BN-pairs of finite Morley rank where B is solvable

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Abstract

Let G be a simple group of finite Morley rank with a definable irreducible BN-pair of (Tits) rank 2 where B is solvable and let $\mathfrak{P}$ be the associated generalized n-gon. If n is odd and B connected, then $n = 3$ and G is definably isomorphic to $PSL_3(K)$ for some algebraically closed field K. Furthermore, $n \leq 14$ if $T = B \cap N \neq 1$. We also give sufficient conditions for G to be a simple algebraic group.

1 Introduction

The classification of simple groups of finite Morley rank is unavoidable if one is concerned with the classification of first-order theories having few models. The starting point of what is called stability theory was Morley's result that a theory having only one model up to isomorphism in some uncountable cardinality has exactly one model up to isomorphism in *every* uncountable cardinality. Such theories are therefore called uncountably categorical. According to Shelah's classification program, these theories should certainly be classifiable. One of the tools in this context is the existence of a model theoretic notion of dimension on definable sets, the Morley rank. Zilber proved that uncountably categorical theories are either *almost strongly minimal*, i.e. in the algebraic closure of a set of Morley rank 1, or there must be infinite definable groups describing the relation between different definable subsets

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(see [Po] 2.25). These groups are uncountably categorical if equipped with the full induced structure, hence of finite Morley rank in the plain group language; if they are simple, they will however remain uncountably categorical in the plain group language. This shows that in order to get a good understanding of the models of an uncountably categorical theory it is necessary to understand both the sets of Morley rank 1 and the groups of finite Morley rank. While (plain) abelian groups of finite Morley rank were classified by Macintyre in the early 70'es, much less is known about other types of groups or strongly minimal sets. Zilber's conjecture that strongly minimal sets are all equality-like, vector space-like or field-like, turned out to be wrong. In producing counterexamples Hrushovski showed that strongly minimal sets can be more exotic than first assumed. However, these exotic examples are not wild enough to influence the algebraic properties of the structure, so that the following conjecture due to Cherlin and Zilber, which is in fact a special case of Zilber's original conjecture, remains open.

The Cherlin-Zilber Conjecture states that an infinite simple group of finite Morley rank is an algebraic group over an algebraically closed field. If the conjecture is true, then any such group must contain a definable (and split) BN-pair. Therefore, it is a natural problem to classify BN-pairs of finite Morley rank (see Section 2).

In several papers, we have obtained classification results for BN-pairs of finite Morley rank satisfying certain additional conditions: If the *Tits rank* of the BN-pair is at least 3 or the BN-pair has Tits rank 2 and is split, then the group satisfies the conjecture, see [KTVM, Te2, Te4]. We here consider the case of Tits rank 2 where B is solvable and prove the following results:

Theorem 3.3, 3.4 *Let G be an infinite group of finite Morley rank with a definable irreducible BN-pair of Tits rank 2 and let $|W| = |N/(B \cap N)| = 2n$. Assume that B is solvable and connected. If n is odd, then $n = 3$ and G/M is definably isomorphic to $PSL_3(K)$ for some algebraically closed field K and some normal subgroup M of G. If n is even and $T = B \cap N \neq 1$, then $4 \leq n \leq 14$.*

This is a first step towards an analog of the result by Feit and Higman [FH] showing that, in the finite case, $n \in \{3, 4, 6, 8\}$. This was crucial in the classification of the finite simple groups.

Theorem 5.7, 5.4 *Let G be an infinite group of finite Morley rank with a definable irreducible BN-pair of Tits rank 2 where B is solvable. If either G acts transitively on the set of ordered ordinary $(n+1)$-gons in the corresponding generalized n-gon $\mathfrak{P}$ or $fix(t) = fix(T^0)$ is finite for some $t \in T^0$, then G/M is definable isomorphic to $PSL_3(K), PSp_4(K)$ or $G_2(K)$ for some algebraically closed field K and some normal subgroup M of G.*

Furthermore, we show that for odd n, $T = B \cap N$ always has finite index in its normalizer.

Notice that these results contrast sharply with the examples in [Te1] where generalized n-gons of finite Morley rank were constructed for arbitrary $n \geq 3$ whose automorphism group acts transitively on the set of ordered ordinary $(n+1)$-gons. However, these generalized n-gons do not allow the definition of any infinite group, and in fact it follows from [TVM1] that their automorphism group does not have finite Morley rank. Thus, the assumption that the group acting on the generalized

n-gon be of finite Morley rank yields a dividing line between the 'wild' examples and the tame ones, even if we make no further assumptions on B (see also [TVM1]).

2 Background and definitions

Background on BN-pairs and generalized polygons with emphasis on their model theory has been presented in [Te2, KTVM, TVM2]. We recall definitions and properties to make the paper self-contained.

In a graph, a sequence $(x_0, x_1, \ldots, x_k)$ of vertices is called a *simple path of length* k, or a (simple) *k-path* if x_{i-1} is adjacent to x_i, for all $i \in \{1, 2, \ldots, k\}$, and if $x_{i-1} \neq x_{i+1}$, for all $i \in \{1, 2, \ldots, k-1\}$. For a connected graph, we can define a natural distance function $d(x, y)$ as the smallest k for which there is a k-path joining x and y.

2.1 Polygons Let $n \geq 2$ be an integer. A bipartite graph $\mathfrak{P}(\mathcal{P}, \mathcal{L}, \mathcal{F})$ is called a *generalized n-gon* (or just n-gon) if it satisfies the following three axioms:

(i) For all elements $x, y \in \mathcal{P} \cup \mathcal{L}$ we have $d(x, y) \leq n$.

(ii) If $d(x, y) = k < n$, then there is a *unique* k-path $(x_0 = x, x_1, \ldots, x_k = y)$ joining x and y.

(iii) $\mathfrak{P}$ is *thick*, i.e. every element $x \in \mathcal{P} \cup \mathcal{L}$ is adjacent to at least three other elements.

In other words, $\mathfrak{P}$ is a bipartite graph of diameter n, girth $2n$ and valency ≥ 3. It is easy to see that for $n = 2$, these axioms just define a complete bipartite graph of valency ≥ 3, and for $n = 3$ these are precisely the axioms of a projective plane. Obviously, the definition is completely symmetric in $\mathcal{P}$ and $\mathcal{L}$ and the generalized n-gon obtained from $\mathfrak{P}$ by exchanging $\mathcal{P}$ and $\mathcal{L}$ is called the *dual* of $\mathfrak{P}$. elements in If we drop axiom (iii), we obtain the notion of a *weak* generalized n-gon.

The vertices $\mathcal{P} \cup \mathcal{L}$ are called the *elements* of $\mathfrak{P}$. A pair of elements (x, y) is called a *flag* if x and y are adjacent. The set of neighbours of an element x is denoted by $D_1(x)$, and, more generally, the set of elements at distance i from x, $0 \leq i \leq n$, is denoted by $D_i(x)$.

Two elements at distance n from each other are called *opposite*. Also, two flags are called opposite if they consist of pairwise opposite elements. A simple path of length $2n$ with $x_0 = x_{2n}$ in $\mathfrak{P}$ is called an *ordered ordinary n-gon* or simply *ordinary n-gon* if we do not want to distinguish one specific flag. Note that any two opposite flags determine an ordinary n-gon, and given any flag (x_0, x_1) of an ordinary n-gon Γ, there is a unique flag of Γ opposite (x_0, x_1).

Let p and q be opposite elements of $\mathfrak{P}$. Axiom (ii) says that for any element $y \in D_1(p)$ there is a unique shortest path from y to q; this path contains a unique element $y' \in D_1(q)$. In this way we obtain a definable bijection between $D_1(p)$ and $D_1(q)$. Composing such bijections one obtains definable bijections between $D_1(x)$ and $D_1(y)$ for any two elements x and y of the same type, either point or line. If n is odd, this is true for all elements, see [KTVM] 2.3 for details.

As in the case of projective planes, one can use these bijections to show that any element of $\mathfrak{P}$ is definable from an ordinary n-gon $(x_0, \ldots, x_{2n-1}, x_{2n} = x_0)$ together with $D_1(x_0) \cup D_1(x_1)$, see again [KTVM] 2.3. In other words, $\mathfrak{P}$ is in the *definable closure* of $\{x_0, x_1, \ldots, x_n\} \cup D_1(x_0) \cup D_1(x_1)$. Using the definable bijections between $D_1(x)$ and $D_1(y)$ for $d(x, y) = n$, it is easy to see that the same is true for $\{x_0, x_1, \ldots, x_n\} \cup D_1(x_0) \cup D_1(x_{n-1})$.

2.2 Remark: This immediately implies that any automorphism of $\mathfrak{P}$ (in the sense of a first order structure, or equivalently, in the sense of incidence geometry) which fixes $\{x_0, x_1, \ldots, x_n\} \cup D_1(x_0) \cup D_1(x_j)$ for a simple path $(x_0, x_1, \ldots, x_n)$ and $j = 1$ or $n - 1$ is the identity. This fact is crucial in most of the proofs.

2.3 Definition Let $\mathfrak{P}$ be a generalized n-gon and let $\alpha = (x_0, x_1, \ldots, x_{n-1}, x_n)$ be a simple path. An automorphism of $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ fixing all elements incident with the elements of $(x_1, \ldots, x_{n-1})$ is called *root elation*, (or simply *elation*). The group U_α of all elations for α is called the *root group* corresponding to α.

For $n \geq 4$ an elation is called *central* if it fixes $D_i(x)$, for some element x, and for all positive $i \leq n/2$.

2.1 BN-pairs

Recall that given a group G, the subgroups B and N form a BN-pair if they satisfy the following conditions:

(i) $BNB = G$.

(ii) $T := B \cap N \trianglelefteq N$.

(iii) The *Weyl group* $W := N/T$ is generated by a distinguished set of involutions S.

(iv) For all $w_i \in S$ and $v, v_i \in N$ with $v_iT = w_i$, then $vBv_i \subseteq BvB \cup Bvv_iB$.

(v) $B^v \neq B$ for all $v \in N$ with $vT \in S$.

The standard example of a BN-pair is the one coming from the group of K -rational points of a simple algebraic group, K an algebraically closed field. Namely, let B be a *Borel* subgroup, viz. a maximal connected solvable subgroup of G, let T be the maximal split torus contained in B and let N be its normalizer. Then it is well known that B and N form a BN-pair where the Weyl group W is finite. Furthermore, B is a semidirect product $U \rtimes T$ of a nilpotent subgroup U, the *unipotent radical* of B, and the torus T.

A (not necessarily algebraic) BN-pair is called *split* if there exists (like in the algebraic case) a normal nilpotent subgroup U of B with $B = UT$. We do not require $U \cap T = 1$. The *(Tits) rank* of the BN-pair is by definition the number of generators in S. If the Weyl group is finite, the BN-pair is called *spherical*. The BN-pair is *irreducible* if there is no partition of S into subsets I, J such that

$W = W_I \times W_J$, where W_I, W_J denote the subgroups of W generated by I and J, respectively. Clearly, a BN-pair of Tits-rank 2 with $|W| = 4$ is necessarily reducible.

The only part of the definition of a BN-pair which might not be expressible by a first order statement is condition (iii). Consequently, when W is finite, the fact that we have a BN-pair is first-order expressible. It was shown in [TVM1] 2.8 that for groups of finite Morley rank (and more generally, for *stable* groups) the Weyl group is necessarily finite. This allows us to talk about *definable BN-pairs* in groups of finite Morley rank.

In this paper we will be concerned with groups of finite Morley rank having a definable irreducible BN-pair of (Tits) rank 2; so the associated Weyl group is a finite group generated by two involutions and hence is the dihedral group of order $2n$ for some $n \geq 3$. Such groups have a nice geometric interpretation as automorphism groups of generalized n-gons, which we now recall.

2.2 Geometric interpretation

The general reference for this section is [Ti]. From now on let G be a group with an irreducible BN-pair of Tits-rank 2, and suppose that the associated Weyl group $W = N/(B \cap N)$ is finite of order $2n$ for $n \geq 3$ and generated by w_1, w_2.

Let $P_1 = \langle B, w_1 \rangle$ and $P_2 = \langle B, w_2 \rangle$. These are the only proper subgroups of G properly containing B and they are definable if B is since $P_1 = B \cup Bw_1B$ and $P_2 = B \cup Bw_2B$. (Any subgroup containing a conjugate of B is called *parabolic.*) We define an incidence structure on the coset spaces $\mathcal{P} = G/P_1$ and $\mathcal{L} = G/P_2$ by defining a point gP_1 to be incident with a line $g'P_2$ if and only if $gP_1 \cap g'P_2 \neq \emptyset$. The axioms of a BN-pair yield that the incidence structure defined in this way is a generalized n-gon $\mathfrak{P}$, where $n = |W|/2$. Let x_0 denote the point $1_G P_1 \in \mathcal{P}$ and x_1 denote the line $1_G P_2 \in \mathcal{L}$. Since $P_1 \cap P_2 = B$, the elements x_0 and x_1 are incident in this structure. In a natural way, G acts on $\mathfrak{P}$ as a group of automorphisms and it can be shown that it acts transitively on the set of ordered ordinary n-gons contained in $\mathfrak{P}$ (see [VM1] Section 4.7). Clearly, in this action P_1 and P_2 are the respective stabilizers of x_0 and x_1 in G, and B is the subgroup fixing the flag (x_0, x_1).

The transitivity of G on ordered ordinary n-gons also implies that B acts transitively on the flags opposite (x_0, x_1), or, equivalently, on the ordinary n-gons containing (x_0, x_1). In this action $T = B \cap N$ is the (pointwise) stabilizer of some ordinary n-gon Γ containing (x_0, x_1) and N is the setwise stabilizer in G of Γ. Clearly, $N \leq N_G(T)$ is acting transitively on the flags of Γ. Thus the Weyl group $W = N/T$, which is just the dihedral group of order $2n$, acts in this way as the group of automorphisms of the incidence graph of Γ. (Note that G might have another BN-pair yielding a different polygon.) If the BN-pair splits as $B = UT$, then since T stabilizes Γ, the subgroup U acts transitively on flags opposite (x_0, x_1).

Conversely, let $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a generalized n-gon and suppose that a group $G \leq Aut(\mathfrak{P})$ acts transitively on the set of ordered ordinary n-gons. The BN-pair of G corresponding to this action can be seen as follows: Let (x_0, x_1) be a flag, and Γ an ordinary n-gon containing (x_0, x_1). Let B be the stabilizer of (x_0, x_1), and N the setwise stabilizer of Γ. Then B and N form a BN-pair of Tits-rank 2, and $\mathfrak{P}$ is isomorphic to the coset geometry $(G/G_{x_0}, G/G_{x_1}, \{(gG_{x_0}, gG_{x_1})|\ g \in G\})$, where G_{x_0} and G_{x_1} denote the stabilizer in G of the elements x_0 and x_1, respectively. As before

G_{x_0} and G_{x_1} then form the parabolic subgroups of G containing B $(= G_{x_0} \cap G_{x_1})$, and the Weyl group $W = N/(B \cap N)$ acts as the dihedral group of order $2n$ on Γ. Starting with a different flag, we obtain a BN-pair consisting of subgroups B' and N' which are conjugate in G to B and N, respectively.

We have thus seen that a BN-pair of Tits rank 2 with $|W| = 2n$ is equivalent to a generalized n-gon $\mathfrak{P}$ with an automorphism group transitive on ordered ordinary n-gons. Rather than working with the group theoretic definition we will in the sequel always be working with this geometric interpretation of the BN-pair.

2.3 Prerequisites

We will use the following results:

2.4 Fact [Te2, Te4] *Let G be an infinite group of finite Morley rank with a definable irreducible split BN-pair of (Tits) rank 2. Then G/M is interpretably isomorphic to either $PSL_3(K), PSp_4(K)$ or $G_2(K)$ for some algebraically closed field K and some normal subgroup M of G.*

Except for the case $|W| = 16$ this is contained in [Te2]. The remaining case was handled in [Te4] where it was shown that if G has a split BN-pair of Tits rank 2 with $|W| = 16$, then the corresponding generalized octagon is a so-called Moufang octagon. By [KTVM] Theorem A, there are no Moufang octagons of finite Morley rank, establishing the fact above. Here, as in the following results the normal subgroup M can be chosen as the kernel of the action of G on the associated generalized polygon.

2.5 Fact [Te2] *If G is a group of finite Morley rank with a definable BN-pair of Tits-rank 2 with $|W| = 6$ such that in the natural action on the associated projective plane G contains some nontrivial elation, then G/M is definably isomorphic to $PSL_3(K)$ for some algebraically closed field K and some normal subgroup M of G.*

2.6 Fact [GVM, VM2] *Let G be a group acting regularly on the set of ordered ordinary n-gons of some generalized n-gon $\mathfrak{P}$ where n is odd. Then $n = 3$, and $\mathfrak{P}$ is the projective plane of order 2.*

Translated into the language of BN-pairs this says that if a group G has a BN-pair of Tits-rank 2 with $|W| = 2n$ for odd n and $T = B \cap N = 1$, then $n = 3$, and the associated projective plane is finite (and the smallest possible). Unfortunately, for even n no result of this type is known.

2.7 Corollary *Let G be an infinite group of finite Morley rank with a definable BN-pair of Tits-rank 2 with $|W| = 2n$ where n is odd. Then $T = B \cap N$ is not contained in the kernel of the action of G on the associated generalized n-gon. In particular, $T \neq 1$.*

2.8 Notation We use the following convention: $[a,b] = a^{-1}b^{-1}ab$, $a^b = b^{-1}ab$ and $a^{-b} = (a^{-1})^b = b^{-1}a^{-1}b$, and we let group elements act from the right. When there is no confusion we also write $g(x)$ for the image of x under g. Otherwise, especially for products of group elements, we write x^g. For $g \in G$, we let $fix(g)$ denote the set $\{x \in \mathcal{P} \cup \mathcal{L};\ x^g = x\}$ and for a subgroup $H \leq G$ we denote $fix(H) = \bigcap_{g \in H} fix(g)$. We write $H_x^{[i]} = \{g \in H \mid D_i(x) \subseteq fix(g)\}$ and $H_{x_1,\dots x_k}^{[i]} = \bigcap_{j=1}^k H_{x_j}^{[i]}$.

We also use the following standard notation: If H is a group acting on the set X, and if $x \in X, Y \subseteq X$ we denote by H_x the stabilizer of x in H, and write H_Y for the pointwise stabilizer and $H_{\{Y\}}$ for the setwise stabilizer of Y in H.

The following well-known observations are at the heart of many arguments.

2.9 Lemma *Let H be a group acting on a set X.*
(i) Let g and h be commuting elements of H. Then g fixes some $x \in X$ if and only if it fixes $h(x)$. In particular, if $g \in G_x^{[k]}$ and $[g,h] = 1$, then $g \in G_{x,h(x)}^{[k]}$.
(ii) If $g \in H_{\{A\}}$ and $h \in H_A$ for $A \subseteq X$, then $[g,h] \in H_A$. In particular, if $g \in G_x$ and $h \in G_x^{[k]}$, then $[g,h] \in G_x^{[k]}$ since g leaves $D_k(x)$ invariant for any k. ∎

3 When B is solvable and connected

If the Cherlin-Zilber Conjecture is true, the BN-pair of a simple group of finite Morley rank will arise from a Borel subgroup, i.e. a maximal solvable and connected subgroup. In this section we therefore consider the situation where G has finite Morley rank and a definable BN-pair of Tits rank 2 with B solvable and connected. One should expect that the BN-pair is split, as any connected solvable group of finite Morley rank can be written as a product of its commutator subgroup, which is nilpotent, and a *Carter subgroup*, i.e. a nilpotent self-normalizing subgroup (see [Wa]). However, it is not clear that the Carter subgroup is contained in $T = B \cap N$ a necessary condition for the splitting of the BN-pair. In the situation of algebraic groups over algebraically closed fields, this is exactly what happens: the Borel subgroup B is solvable and connected, the torus T is abelian, connected and self-normalizing and the unipotent radical U is the commutator subgroup of B. In Section 5 we will give criteria that ensure the splitting of B, even if B is not necessarily connected.

However, if $|W| = |N/(B \cap N)| = 2n$ with n odd, it is not necessary to establish the splitting first since we can classify directly.

From now on, throughout the rest of the paper we keep the interpretation of G as an automorphism group of the associated generalized n-gon $\mathfrak{P}$, so $B = G_{x_0,x_1}$, $N = G_{\{\Gamma\}}$, $T = B \cap N = G_\Gamma$ etc. as explained in 2.2.

We will need the following fact:

3.1 Fact *For $t \in T$, $fix(t)$ is a (possibly weak) generalized n-gon (see [VM1] Theorem 4.4.2).*

3.2 Lemma *Let G be an infinite group of finite Morley rank with a definable BN-pair of Tits rank 2 where B is solvable and connected, and let H denote the commutator subgroup of B. Let k be minimal with the property that $Z(H) \not\leq G_{x_i}^{[k]}$ for either $i = 0$ or 1. If $k < n/2$, then $T \leq G_{x_{n+i},x_i}^{[1]}$ for some $x_{n+i} \in D_n(x_i)$.*

Proof. We keep the set-up introduced in 2.2, so G acts on $\mathfrak{P}$, $B = G_{x_0,x_1}$ for the flag (x_0, x_1) of $\mathfrak{P}$, $T = G_\Gamma$ for the ordinary n-gon Γ, which contains (x_0, x_1) etc. Since B is solvable and connected, the commutator subgroup H of B is nilpotent (see [Po] 3.19). Since the situation is symmetric in x_0 and x_1, we may assume that $Z(H) \not\leq G_{x_1}^{[k]}$ for $k < n/2$ minimal. So there is some $i \in Z(H)$ and a path $\gamma = (x_0, x_1, \dots x_n)$ of length n with $x_{k+1} \notin fix(i)$. Since $i \in Z(H)$, by 2.9(i) $H_\gamma \leq H_{i(\gamma)}$. The path $\gamma' = (x_n, \dots x_{k+1}, x_k, i(x_{k+1}), \dots i(x_n))$ is a path of length $2n - 2k > n$. The flags (x_n, x_{n-1}) and $(i(x_{2k}), i(x_{2k+1}))$ are opposite each other and hence determine some ordinary n-gon Γ' which is thus fixed under H_γ. But some flag (x, y) of Γ' is opposite (x_0, x_1) showing that H_γ fixes in fact the ordinary n-gon Γ'' determined by (x, y) and (x_0, x_1). By conjugating if necessary, we may assume $H_\gamma \leq T \leq B_\gamma$. But clearly we then have $H_\gamma \subseteq T^g$ for all $g \in B_\gamma$. Since B_γ is transitive on $D_1(x_n) \setminus \{x_{n-1}\}$ and $D_1(x_0) \setminus \{x_1\}$, we conclude that $H_\gamma \leq H_{x_n,x_0}^{[1]}$.

As H_γ contains the commutator subgroup of B_γ, this implies that B_γ acts as a regular abelian group on $D_1(x_n) \setminus \{x_{n-1}\}$ and $D_1(x_0) \setminus \{x_1\}$. So $T \leq B_\gamma \cap G_{x_n,x_0}^{[1]}$. ■

3.3 Theorem *Let G be an infinite group of finite Morley rank with a definable BN-pair of Tits rank 2 where B is solvable and connected. Assume that $|W| = |N/(B \cap N)| = 2n$ with n odd. Then $n = 3$ and G/M is definably isomorphic to $PSL_3(K)$ for some algebraically closed field K and some normal subgroup M of G.*

Proof. Let H denote the commutator subgroup of B. We claim that $Z(H) \leq G_{x_0,x_1}^{[k]}$ for all $k < n/2$. Otherwise, we conclude by 3.2 that $T \leq G_{x_n,x_0}^{[1]}$. The group N acts transitively on the flags of Γ and T is invariant under N. Since n is odd, we see that $T \leq G_{x_0,\dots x_n}^{[1]} = 1$. By Fact 2.6, $n = 3$ and $\mathfrak{P}$ is the projective plane of order 2. But B is connected and acts transitively on the set of flags opposite (x_0, x_1). This is impossible if that set is finite. Thus we have reached a contradiction.

Thus $Z(H) \leq G_{x_0,x_1}^{[k]}$ for all $k < n/2$. Now Theorem 1 of [TVM1] shows that $n = 3$. For $n = 3$, $Z(H) \leq G_{x_0,x_1}^{[1]}$ shows that G contains elations. Fact 2.5 finishes the proof of this theorem. ■

For the even case, we obtain the following restrictions:

3.4 Theorem *Let G be an infinite group of finite Morley rank with a definable BN-pair of Tits rank 2 where B is solvable and connected. Assume that $|W| = |N/(B \cap N)| = 2n$ with n even. If $T \neq 1$, then $n \in \{4, 6, 8, 10, 12, 14\}$.*

Proof. Let again H denote the commutator subgroup of B. By 3.2, if for some $k < n/2$ we have $Z(H) \not\leq G_{x_0}^{[k]}$ and $Z(H) \not\leq G_{x_1}^{[k]}$ then $T \leq T \cap G_{x_0,x_1}^{[1]} = 1$.

So we may assume that $Z(H) \leq G_{x_0}^{[k]}$ for $k < n/2$. We may also assume that $Z(H)$ does not contain central elations, as otherwise the proof of Theorem 1 in [TVM1] shows that $n \in \{4, 6, 8, 12\}$. So for every $g \in Z(H)$ there is some $x \in D_{\frac{n}{2}}(x_0) \setminus fix(g)$.

First assume that $n = 4m + 2$, so $Z(H) \leq G_{x_0}^{[2m]}$. Let $g \in Z(H)$, let $x_{2m+1} \in D_{2m+1}(x_0) \setminus fix(g)$ and let $x_{2m+2} \in D_{2m+2}(x_0) \cap D_1(x_{2m+1})$. Let $\gamma = (x_0, x_1, \dots, x_{2m+1}, , x_{2m+2})$ be a path connecting x_0 and x_{2m+2} and let $h \in G_{x_{2m+2}}^{[2m]}$

not fixing x_1. (By the homogeneity of the generalized n-gon $\mathfrak{P}$ we may take a conjugate of g, so such an element exists.) As $g \in G_{x_{2m}} \cap G_{x_2}^{[2m-2]}$, by Lemma 2.9(ii) the commutator $[g,h] \in G_{x_2,x_{2m}}^{[2m-2]}$. By Remark 2.2, $[g,h]=1$ if $m \geq 5$ (and $n \geq 22$) and for $n=18$, $[g,h]$ is an elation fixing $D_6(x_2) \cup D_6(x_8)$. If $[g,h]=1$, then by Lemma 2.9(i) $g \in G_{x_0,h(x_0)}^{[2m]}$. Since $d(x_0,h(x_0))=4$, this contradicts Remark 2.2 if $n \geq 22$. Thus, $n \leq 18$ and if $n=18$, then $\alpha=[g,h]$ is a nontrivial elation.

So suppose now that $n=18$, so $m=4$ and $x_{2m+2}=x_{10}$. Choose an element $x_0'' \in D_4(x_{10}) \cap D_6(x_8)$ and some $h' \in G_{x_0''}^{[8]}$ not fixing x_3. Then by Lemma 2.9(ii) the commutator $[\alpha,h'] \in G_{x_0''}^{[8]} \cap G_{x_6}^{[4]} = 1$ since $\alpha \in G_{x_0''} \cap G_{x_6}^{[4]}$ and $h' \in G_{x_6} \cap G_{x_0''}^{[8]}$. But this implies by Lemma 2.9(i) and Remark 2.2 that $\alpha \in G_{x_2,h'(x_2)}^{[6]} = 1$ since $d(x_2,h'(x_2))=8$. Hence $n \neq 18$ and so in this case $n \leq 14$.

Next assume that $n=4m$. First suppose that G contains a nontrivial element $g \in G_{x_0,y}^{[2m-1]}$ for some $y \in D_2(x_0)$. Let $x_{2m} \in D_{2m}(x_0) \cap D_{2m+2}(x_2)$ and let $(x_0,x_1,x_2,\ldots x_{2m})$ be a path connecting x_0 and x_{2m}. Let $h \in G_{x_{2m},y'}^{[2m-1]}$ be a nontrivial element for some $y' \in D_2(x_{2m}) \cap D_{2m+2}(x_0)$, so $d(x_0,h(x_0))=2$. By Lemma 2.9(ii) the commutator $[g,h] \in G_{x_1,x_{2m-1}}^{[2m-2]}$. If $m \geq 4$ (and $n \geq 16$), by Remark 2.2 we have $[g,h]=1$. This implies by Lemma 2.9(i) that $g \in G_{x_0,y,h(x_0),h(y)}^{[2m-1]} = 1$, a contradiction.

Now let $g \in Z(H) \leq G_{x_0}^{[2m-1]}$, let $x_{2m} \in D_{2m}(x_0) \backslash fix(g)$. Let $\gamma=(x_0,x_1,\ldots,x_{2m})$ be a path connecting x_0 and x_{2m} and let $h \in G_{x_{2m}}^{[2m-1]}$ be an element not fixing x_0. By 2.9(ii), we have $[g,h] \in G_{x_1,x_{2m-1}}^{[2m-2]}$. If $m \geq 4$ (and $n \geq 16$), by Remark 2.2 we have $[g,h]=1$. This implies by Lemma 2.9(i) that $g \in G_{x_0,h(x_0)}^{[2m-1]}$. But $d(x_0,h(x_0))=2$, contradicting our previous argument. So $n \leq 12$ in this case and if $n=12$, either g or $[g,h]$ is a nontrivial elation. ∎

While we should have $n \in \{3,4,6\}$ the remaining cases are harder to deal with. The case that $G_x^{[6]}$ is nontrivial may occur for $n=12$ or 14. It was shown in [Te3] that this leads to a contradiction if $n=12$ and assuming that the group $H=B'$ acts transitively on the flags opposite (x_0,x_1). So the cases $n=12$ and 14 have to be handled here again with less information. Notice that the case $n=12$ could not be excluded by the weaker assumptions of [TVM2]. Similar statements hold for the case that $G_x^{[4]}$ is nontrivial and $n=8$ or 10.

4 Some preliminary results

Recall that we keep the interpretation of G as an automorphism group of $\mathfrak{P}$ as explained in 2.2.

4.1 Lemma *Let G be a group with a spherical BN-pair of Tits-rank 2, and let $\mathfrak{P}$ be the associated generalized polygon, where $B=G_{x_0,x_1}$ in the natural action defined above. Then the BN-pair is split if and only if there is a normal nilpotent subgroup U of B acting transitively on the flags opposite (x_0,x_1).*

Proof. As pointed out in 2.2, if B splits as $B=UT$, then U has to act transitively on the flags opposite (x_0,x_1). Conversely, suppose that there is a normal nilpotent subgroup U of B acting transitively on the flags opposite (x_0, x_1). We have to show that $B = UT$. Let $g \in B$ and let $(x, y) \subset \Gamma$ be the unique flag of Γ opposite (x_0, x_1). By assumption there is some $h \in U$ with $h(x, y) = g(x, y)$. But then clearly, hg^{-1} fixes (x, y) and hence Γ. Thus $hg^{-1} \in T$ as required. ■

4.2 Lemma *If G has finite Morley rank, $fix(T) = \bigcup_{g \in N_G(T)} \Gamma^g$ and $fix(T^0) = \bigcup_{g \in N_G(T^0)} \Gamma^g$. Hence, in particular $fix(T) = \Gamma$ if and only if $N = N_G(T)$.*

Proof. We prove the statement for T^0, the proof for T being completely similar. By definition, $\Gamma \subseteq fix(T^0)$, and hence $\Gamma^g \subseteq fix((T^0)^g) = fix(T^0)$ for any $g \in N_G(T^0)$. For the converse, suppose that $x \in fix(T^0)$. Since $fix(T^0)$ is a (possibly weak) generalized n-gon, x is contained in some ordinary n-gon $\Gamma' \in fix(T^0)$. There is some $g \in G$ with $\Gamma' = \Gamma^g$. Since $\Gamma' \in fix(T^0)$, $T^0 \leq G_{\Gamma'} = T^g$. So $T^0 = (T^g)^0 = (T^0)^g$ and hence $g \in N_G(T^0)$, as claimed. The last part of the lemma follows easily since $N = G_{\{\Gamma\}} \subseteq N_G(T)$. ■

4.3 Corollary *If G has finite Morley rank, we have $fix(T) = fix(T^g)$ if and only if $g \in N_G(T)$ and $fix(T^0) = fix((T^0)^g)$ if and only if $g \in N_G(T^0)$.*

Proof. For the nontrivial direction suppose $g \notin N_G(T)$. Then $\Gamma^g \in fix(T^g) \setminus fix(T)$ by 4.2. Similarly for T^0. ■

4.4 Corollary *Assume G has finite Morley rank. If $fix(T^0) = fix(T)$, then $N_G(T) = N_G(T^0)$.*

Proof. Suppose $g \in N_G(T^0)$. Then $fix(T) = fix(T^0) = fix(((T^0)^g) = fix(T^g)$, so $g \in N_G(T)$. The other direction is clear. ■

4.5 Proposition *Let G be a group of finite Morley rank with a definable BN-pair of Tits-rank 2 with $|W| = 2n$ for odd n. Then $T = B \cap N$ has finite index in its normalizer. In fact, either $n = 3$ or $N = N_G(T)$.*

Proof. If $fix(T) = \Gamma$, the claim follows from Lemma 4.2. Otherwise $fix(T)$ is a thick generalized n-gon. We claim that $N_G(T)$ induces a regular action on the ordinary n-gons contained in $fix(T)$. Namely, let $\Gamma' \subseteq fix(T)$ be an ordinary n-gon. By Lemma 4.2, there is some $g \in N_G(T)$ with $\Gamma^g = \Gamma'$. Suppose $h \in N_G(T)$ also satisfies $\Gamma^h = \Gamma'$, then $gh^{-1} \in T$ fixes $fix(T)$. By Fact 2.6, $fix(T)$ and hence $N_G(T)/T$ is finite and $n = 3$. ■

5 Splitting B

Suppose now that G has finite Morley rank and a definable BN-pair of Tits rank 2 where B is solvable and not necessarily connected. We will now give some criteria that ensure that the BN-pair splits. *Recall the interpretation of G as an automorphism group of $\mathfrak{P}$*, so $B = G_{x_0,x_1}$, $N = G_\Gamma$, $T = B \cap N = G_\Gamma$ etc. as explained in 2.2.

Notice that while in algebraic groups over algebraically closed fields both B and $T = B \cap N$ are always connected groups, this is not true anymore for Lie-groups or more generally for simple semi-algebraic groups over real closed fields. However, even in these classes of examples T^0 has finite index in its normalizer and in fact we have $fix(T) = fix(T^0) = \Gamma$.

The following well-known lemma does not make any model theoretic assumptions. We include a proof of it here since there does not seem to be a reference for it.

5.1 Lemma *Let G be a group with a BN-pair of Tits rank 2. If G acts transitively on the set of ordered ordinary $(n+1)$-gons of the corresponding generalized n-gon $\mathfrak{P}$, then — with the notation of 2.2 — T fixes $\Gamma = (x_0, x_1, x_2, \ldots, x_{2n-1}, x_{2n} = x_0)$ and acts transitively on the $D_1(p) \setminus \{x_1, x_{2n-1}\} \times D_1(x_{n-1}) \setminus \{x_n, x_{n-2}\}$.*

Proof. First we note that given the ordinary n-gon $\Gamma = (x_0, x_1, x_2, \ldots, x_{2n-1}, x_{2n} = x_0)$, any two elements (x, y) with $x \in D_1(p) \setminus \{x_1, x_{2n-1}\}$ and $y \in D_1(x_{n-1}) \setminus \{x_n, x_{n-2}\}$ determine an ordinary $(n+1)$-gon: since $d(x, y) = n - 1$, there is a unique path of length $n-1$ joining them. This path together with the path $(x_0 = x_{2n}, x_{2n-1}, \ldots, x_{n-1})$ forms an ordinary $n+1$-gon Θ. Given another pair $(x', y') \in D_1(p) \setminus \{x_1, x_{2n-1}\} \times D_1(x_{n-1}) \setminus \{x_n, x_{n-2}\}$ and corresponding $(n+1)$-gon Θ' containing $(x_0 = x_{2n}, x_{2n-1}, \ldots, x_{n-1})$ and the path determined by x' and y', by the transitivity of G on the set of ordered ordinary $n+1$-gons there is some $g \in G$ fixing $(x_0, x_{2n-1}, \ldots, x_{n-1})$ with $\Theta^g = \Theta'$. So $g \in T$, implying that the group T acts transitively on the set $D_1(x_0) \setminus \{x_1, x_{2n-1}\} \times D_1(x_{n-1}) \setminus \{x_n, x_{n-2}\}$. ∎

5.2 Remark (i) Note that since $D_1(x)$ has Morley degree 1 for all x (see e.g. [Te2] 2.6), also T^0 is transitive on this set.

(ii) Any ordinary n-gon $\Gamma' \neq \Gamma$ determines some element in $D_1(x_0) \setminus \{x_1, x_{2n-1}\}$ or $D_1(x_{n-1}) \setminus \{x_n, x_{n-2}\}$. Thus, $fix(T^0) = fix(T) = \Gamma$ and hence $N = N_G(T)$. In particular, T and T^0 have finite index in their normalizers.

(iii) Since T is invariant under N and N acts transitively on the flags of Γ, T is also transitive on $D_1(x_i) \setminus \{x_{i-1}, x_{i+1}\}$ for all $1 \leq i < 2n$.

The following general lemma might be useful even outside the context of finite Morley rank.

5.3 Lemma *Let G be a group with a BN-pair of Tits rank 2. Let (y_0, y_1) be a flag opposite (x_0, x_1). For any $t \in T = B \cap N$, there exists some $g \in B' = [B, B]$ with $(y_0, y_1)^g = (y_0, y_1)^t$. If G has finite Morley rank and $t \in T^0$, we can choose $g \in U := [B^0, B^0]$.*

Proof. If t fixes (y_0, y_1), there is nothing to prove. Otherwise, let (z_0, z_1) be the flag of Γ opposite (x_0, x_1) and let $h \in B$ be such that $(z_0, z_1)^h = (y_0, y_1)$. Then $[h, t] \in B'$ is the required element. The last sentence follows from the fact that $T^0 \leq B^0 \cap N$ and we can choose $h \in B^0$ since this group is still transitive on the flags opposite (x_0, x_1). ■

Clearly, Lemma 5.3 holds also for any conjugate T^b of T for $b \in B$. Therefore, it suffices to prove that the conjugates of T generate B in order to show that B' is transitive on the flags opposite (x_0, x_1).

5.4 Theorem *Let G be an infinite group of finite Morley rank with a definable BN-pair of Tits rank 2 where B is solvable. If G acts transitively on the set of ordered ordinary $(n+1)$-gons in the corresponding generalized n-gon $\mathfrak{P}$, then G/M is definably isomorphic to $PSL_3(K)$, $PSp_4(K)$ or $G_2(K)$ for some algebraically closed field K and some normal subgroup M of G.*

Proof. By Fact 2.4 it suffices to prove that the BN-pair splits. We use Lemma 5.3 to prove that $U := [B^0, B^0]$ acts transitively on the ordinary n-gons containing (x_0, x_1). Since U is nilpotent ([Po] 3.19), the splitting then follows from 4.1.

Let $\Gamma = (x_0, x_1, x_2, \ldots, x_{2n-1}, x_{2n} = x_0)$ and $\Gamma' = (y_0 = x_0, y_1 = x_1, y_2, \ldots, y_{2n-1}, y_{2n} = y_0)$ be two distinct ordinary n-gons. Using Lemma 5.1, we will prove by induction on the length of the path where Γ and Γ' differ that there are elements in respective conjugates of T^0 that transform Γ into Γ' (The corresponding proof in [Te3] is correct only for $n \leq 4$.) Then U is as required by Lemma 5.3.

Let i, j be such that for $k < i$ we have $x_k = y_k$ and for $k < j$ we have $x_{2n-k} = y_{2n-k}$. Clearly, $2 \leq i \leq n$ and $1 \leq j \leq n-1$. So Γ and Γ' differ on a path of length $2n-2 \geq k = 2n-j-i \geq n$. First assume that they differ on a path of length $k = n$. By relabeling if necessary, we may assume that Γ and Γ' agree on the path $(x_0, x_1, \ldots, x_n)$. Let $\Theta = (x_0, x_1, \ldots, x_n, z_{n+1}, \ldots, z_{2n-1}, x_0)$ be an ordinary n-gon with z_{2n-1} different from x_{2n-1}, y_{2n-1}. By Lemma 5.1, G^0_Θ acts transitively on $D_1(x_0) \setminus \{x_1, z_{2n-1}\}$, so there is some $t \in G^0_\Theta$ with $t(x_{2n-1}) = y_{2n-1}$, and hence $\Gamma^t = \Gamma'$ as they agree on a path of length strictly greater than n.

Now assume that $k > n$, so Γ and Γ' agree on a path of length $2n-k < n$. Without loss of generality we may assume they agree on $(x_0, x_1, \ldots, x_{2n-k})$. Let $\Theta = (x_0, x_1, \ldots, x_{2n-k}, z_{2n-k+1}, \ldots, z_{2n-1}, x_0)$ be an ordinary n-gon with z_{2n-1} different from x_{2n-1}, y_{2n-1}. By Lemma 5.1, G^0_Θ acts transitively on $D_1(x_0) \setminus \{x_1, z_{2n-1}\}$, so $t(x_{2n-1}) = y_{2n-1}$ for some $t \in G_\Theta$. Then Γ^t and Γ' differ on a path of length strictly less than k. Now our induction hypothesis finishes the proof. ■

In order to prove in a more general context that the BN-pair splits, by Lemma 5.3 it suffices to prove that the B-conjugates of T generate B. For this we use the following criterion, which is always satisfied in simple algebraic groups of Tits rank 2:

5.5 Lemma *If G has finite Morley rank and $fix(t) = fix(T^0)$ for some $t \in T^0$, then $fix(t) = fix(T^0)$ for any generic t of T^0.*

Proof. The set $X = \{t \in T^0 : \ fix(t) = fix(T^0)\}$ is nonempty and definable. Clearly, $Stab(X) = \{t \in T : \ tt' \in X \text{ for all } t' \in X\}$ contains T^0, showing this set to be generic (see [Po] 2.2). ∎

5.6 Lemma *Let G be an infinite group of finite Morley rank with a definable BN-pair of Tits rank 2 where B is solvable. If T^0 has finite index in its normalizer and $fix(T^0) = fix(t)$ for some $t \in T^0$, then $\langle (T^0)^b : b \in B\rangle = B^0$.*

Proof. By Lemma 5.5 we have $fix(T^0) = fix(t)$ for $t \in T^0$ generic. Thus the set $X = \bigcup_{g \in B}\{t^g : \ t \in T^0 : fix(t) = fix(T^0)\}$ is definable and contains all generics of $(T^0)^g$ for $g \in B$. We claim that the Morley rank of X equals the Morley rank of B. Define the following map

$$\varphi : X \longrightarrow B/N_B(T^0); \ t \mapsto gN_B(T^0)$$

where $fix(t) = fix((T^0)^g)$. Clearly, φ is interpretable and surjective. It is well-defined because $fix((T^0)^g) = fix((T^0)^h)$ implies $gh^{-1} \in N_B(T^0)$ by Cor. 4.3. For $gN_B(T^0) \in B/N_B(T^0)$, $\varphi^{-1}(gN_B(T^0))$ contains the generics of $(T^0)^g$ and hence

$$RM(\varphi^{-1}(gN_B(T))) \geq RM(T^0) = RM(N_B(T^0)).$$

By the additivity of Morley rank (see [Po] 2.14), we hence see that

$$RM(B) \geq RM(X) \geq RM(B/N_B(T^0)) + RM(N_B(T^0)) = RM(B).$$

By Zilber's Indecomposability theorem (see [Po] 2.9), the group $\langle (T^0)^b : b \in B\rangle$ is definable and connected. As it contains X, we must have $\langle T^b : b \in B\rangle = B^0$. ∎

5.7 Theorem *Let G be an infinite group of finite Morley rank with a definable BN-pair of Tits rank 2 where B is solvable. If T^0 has finite index in its normalizer and $fix(T^0) = fix(t)$ for some $t \in T^0$, then the BN-pair is split. Thus, G/M is definably isomorphic to $PSL_3(K), PSp_4(K)$ or $G_2(K)$ for some algebraically closed field K and some normal subgroup M of G.*

Proof. Since B is transitive on the set of flags opposite (x_0, x_1) and this set has Morley degree 1 (see eg. [KTVM] 2.11), B^0 is still transitive on this set. By Lemma 5.6, we know that B^0 is generated by the B-conjugates of T^0. Lemma 5.3 now implies that $U := [B^0, B^0]$ is also transitive on the set of flags opposite (x_0, x_1). As U is nilpotent (see [Po] 3.19), Lemma 4.1 yields that the BN-pair is split.

The last sentence of the theorem now follows from Fact 2.4. ∎

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