(A)$)
- $j(\bigcap \mathcal{A}) = j(\bigcap_{A \in \mathcal{A}} j(A))$ (which is $\bigcap_{A \in \mathcal{A}} j(A)$)
- $j(\bigcup \mathcal{A}) = j(\bigcup_{A \in \mathcal{A}} j(A))$

where $\sim_\Omega$ denotes the transposed complement in Ω (so that $\sim_j = j \circ \sim_\Omega$). All cases are straightforward (use the density of the $\prec$ order for $\sim_\Omega$). To complete the proof check that the topological closure Cl in Ω_j behaves nicely – more precisely, that for any $\mathcal{A} \subseteq \Omega_j$,

$$\cap \operatorname{Cl} \mathcal{A} = \cap \mathcal{A} \qquad \text{and} \qquad j(\cup \operatorname{Cl} \mathcal{A}) = j(\cup \mathcal{A})$$

∎

[9] For a binary relation R and any x in its domain we let $R[x] = \{y \mid (x, y) \in R\}$. Note that we consider symmetrical relations here. When R is an entourage of a uniformity, $R[x]$ is a neighbourhood of x in the induced topology.

Proof of lemma 8. Again, the result is clear if we ignore all occurrences of j. The complete result is proved from the equalities found in the proof of the previous lemma; for example, the $\sim_j$ case is as follows: for any $A \in \Omega_j$ we have

$$\sim_j \sim_j A = \sim_j j \sim_\Omega A = j \sim_\Omega \sim_\Omega A = jA = A$$

The other cases are similar. For both quantifiers we moreover have to check that

$$\mathrm{Cl}\,\{\sim_j A \mid A \in \mathcal{A}\} = \{\sim_j A \mid A \in \mathrm{Cl}\,\mathcal{A}\}$$

∎

Proof of lemma 9. Just check that for any $A \in \Omega$, we have $\pi_2[A] = \mathcal{E}$ if, and only if, $\pi_2[j(A)] = \mathcal{E}$. ∎

This concludes the proof of theorem 3.

5 Applications and complements

The present theory was developed to study the relationships (both the similarities and the differences) between a given first-order structure and a second one obtained by Cauchy-completing the former. It actually applies as soon as we have a first-order structure with a compact uniformity and consider a substructure which is dense. The notion of formula balancing does not change when restricted to a dense subset, so that it can be used to approximate the theory of the compact structure by computations in its dense substructure.

We know quite well which classes of formulas are preserved across various operations (see e.g. [2]; for Cauchy-completions [1]). Hinnion studied the Cauchy-completion of arbitrary structures in [7]; as the paper is unpublished, his results of interest to us will be recalled in 5.2.

Formula balancing takes another approach: if the negation of an arbitrary formula φ is *not balanced* in the substructure, then φ is automatically true in the compact structure by theorem 4. This result can be completed by converse implication results stating that certain classes of formulas, if balanced in the substructure, are true in the full structure.

5.1 Converse implications

Proposition 10. *Let M be a first-order structure endowed with a compatible uniformity and $\varphi(\vec{x})$ a* positive *or* negative *formula (i.e. respectively, a formula with no negation sign or the negation of such a formula). Let $\vec{x} \in M^n$.*

1. *φ and $\neg\varphi$ cannot be both balanced at $\vec{x}$;*

2. *if M is compact and φ is balanced at $\vec{x}$, then $M \vDash \varphi(\vec{x})$.*

Proposition 11. *Let M be any first-order structure with a compatible uniformity and $\varphi(\vec{x})$ a* universal *formula with n free variables. Then*

$$\text{for any } \vec{x} \in M^n, \text{ if } \varphi \text{ is balanced at } \vec{x}, \text{ then } M \vDash \varphi(\vec{x}).$$

Proposition 10 is an easy exercise. Proposition 11 is shown by a straightforward induction on φ. As a corollary, if M is compact, a universal formula is true if and only if it is balanced. These two cases are of particular interest for algebraic structures, in which a lot of axioms are expressed as either positive or universal sentences (e.g. groups, rings, fields, etc.).

5.2 Cauchy-completions

In this section we denote by X a first-order structure endowed with a compatible uniformity and $\overline{X}_{\mathcal{E}}$ its Cauchy-completion.[10] As usual we assume that the extension of all relations in X is closed, and functions are interpreted by uniformly continuous functions of X. Both the relations and the functions are uniquely extended to $\overline{X}_{\mathcal{E}}$ by the universal property of the Cauchy-completion.

As seen above, as X is dense in $\overline{X}_{\mathcal{E}}$, a sentence is balanced in X if and only if it is balanced in $\overline{X}_{\mathcal{E}}$. Here is a summary of all possible cases for a sentence φ, showing that there is no completely general relationship between balancing, truth in X, and truth in $\overline{X}_{\mathcal{E}}$ other than that given by theorem 4.

For the compact examples, we take $X = \{x \in \mathbb{Q} \mid 0 < x < 5\}$ and $\overline{X}_{\mathcal{E}} = [0, 5]$ with the usual uniformities.

$X \vDash \varphi$	balanced	$\overline{X}_{\mathcal{E}} \vDash \varphi$	example
true	only φ	true	$\top$
true	φ and $\neg\varphi$	true	$(\forall x, y)(x = y \Rightarrow x = y)$
true	φ and $\neg\varphi$	false	$(\forall x)(\exists y)(y \leqslant x \wedge y \neq x)$
true	only $\neg\varphi$	false	$(\forall x)(\exists y)(y \nleqslant x)$

Another interesting example for the third case is the formula that states that the predicate $P(x) \equiv x \leqslant \pi$ is not defined in the structure, i.e. $\neg(\exists x)(\forall y)(y \leqslant x \iff P(y))$. Of course, examples that are false in X are deduced from the given ones by taking their negation. The next two examples are the cases that are impossible if $\overline{X}_{\mathcal{E}}$ is compact, so we take X to be all positive rationals and $\overline{X}_{\mathcal{E}}$ all non-negative reals. $G(x, y)$ is the predicate $y = x^2$; see section 5.3 for details about functions represented as their underlying graph.

$X \vDash \varphi$	balanced	$\overline{X}_{\mathcal{E}} \vDash \varphi$	example
true	only $\neg\varphi$	true	"G is the graph of a function"
true	only φ	false	$(\forall x)(\exists y)(xy = 1)$

Note that proposition 11 holds for *any* universal formula, even ones with negations; by contrast, the study of elementary (direct) preservation properties from X to $\overline{X}_{\mathcal{E}}$ (Hinnion [7]) shows that "$X \vDash \varphi \;\Rightarrow\; \overline{X}_{\mathcal{E}} \vDash \varphi$" holds for formulas φ inductively built from:

[10] Again, most of this applies to any "compactification", i.e. any pair of structures $X \subseteq X'$ where X is dense in X' and X' is compact.

- atomic formulas, $\vee$, $\wedge$, $\forall$;
- the $\exists$ quantifier if $\overline{X}_{\mathcal{E}}$ is compact.

The common point between Hinnion's and our results is that the treatment of the $\exists$ quantifier requires compactness.

5.3 Functions

Comparing a first-order structure X with one in which the functions have been replaced by their underlying relations (call it X'), we see that the condition we assumed on the uniformities of X (making the functions uniformly continuous) is stronger than the condition on X' (making the graphs closed). The following proposition addresses this problem:

Proposition 12. *With the notations above, assume that $R^{X'}$ is the closed graph of a function f^X. Consider the axiom σ in the language of X' that states that R satisfies the "unique image" condition. Then f^X is uniformly continuous if and only if σ is balanced in X'.*

The proof is a tedious but straightforward verification on σ. As a corollary, if the Cauchy-completion $\overline{X}_{\mathcal{E}}$ of a structure X is compact, then a function on X extends to a function on $\overline{X}_{\mathcal{E}}$ if *and only if* it was a uniformly continuous function on X. Both directions of the equivalence can fail without compactness.

Remark 5. Proposition 12 could be the starting point for a uniform model theory in which uniformities are required to be compatible with the axioms of a theory T in the sense that these axioms are balanced[11]. With such a definition, the presentations as functions or as relations with graph-of-a-function axioms are again equivalent. Propositions 10, 11 and 12 are powerful tools in algebraic settings; for example, it is immediate that a structure of group, ring, field, module, ... is carried to a compactification if, and only if, the uniformity is compatible in the above sense.

Remark 6. Balancing is not closed under logical consequences, i.e. it is possible that φ and $\varphi \Rightarrow \psi$ are both balanced formulas, but ψ is not. Trivial examples are given by all formulas φ such that φ and $\neg\varphi$ are both balanced: in this case, φ and $\varphi \Rightarrow \perp$ are balanced, but $\perp$ is never balanced. In this respect, paraconsistent logics seem more natural for balancing, as balancing can be made closed under their logical consequence relation. As explained in section 2 the notion of "CL-reduction filter" can be extended to non-classical logics; a direct adaptation of $\mathcal{F}_{Pd}$ can turn it into a Pd- or Pt-reduction filter, where Pd stands for Paradoxical logic ("true", "false", "both") and Pt means Partial logic ("true", "false", "neither"). This is used in [14] to build new models in these logics.

[11] Whether and how much this definition depends on the chosen axiomatic presentation of T should be investigated.

5.4 Higher cardinals and models of set theory

The notions presented in the present paper were actually first developed in the frame of large cardinal topology, where significant set-theoretical consistency problems have been solved by building Cauchy-completions of relatively simple structures and exhibiting their new properties. Formula balancing can be seen as a way to predict which new properties can appear by this kind of process. This line of work originates from a paper of Malitz ([12]) subsequently developed by Weydert ([16]), Forti, Hinnion and Honsell ([5], [6]) and Esser ([3], [4]).

Accordingly, all the results above immediately generalize to κ-uniform spaces, where κ is any infinite regular cardinal. See e.g. [9] and [15] for large cardinals and their use in general topology. We recall the basic definitions: we call κ-uniform space a uniform space whose filter of entourages is κ-complete, i.e. closed under κ-finite intersections (i.e. intersections of cardinality less than κ). By κ-compact we mean that any open cover admits a sub-cover with less than κ pieces, or equivalently that any κ-complete filter on the space has got an adherence point. The classical case is of course $\kappa = \aleph_0$.

If X is a uniform space, then it is well-known that its Cauchy-completion is compact if and only if it is bounded [10]. In general, however, a κ-uniform space with a κ-compact completion is always κ-bounded (i.e. for any entourage V there is a κ-finite cover of X with V-small pieces), but the converse is not true. One might wonder if the main theorems developed here would also work if one assumed only that X is κ-bounded. We will show now that it is not the case.

We are about to build an $\aleph_1$-bounded $\aleph_1$-uniform model whose completion is not $\aleph_1$-compact. The universe of this model is any tree of height $\aleph_1$, whose levels are all $\aleph_1$-finite (i.e. finite or countable), but with no branch of length $\aleph_1$. This exists because $\aleph_1$ is not a ramifiable cardinal[12]; this is a result of Aronszajn (1934). See for example [9] (theorem 7.10) for the construction.

Let us call X this tree. We call the *α-root of a point* $x \in X$ the following point:

- x itself if x is of level less than or equal to α;
- the unique x' of level α below x, otherwise.

Note that the α-root is a point of level at most (but sometimes less than) α. We put on X the uniformity generated by the basis $\mathcal{F} = \{\sim_\alpha \mid \alpha < \aleph_1\}$, where two points $x, y \in X$ are said to satisfy $x \sim_\alpha y$ if and only if they have the same α-root. This generates an $\aleph_1$-uniformity and each $\sim_\alpha$ cuts the tree into one (singleton) class per item under the level α, plus one large class above each point of level α. Clearly X is $\aleph_1$-bounded.

It can be shown that X itself is already Cauchy-complete, so $X = \overline{X}_{\mathcal{E}}$ is an $\aleph_1$-bounded $\aleph_1$-uniform space, but it is not $\aleph_1$-compact (because the induced topology is discrete).

We now put on X the binary relation R defined as $R(x,y) \iff x \not\leq y$, i.e. "x is not (non-strictly) below y in the tree". R is a closed subset of $X \times X$ (because it

[12] The metatheory is ZFC.

has got the discrete topology!). We are now ready to check that the sentence

$$\varphi \quad : \quad (\forall x)(\exists y)\ \neg R(x,y)$$

is true in X but not balanced.

φ is obviously true in X: just choose $y = x$. But φ cannot be balanced because $[\![\varphi]\!] = \emptyset$. Indeed, let $W \in \mathcal{E}$. We must check that $X^{\sim} \nvDash (\forall x)(\exists y)\ \neg R_W(x,y)$. As $\{\sim_\alpha \mid \alpha < \aleph_1\}$ is a basis of $\mathcal{E}$, there exists $\alpha < \aleph_1$ such that $\sim_\alpha \subseteq W$. Let x be a point of level $\alpha+1$. We claim that for any $y \in X$ we have $X^{\sim} \vDash R_{\sim_\alpha}(x,y)$. Consider a given $y \in X$. Let y' be the α-root of y; then y' is at most of level α so that x cannot be below y': we have $X \vDash R(x,y')$. As $y' \sim_\alpha y$ the claim is verified.

References

[1] Ball, R. *Cauchy completions are homomorphic images of submodels of ultrapowers*, Proceedings of the Conference on Convergence Structures, Cameron Univ., Lawton, Okla., pages 1-7 (1980).

[2] Chang C.C. and Keisler H.J. *Model theory*, Studies in Logic and the Foundations of Mathematics, 73. North-Holland Publishing Co., Amsterdam-New York (1990).

[3] Esser O. *Un modèle topologique d'ensembles non bien fondés*, Mémoire de Licence, Faculté des Sciences, U.L.B. (1993)[1].

[4] Esser O.[1] *Mildly ineffable cardinals and hyperuniverses*, Reports on Mathematical Logic, volume 37 (2003).

[5] Forti M. and Hinnion R. *The consistency problem for positive comprehension principles*, The Journal of Symbolic Logic, volume 54, pages 1401-1418 (1989).

[6] Forti M. and Honsell F. *Weak foundation and Anti-foundation properties of positively comprehensive hyperuniverses*, Cahiers du Centre de Logique, Université Catholique de Louvain, volume 7, pages 31-43 (1992).

[7] Hinnion R. *A general Cauchy-completion process for arbitrary first-order structures*, unpublished. (See [8] for the main definitions and results.)

[8] Hinnion R. *Directed Sets and Malitz-Cauchy-Completions*, Mathematical Logic Quarterly 43 (1997).

[9] Kanamori A. *The Higher Infinite*, Springer-Verlag, Berlin-Heidelberg (1994).

[10] Kelley J.L. *General Topology*, Van Nostrand (1955).

[11] Mac Lane, S. and Moerdijk, I. *Sheaves in geometry and logic, A first introduction to topos theory*, Springer-Verlag, New York (1994).

[1] http://homepages.ulb.ac.be/~oesser

[12] Malitz R.J. *Set theory in which the axiom of foundation fails*, Ph.D Thesis, University of California (1976) (unpublished; available from University Microfilms International, Ann Arbor, Michigan 48106).

[13] Michael E. *Topologies on spaces of subsets*, Transactions of the American Mathematical Society 71, pages 152-182 (1951).

[14] Rigo A. *Continuously valuated models for paraconsistent and paracomplete theories*, submitted[2].

[15] Stevenson F.W., Thorn W.J. *Results on ω_κ-metric spaces*, Fundam. Math. 65, pages 317-324 (1969).

[16] Weydert E. *How to approximate the naive comprehension scheme inside of classical logic*, Bonner Math. Schriften, Nr. 194, pages 1-40 (1989).

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