Semipartial geometries, arising from locally hermitian 1-systems of $W_5(q)$

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Abstract

It is known that every 1-system of $W_5(q)$ is an SPG regulus and thus defines a semipartial geometry. In this paper, the semipartial geometries arising from locally hermitian 1-systems of $W_5(q)$, q even, will be investigated. It will be shown that non-isomorphic locally hermitian 1-systems of $W_5(q)$ yield non-isomorphic semipartial geometries, which implies the existence of new semipartial geometries.

1 Definitions

1.1 1-systems of $W_5(q)$

A *1-system* of $W_5(q)$ is a set $\mathcal{M}$ of q^3+1 lines $L_0, L_1, \ldots, L_{q^3}$ of $W_5(q)$ with the property that every generator of $W_5(q)$ which contains an element $L_i \in \mathcal{M}$, is disjoint from all lines $L_j \in \mathcal{M} \setminus \{L_i\}$. The set of all points on the lines of $\mathcal{M}$ will be denoted by $\widetilde{\mathcal{M}}$; so $\widetilde{\mathcal{M}}$ is the union of the elements of $\mathcal{M}$.

If q is odd, then the symplectic polar space $W_5(q)$ does not contain reguli of totally isotropic lines, the opposite regulus of which entirely consists of totally isotropic lines. If q is even, such reguli do exist, as can be seen as follows. For even values of q, the polar spaces $W_5(q)$ and $Q(6,q)$ are isomorphic, as one obtains $W_5(q)$ by

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Received by the editors November 2002.

Communicated by H. Van Maldeghem.

1991 *Mathematics Subject Classification* : 51A45, 51A50, 51E14, 51E20, 51E30.

Key words and phrases : semipartial geometries, SPG reguli, m-systems, polar spaces.

projecting $Q(6,q)$ from its nucleus n onto a $\mathsf{PG}(5,q)$, not containing the nucleus. Consider two arbitrary skew lines M and N of $W_5(q)$, which are the projections from n onto $\mathsf{PG}(5,q)$ of the disjoint lines M' and N' of $Q(6,q)$. Then $\langle M', N'\rangle$ is 3-dimensional and it intersects $Q(6,q)$ in a hyperbolic quadric $Q^+(3,q)$, which consists in fact of two opposite reguli. Hence it is projected from n onto a hyperbolic quadric $Q^+(3,q)$ consisting of two opposite reguli of lines of $W_5(q)$.
In this paper, we will call a regulus of lines of $W_5(q)$ a *strong regulus* if and only if its opposite regulus also consists entirely of totally isotropic lines of $W_5(q)$. From the above, it follows that every two disjoint lines of $W_5(q)$, q even, determine a unique strong regulus of lines of $W_5(q)$, containing both of them. Keeping these observations in mind, one can define locally hermitian 1-systems of $W_5(q)$.

Definition
Let $\mathcal{M}$ be a 1-system of the symplectic polar space $W_5(q)$, q even, in $\mathsf{PG}(5,q)$. We say that $\mathcal{M}$ is *locally hermitian* at some line $L \in \mathcal{M}$ if and only if for every line $M \in \mathcal{M} \setminus \{L\}$, the unique strong regulus of $W_5(q)$ which contains L and M, is completely contained in $\mathcal{M}$.

In [5], a class of locally hermitian 1-systems of $W_5(q)$ has been discovered for q even and $q > 2$. Moreover, it is shown that this class contains $\frac{q-2}{2}$ elements which are pairwise non-isomorphic for the stabilizer of $W_5(q)$ in $\mathsf{PGL}(6,q)$. Under the action of the stabilizer of $W_5(q)$ in $\mathsf{P\Gamma L}(6,q)$, the number of orbits in this set of 1-systems of $W_5(q)$ equals the number of orbits of $\mathrm{Aut}(\mathsf{GF}(q))$ in the set of all elements of $\mathsf{GF}(q) \setminus \{0\}$ with trace zero; see also [5].

1.2 Semipartial geometries

A *semipartial geometry* is an incidence structure $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of points and lines satisfying the following axioms:

spg1 Each point is incident with $t+1$ ($t \geq 1$) lines and two distinct points are incident with at most one line.

spg2 Each line is incident with $s+1$ ($s \geq 1$) points and two distinct lines are incident with at most one point.

spg3 If two points are not collinear, then there are μ ($\mu > 0$) points collinear with both.

spg4 If a point x and a line L are not incident, then there are 0 or α ($\alpha \geq 1$) points which are collinear with x and incident with L.

In [4], it is shown that every 1-system of $W_5(q)$ is an SPG regulus in $\mathsf{PG}(5,q)$ with parameters $m=1$, $r=q^3+1$, $\alpha=q$, and $\theta=q+1$, which means the following. An *SPG regulus* of $\mathsf{PG}(n,q)$ is a set $\mathcal{R}$ of m-dimensional subspaces $\pi_1, \pi_2, \ldots, \pi_r$, $r>1$, of $\mathsf{PG}(n,q)$, satisfying:

SPG1 $\pi_i \cap \pi_j = \emptyset$ for all $i \neq j$.

SPG2 If $\mathsf{PG}(m+1,q)$ contains $\pi_i \in \mathcal{R}$, then it has a point in common with either 0 or α ($\alpha > 0$) spaces in $\mathcal{R} \setminus \{\pi_i\}$. If $\mathsf{PG}(m+1,q)$ has no point in common with $\pi_j \in \mathcal{R}$ for all $j \neq i$, then it is called a *tangent* $(m+1)$-space of $\mathcal{R}$ at π_i.

SPG3 If the point x of $\mathsf{PG}(n,q)$ is not contained in an element of $\mathcal{R}$, then it is contained in a constant number θ ($\theta \geq 0$) of tangent $(m+1)$-spaces of $\mathcal{R}$.

As SPG reguli yield semipartial geometries by Thas [7], it is clear that every 1-system of $W_5(q)$ defines a semipartial geometry. This semipartial geometry is constructed as follows. Let $\mathcal{M}$ be a 1-system of a symplectic polar space $W_5(q)$ in $\mathsf{PG}(5,q)$ and embed $\mathsf{PG}(5,q) := H$ as a hyperplane in $\mathsf{PG}(6,q)$. Define an incidence geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with the set of all points of $\mathsf{PG}(6,q) \setminus H$ as point set $\mathcal{P}$, the set of all planes of $\mathsf{PG}(6,q)$ not in H which meet H in a line of $\mathcal{M}$ as line set $\mathcal{L}$, and with the natural incidence I. Then Γ is a semipartial geometry with parameters $s = q^2 - 1$, $t = q^3$, $\alpha = q$ and $\mu = q^2(q^2 - 1)$. This semipartial geometry will further be denoted by $\mathsf{SPG}(\mathcal{M})$. Since the semipartial geometries $\mathsf{SPG}(\mathcal{M})$, with $\mathcal{M}$ locally hermitian, will appear to have many subnets as subgeometries, we also mention the definition of a net.

A *net of order* $s+1$ *and degree* $t+1$ is an incidence geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of points and lines satisfying the axioms **spg1** and **spg2** above and the following third axiom:

N If a point x and a line L are not incident, then there exists exactly one line which is incident with x and not concurrent with L.

In a net of order $s+1$ and degree $t+1$, two distinct lines are called *parallel* if and only if they have no point in common. For every non-incident point-line pair (x, L), the unique line through x and not concurrent with L is then the unique line incident with x and parallel to L.

For later purposes, we give two constructions of a net of order q^2 and degree $q+1$.

Consider a regulus R of lines in $\mathsf{PG}(3,q)$, embed $\mathsf{PG}(3,q)$ as a hyperplane in a projective space $\mathsf{PG}(4,q)$, and let $\mathsf{AG}(4,q)$ denote the affine space $\mathsf{PG}(4,q) \setminus \mathsf{PG}(3,q)$. Suppose that $\mathcal{P}$ is the set of all points of $\mathsf{AG}(4,q)$, let $\mathcal{L}$ consist of the planes of $\mathsf{AG}(4,q)$, the extensions of which to $\mathsf{PG}(4,q)$ meet $\mathsf{PG}(3,q)$ in a line of R, and let incidence be the incidence of $\mathsf{AG}(4,q)$. Then $\mathcal{N} := (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a net of order q^2 and degree $q+1$. In Johnson [3], a net of this kind is called a *regulus net*.

Consider the 3-dimensional projective space $\mathsf{PG}(3,q)$ and let N be a fixed line of $\mathsf{PG}(3,q)$. Define $\mathcal{P}$ to be the set of all lines of $\mathsf{PG}(3,q)$, skew to N, and let $\mathcal{L}$ be the set of points of $\mathsf{PG}(3,q)$, not on N. Then $\mathcal{N} := (\mathcal{P}, \mathcal{L}, \mathrm{I})$, where I is the incidence of $\mathsf{PG}(3,q)$, is a net of order q^2 and degree $q+1$. A net of this kind is denoted by H_q^3 and called a *co-dimension 2 net* in Johnson [3].

In the book *"Subplane covered nets"* by Johnson ([3]), in which the reader can find a wealth of information concerning nets and related topics, it is shown that every regulus net is isomorphic to a co-dimension 2 net and conversely.

Suppose that $\mathcal{M}$ is a locally hermitian 1-system of $W_5(q)$, q even. Then $\mathcal{M}$ consists of q^2 strong reguli $R_1, R_2, \ldots, R_{q^2}$ through a common line $L \in \mathcal{M}$. Consider an arbitrary point x of $\mathsf{SPG}(\mathcal{M})$. For every regulus R_i, $i \in \{1, 2, \ldots, q^2\}$, it then holds that the subgeometry of $\mathsf{SPG}(\mathcal{M})$, induced in $\langle R_i, x\rangle$, is a subnet of order q^2 and degree $q+1$ of $\mathsf{SPG}(\mathcal{M})$. In particular, it is a regulus subnet of $\mathsf{SPG}(\mathcal{M})$. We conclude that $\mathsf{SPG}(\mathcal{M})$ contains a lot of (regulus) subnets of order q^2 and degree $q+1$.

The following lemma, which has been shown in [6], gives information on the structure of subnets of $\mathsf{SPG}(\mathcal{M})$ of order q^2 and degree $q+1$. It will play an important role in the next section.

Lemma 1.1. *Let $\mathcal{M}$ be a 1-system of a symplectic polar space $W_5(q)$, $q > 2$, in $\mathsf{PG}(5, q) := H$. A subnet of order q^2 and degree $q+1$ in $\mathsf{SPG}(\mathcal{M})$ is always the subgeometry induced by $\mathsf{SPG}(\mathcal{M})$ in a subspace $\mathsf{PG}(4, q)$ of the ambient space $\mathsf{PG}(6, q)$ of $\mathsf{SPG}(\mathcal{M})$, where $\mathsf{PG}(4, q)$ meets H in a $\mathsf{PG}(3, q)$ containing exactly $q+1$ lines of $\mathcal{M}$.*

2 Non-isomorphic locally hermitian 1-systems yield non-isomorphic semipartial geometries

In this section, we focus on the question whether the semipartial geometries that arise from non-isomorphic, locally hermitian 1-systems $\mathcal{M}_1$, respectively $\mathcal{M}_2$, of $W_5(q)$ with q even, are isomorphic or not. We first show that an isomorphism between $\mathsf{SPG}(\mathcal{M}_1)$ and $\mathsf{SPG}(\mathcal{M}_2)$ is induced by an element of $\mathsf{P\Gamma L}(7, q)$ which maps $\mathcal{M}_1$ onto $\mathcal{M}_2$.

Theorem 2.1. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be locally hermitian 1-systems of a symplectic polar space $W_5(q)$ in $\mathsf{PG}(5, q) := H$, with q even and $q > 2$. If θ is an isomorphism between $\mathsf{SPG}(\mathcal{M}_1)$ and $\mathsf{SPG}(\mathcal{M}_2)$, then θ is induced by an element $\vartheta \in \mathsf{P\Gamma L}(7, q)$ which maps $\mathcal{M}_1$ onto $\mathcal{M}_2$.*

Proof.
As $\mathcal{M}_1$ and $\mathcal{M}_2$ are locally hermitian, they both consist of q^2 strong reguli through some line, say $L_1 \in \mathcal{M}_1$, respectively $L_2 \in \mathcal{M}_2$. Every such strong regulus defines a collection of subnets of order q^2 and degree $q+1$ in $\mathsf{SPG}(\mathcal{M}_i)$, $i = 1, 2$. Clearly, θ must map a subnet $\mathcal{N}_1$ of order q^2 and degree $q+1$ of $\mathsf{SPG}(\mathcal{M}_1)$ onto a subnet $\mathcal{N}_2$ of the same order and degree in $\mathsf{SPG}(\mathcal{M}_2)$. Suppose that $\mathcal{N}_1$ is a regulus net, determined by a regulus R of lines of $\mathcal{M}_1$. Then by Lemma 1.1, the net $\mathcal{N}_2 = \mathcal{N}_1^\theta$ is the subgeometry of $\mathsf{SPG}(\mathcal{M}_2)$, induced in a $\mathsf{PG}(4, q)$ which intersects H in a $\mathsf{PG}(3, q)$, containing $q+1$ lines of $\mathcal{M}_2$. Now, since $\mathcal{N}_2$ is isomorphic to the regulus net $\mathcal{N}_1$ by assumption, a result of Johnson, see [2], implies that these $q+1$ lines of $\mathcal{M}_2$ must be the lines of a regulus R' in $\mathsf{PG}(3, q)$. Hence $\mathcal{N}_2$ is also a regulus net and we know that θ maps every regulus subnet of $\mathsf{SPG}(\mathcal{M}_1)$ onto a regulus subnet of $\mathsf{SPG}(\mathcal{M}_2)$.

Let $\mathcal{N}_1$ be an arbitrary regulus subnet of $\mathsf{SPG}(\mathcal{M}_1)$ and set $\mathcal{N}_2 := \mathcal{N}_1^\theta$. As has been mentioned in Section 1.2, the net $\mathcal{N}_1$ is isomorphic to a co-dimension 2 net H_q^3, which we consider to be embedded in a $\mathsf{PG}(3, q) \setminus N$ as described in Section 1.2. By Theorem 11.1 of Johnson [3], the full collineation group of H_q^3 is isomorphic to the

stabilizer $\mathsf{P\Gamma L}(4,q)_N$ of the line N in $\mathsf{P\Gamma L}(4,q)$.
If $\mathcal{N}_i$, $i=1,2$, is the subgeometry of $\mathsf{SPG}(\mathcal{M}_i)$, induced in the 4-dimensional subspace δ_i of $\mathsf{PG}(6,q)$, where $\delta_i\cap H$ contains a regulus $R^{(i)}$ of lines of $\mathcal{M}_i$, then every element ζ of $\mathsf{P\Gamma L}(7,q)$ which maps δ_1 onto δ_2 and $R^{(1)}$ onto $R^{(2)}$, clearly induces an isomorphism between $\mathcal{N}_1$ and $\mathcal{N}_2$. Now the number of such elements ζ equals the number of elements of $\mathsf{P\Gamma L}(7,q)$ stabilizing a given 4-dimensional subspace of $\mathsf{PG}(6,q)$, multiplied with the number of collineations of $\mathsf{PG}(4,q)$, stabilizing a regulus of lines in some hyperplane of $\mathsf{PG}(4,q)$. This last number can easily be calculated; it is equal to $hq^6(q-1)(q^2-1)^2$, where $q=p^h$ with p prime. But $hq^6(q-1)(q^2-1)^2$ is also the order of the group $\mathsf{P\Gamma L}(4,q)_N$, which implies that every isomorphism between $\mathcal{N}_1$ and $\mathcal{N}_2$ must be induced by an element $\zeta\in\mathsf{P\Gamma L}(7,q)$ which maps δ_1 onto δ_2 and $R^{(1)}$ onto $R^{(2)}$.
Let $R_1,R_2,\dots,R_{q^2}$ be the q^2 strong reguli of $\mathcal{M}_1$ through L_1. It then follows from the previous paragraph that θ maps all lines of $\mathsf{AG}(6,q)=\mathsf{PG}(6,q)\backslash H$, the extension of which to $\mathsf{PG}(6,q)$ meets H in a point of some $\langle R_i\rangle$, $i\in\{1,2,\dots,q^2\}$, onto lines of $\mathsf{AG}(6,q)$.

Next, we determine how many points of H are contained in some $\langle R_i\rangle$, $i\in\{1,2,\dots,q^2\}$. If two 3-spaces $\langle R_i\rangle$ and $\langle R_j\rangle$, $i\neq j$, have a plane in common, then this must be a plane through L_1. Hence this plane contains a transversal of R_i and also one of R_j. In case these two transversals coincide, it follows that the elements of $\mathcal{M}_1$ are not pairwise disjoint, a contradiction. On the other hand, if the two transversals are distinct, then the plane $\langle R_i\rangle\cap\langle R_j\rangle$ contains a line of $\mathcal{M}_1$ and at least $2q-1$ points of $\widetilde{\mathcal{M}}_1$, not on this line. The latter contradicts Axiom **SPG2** and the fact that $\mathcal{M}_1$ is an SPG regulus with $\alpha=q$. Consequently, $\langle R_i\rangle\cap\langle R_j\rangle$ is the line L_1 for all $i\neq j$. Thus the union of the 3-spaces $\langle R_i\rangle$, $i=1,2,\dots,q^2$, contains exactly q^5+q^4+q+1 points of H. The 3-space $L_1^\perp$, with $\perp$ the polarity of $W_5(q)$, intersects every $\langle R_i\rangle$, $i\in\{1,2,\dots,q^2\}$, in the line L_1, and as such it yields q^3+q^2 additional points of H. We conclude that a point of H is either contained in some $\langle R_i\rangle$, $i\in\{1,2,\dots,q^2\}$, or it is a point of the 3-space $L_1^\perp$.

Let π be a plane of $\mathsf{PG}(6,q)$, not in H, and assume that π intersects H in a line K which has exactly one point z in common with $L_1^\perp$. Then, by considering all lines of π through two distinct points of $\pi\setminus K$ not on a common line through z, and taken into account that the q affine points of every line of π not through z, are mapped by θ onto the q points of an affine line, it is evident that all points of $\pi\setminus K$ are mapped by θ onto the q^2 points of a plane of $\mathsf{AG}(6,q)$. Now, let M be any line of $\mathsf{PG}(6,q)\setminus H$ through a point $z\in L_1^\perp$. If π and π' are two distinct planes of $\mathsf{PG}(6,q)$ through M, which meet H in distinct lines K and K' through z, but not contained in $L_1^\perp$, then the points of $M\setminus\{z\}$ must be mapped by θ onto the points of the intersection of the affine planes $(\pi\setminus K)^\theta$ and $(\pi'\setminus K')^\theta$, which form a line of $\mathsf{AG}(6,q)$. It follows that θ maps all lines of $\mathsf{AG}(6,q)$ onto lines of $\mathsf{AG}(6,q)$.

From the foregoing and as $q>2$, we conclude that θ is an element of $\mathsf{A\Gamma L}(7,q)$, which implies that it can be extended to an element $\vartheta\in\mathsf{P\Gamma L}(7,q)$. Obviously, ϑ must then map $\mathcal{M}_1$ onto $\mathcal{M}_2$. This proves the theorem. ■

Remarks

1. There are several possible ways to show that θ preserves the collinearity of $\mathsf{AG}(6,q)$ within regulus subnets of $\mathsf{SPG}(\mathcal{M}_1)$.

   An alternative proof also relies on the fact that the full collineation group of H_q^3 is isomorphic to $\mathsf{P\Gamma L}(4,q)_N$, but it would be valid over an infinite field as well. By investigating the isomorphism between a regulus net and the co-dimension 2 net H_q^3 in its representation in $\mathsf{PG}(3,q) \setminus N$, one easily sees that q collinear points in a regulus net which are also collinear in the affine space in which the regulus net is represented, correspond to q lines, disjoint from N, through a point p of $\mathsf{PG}(3,q) \setminus N$, and in a plane π of $\mathsf{PG}(3,q)$ not containing N. Also, q points of the regulus net which are collinear in the affine space but not in the net, correspond to the q lines of a regulus of $\mathsf{PG}(3,q)$ through the special line N. Since every element of $\mathsf{P\Gamma L}(4,q)_N$ preserves both types of configurations of q lines of $\mathsf{PG}(3,q)$, every isomorphism between two regulus nets must preserve the collinearity of the affine space containing the first regulus net.

   A second possible proof does not use the collineation group of H_q^3, but relies on straightforward properties of a regulus net. Let $\mathcal{N}$ be a regulus net of order q^2 and degree $q+1$, embedded in an affine space $\mathsf{AG}(4,q)$ as usually. If a, b and c are three collinear points of $\mathcal{N}$, then one easily sees that there exist $2q^2-q-3$ points of $\mathcal{N} \setminus \{a,b,c\}$, which are collinear in $\mathcal{N}$ with a, b and c, provided that abc is a line of $\mathsf{AG}(4,q)$. If a, b and c are not collinear in $\mathsf{AG}(4,q)$ however, there exist only q^2-3 points of $\mathcal{N} \setminus \{a,b,c\}$, which are collinear in $\mathcal{N}$ with a, b and c. So if abc is a line of $\mathsf{AG}(4,q)$, then the same must hold for $a^\theta b^\theta c^\theta$.
   Similarly, let a, b and c be distinct points of $\mathcal{N}$ and suppose that they are pairwise not collinear in $\mathcal{N}$. If abc is a line of $\mathsf{AG}(4,q)$, then no point of $\mathcal{N} \setminus \{a,b,c\}$ is collinear in $\mathcal{N}$ with a, b and c. If a, b and c are not collinear in $\mathsf{AG}(4,q)$, then it can be shown that there always exists at least one point of $\mathcal{N} \setminus \{a,b,c\}$, collinear in $\mathcal{N}$ with all three of a, b and c. Hence in this case as well, θ must map the line abc of $\mathsf{AG}(4,q)$ onto an affine line.

2. The proof of Theorem 2.1 is not valid if $q=2$. Still, we can draw some conclusions. By the classification of the 1-systems of $W_5(2)$, carried out by Hamilton and Mathon [1], the symplectic polar space $W_5(2)$ has exactly two non-isomorphic 1-systems. One of them is the hermitian spread of a $Q^-(5,2)$ and the other one is obtained from the hermitian spread by reversing a regulus, so it is not locally hermitian by [6, Theorem 2.2]. Since Theorem 2.1 deals with two distinct locally hermitian 1-systems of $W_5(q)$, it does not make sense for $q=2$. Moreover, the fact that the semipartial geometries, arising from the two non-isomorphic 1-systems of $W_5(2)$ are not isomorphic, follows from [6, Theorem 4.4].

The result obtained in Theorem 2.1 will now be used to show that for non-isomorphic locally hermitian 1-systems $\mathcal{M}_1$ and $\mathcal{M}_2$ of $W_5(q)$, q even and $q>2$, the corresponding semipartial geometries $\mathsf{SPG}(\mathcal{M}_1)$ and $\mathsf{SPG}(\mathcal{M}_2)$ are also non-isomorphic.

Theorem 2.2. *Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are two locally hermitian 1-systems of $W_5(q)$, q even and $q > 2$. Then the corresponding semipartial geometries $\mathsf{SPG}(\mathcal{M}_1)$ and $\mathsf{SPG}(\mathcal{M}_2)$ are isomorphic if and only if $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic for the stabilizer of $W_5(q)$ in $\mathsf{P\Gamma L}(6,q)$.*

Proof.

Denote the line at which $\mathcal{M}_i$ is locally hermitian by L_i, for $i = 1, 2$.

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic for the stabilizer of $W_5(q)$ in $\mathsf{P\Gamma L}(6,q)$, with isomorphism α, then it is clear that α can be extended to an element $\beta \in \mathsf{P\Gamma L}(7,q)$ which induces an isomorphism between $\mathsf{SPG}(\mathcal{M}_1)$ and $\mathsf{SPG}(\mathcal{M}_2)$.

Conversely, suppose that θ is an isomorphism between the semipartial geometries $\mathsf{SPG}(\mathcal{M}_1)$ and $\mathsf{SPG}(\mathcal{M}_2)$. Then by Theorem 2.1, θ is induced by an element $\vartheta \in \mathsf{P\Gamma L}(7,q)$, which maps $\mathcal{M}_1$ onto $\mathcal{M}_2$. Without loss of generality, we may assume that $L_1^{\vartheta} = L_2$. If ϑ stabilizes the symplectic polar space $W_5(q)$, then the claim is obviously true. Therefore we assume that ϑ does not stabilize $W_5(q)$, which implies that $\mathcal{M}_2 = \mathcal{M}_1^{\vartheta}$ must be a 1-system of two distinct symplectic polar spaces $W_5(q)$ and $W_5(q)'$, with $W_5(q)'$ the image of $W_5(q)$ under ϑ. Denote the polarity of $W_5(q)$ by ζ and the polarity of $W_5(q)'$ by ξ. We shall prove that $\zeta = \xi$, so that $W_5(q)$ and $W_5(q)'$ coincide and the assumption is false.

For every line $M \in \mathcal{M}_2$, M^{ξ} is a 3-dimensional subspace and contains no points of $\widetilde{\mathcal{M}}_2$, except for the ones on M. Since the union of the tangent planes at M of the SPG regulus $\mathcal{M}_2$ contains $\frac{q^4-1}{q-1}$ points, M^{ξ} must coincide with the union of these tangent planes. But the same holds for M^{ζ}, so that $M^{\xi} = M^{\zeta}$ for all lines $M \in \mathcal{M}_2$.

If x is a point of the line L_2 (at which $\mathcal{M}_2$ is locally hermitian), then x^{ξ} is 4-dimensional and it must contain L_2^{ξ} and all totally isotropic lines of $W_5(q)'$ through x. Now the lines of $\mathcal{M}_2$ are totally isotropic for both ζ and ξ and as q is even, this implies that also the transversals of the q^2 strong reguli of $\mathcal{M}_2$ through L_2 are totally isotropic for ζ and ξ. Thus x^{ξ}, as well as x^{ζ}, contains $L_2^{\xi} = L_2^{\zeta}$ and all transversals on x of the q^2 strong reguli of $\mathcal{M}_2$ through L_2, and it follows that $x^{\xi} = x^{\zeta}$.

For a point $y \in \widetilde{\mathcal{M}}_2$, y on some line $M \in \mathcal{M}_2 \setminus \{L_2\}$, it holds similarly that y^{ξ} contains $M^{\xi} = M^{\zeta}$ and the unique transversal through y of the strong regulus of lines of $\mathcal{M}_2$, determined by L_2 and M. But this transversal is also contained in y^{ζ} and does not lie in $M^{\xi} = M^{\zeta}$, and consequently we may again conclude that $y^{\xi} = y^{\zeta}$.

Finally, let r be a point of $\mathsf{PG}(5,q)$, not in $\widetilde{\mathcal{M}}_2$. Then there exist $q+1$ tangent planes of the SPG regulus $\mathcal{M}_2$ through the point r. These $q+1$ tangent planes are totally isotropic for both ζ and ξ and must hence be contained in r^{ζ} and r^{ξ}. Two such tangent planes through r cannot have a line in common, because in that case the first tangent plane would meet the line of $\mathcal{M}_2$ in the second tangent plane in a point, and vice versa, a contradiction. As a consequence, these $q+1$ tangent planes span at least a 4-dimensional subspace of $\mathsf{PG}(5,q)$. On the other hand, all $q+1$ tangent planes must be contained in r^{ζ} and r^{ξ}, which are both 4-dimensional, so that the subspace generated by the $q+1$ tangent planes of the SPG regulus $\mathcal{M}_2$ through r must be 4-dimensional and coincide with r^{ζ} and r^{ξ}. This yields that $r^{\xi} = r^{\zeta}$ and we can now conclude that the symplectic polarities ζ and ξ are identical.

This proves the theorem. ∎

In combination with the results from [5], Theorem 2.2 implies that there exist d non-isomorphic semipartial geometries $\mathsf{SPG}(\mathcal{M})$, with $\mathcal{M}$ a locally hermitian 1-system of $W_5(q)$, $q > 2$ and even, belonging to the class discovered in [5], and which is not a spread of an elliptic quadric $Q^-(5,q)$. Here d stands for the number of orbits of the automorphism group of $\mathsf{GF}(q)$ in the set of elements of $\mathsf{GF}(q) \setminus \{0\}$ with trace zero. Since none of the considered 1-systems is a spread of a $Q^-(5,q)$ and taking account of the results in [6], the semipartial geometries they yield are new.

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