Calculus in $\mathcal{O}$-algebras with positive squares

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Abstract

Let A be an $\mathcal{O}$-algebra with positive squares and $F(X_1, ..., X_n) \in \mathbb{R}^+[X_1, ..., X_n]$ be a homogeneous polynomial of degree p $(p \in \mathbb{N}^*,\ p \neq 2)$. It is shown that for all $0 \leq a_1, ..., a_n \in A$ there exists $0 \leq a \in A$ such that $F(a_1, ..., a_n) = a^p$.

As an application we show that every algebra homomorphism T from an $\mathcal{O}$-algebra A with positive squares into an Archimedean semiprime f-algebra B is positive. This improves a result of Render [14, Theorem 4.1], who proved it for the case of order bounded multiplicative functional T from an $\mathcal{O}$-algebra A with positive squares into $\mathbb{R}$.

1 Introduction

In this paper we are going to discuss lattice-ordered algebras (briefly, ℓ-algebras) A with the property

$$|a| \wedge |b| = 0 \Rightarrow ab \in N(A), \qquad (\mathcal{O}')$$

so called $\mathcal{O}'$-*algebras*, where $N(A)$ is the set of all nilpotent elements in A. This class of algebras generalizes the class of almost f-algebras (here the set $N(A)$ in $(\mathcal{O}')$ is replaced by the trivial ideal $\{0\}$) and d-algebras. We give examples showing that these algebras are in general not commutative. In section 2, we give a description of the set $N(A)$ of an Archimedean $\mathcal{O}'$-algebra in which the square of every element is positive. It is shown among other things that $N(A)$ is an ℓ-ideal.

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In section 3, we give a generalization of a theorem about homogeneous polynomials on f-algebras (due to Beukers and Huijsmans) to the case of *positive square* $\mathcal{O}'$-algebras. The theorem in question is the following : if A is a relatively uniformly complete $\mathcal{O}'$-algebra in which squares are positive and $F(X_1,...,X_n) \in \mathbb{R}^+[X_1,...,X_n]$ is a homogeneous polynomial of degree p ($p \in \mathbb{N}^*$, $p \neq 2$) then there exists $0 \leq a \in A$ such that $F(a_1,...,a_n) = a$ for every $0 \leq a_1,...,a_n \in A$.

A consequence of this theorem will be the fact that every algebra homomorphism T from a relatively uniformly complete $\mathcal{O}'$-algebra A in which squares are positive into an Archimedean semiprime *f*-algebra B is positive and by the way it becomes a lattice homomorphism. This improves Theorem 4.1 in [14] by Render, who proved it for the case of order bounded multiplicative functional T from a Banach $\mathcal{O}'$-algebra A in which squares are positive into $\mathbb{R}$.

We point out that all proofs are purely order theoretical and algebraic in nature and furthermore do not involve any analytical means.

For the terminology and the concepts on vector lattices and ℓ-algebras that are not explained or proved in this paper we refer to [1,2 and 3]. To avoid unnecessary repetition we assume throughout this paper that all vector lattices and ℓ-algebras under consideration are Archimedean.

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2 Set of all nilpotent elements of positive squares $\mathcal{O}'$-algebras

Let A be a vector lattice. Recall that a vector subspace I of A is called *order ideal* (or *o-ideal*) whenever $|a| \leq |b|$ and $b \in I$ imply $a \in I$. Every o-ideal is a vector sublattice of A. The principal o-ideal generated by $0 \leq e \in A$ is denoted by A_e.

In the next lines, we recall some definitions and basic facts about ℓ-algebras. For more informations about this field, the reader can consult [1,2 and 5]. The (real) algebra A which is simultaneously a vector lattice is called *lattice ordered algebra* (briefly, *ℓ-algebra*) whenever $a, b \in A^+$ implies $ab \in A^+$(equivalently, $|ab| \leq |a|\,|b|$ for all $a, b \in A$). A subset I of an ℓ-algebra A, is called an *ℓ-ideal*, whenever I is an o-ideal and a ring ideal of A. For an ℓ-algebra A we denote the collection of all nilpotent elements of A by $N(A)$. An ℓ-algebra A is called *semiprime* if $N(A) = \{0\}$. An ℓ-algebra A in which the square of every element is positive will be called *positive square algebra*. This kind of algebras received its name by Vismanthan in [19]. In [16], Scheffold introduced the following condition for a Banach ℓ-algebra A

$$|a| \wedge |b| = 0 \text{ implies } r(ab) = 0 \text{ for all } a, b \in A \qquad (\mathcal{O})$$

where $r(.)$ denotes the spectral radius calculated in the complexification of A. Therefore any Banach ℓ-algebra which satisfies the condition $(\mathcal{O})$ is called an *$\mathcal{O}$-algebra*.

Render proved in [13, Theorem 1] that in a *positive square* Banach algebra A, the Condition $(\mathcal{O})$ is equivalent to the following:

$$|a| \wedge |b| = 0 \text{ implies } ab \in N(A) \text{ for all } a, b \in A. \qquad (\mathcal{O}')$$

Similarly, we say that an ℓ-algebra A is an $\mathcal{O}'$*-algebra*, whenever A satisfies condition $(\mathcal{O}')$.

Next we focus on almost f-algebras, d-algebras and f-algebras. Our references are [2, 3, 8 and 12]. An ℓ-algebra A is called an *f-algebra* if $a \wedge b = 0$ and $c \geq 0$ imply $ac \wedge b = ca \wedge b = 0$. An ℓ-algebra A is called an *almost f-algebra* if $ab = 0$ as soon as $a \wedge b = 0$. An ℓ-algebra A is called a *d-algebra* if $ac \wedge bc = ca \wedge cb = 0$ whenever $a \wedge b = 0$ and $c \geq 0$.

Obviously, any almost f-algebra and any f-algebra is a *positive square* $\mathcal{O}'$-algebra and every d-algebra is an $\mathcal{O}'$-algebra. Whereas, the converse is false. We illustrate this by the following examples.

Example 1. *Take $A = \mathbb{R}^4$ with the coordinatewise vector space operations and the cone*
$A^+ = \{a = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4;\ a_1 \geq 0, |a_2| \leq a_1,\ a_3 \geq 0 \text{ and } |a_4| \leq a_3\}$,
supplied with the following multiplication:
$a.b = (0, 0, a_1 b_1, 0)$ *for* $a = (a_1, a_2, a_3, a_4)$ *and* $b = (b_1, b_2, b_3, b_4)$.
A straightforward calculation shows that A is a commutative positive square $\mathcal{O}'$-algebra. However, A is not an almost f-algebra (hence it is not an f–algebra). Indeed, if $e_1 = (1, 0, 0, 0), e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$ and $e_4 = (0, 0, 0, 1)$ are the basis vectors, then $(e_1 + e_2) \wedge (e_1 - e_2) = 0$, but $(e_1 + e_2) . (e_1 - e_2) = e_3$. Moreover, $e_1 (e_1 + e_2) \wedge e_1 (e_1 - e_2) = e_3$ and consequently A is not a d-algebra.

Example 2. *Take $A = \mathbb{R}^4$ be ordered as in Example 1 but the multiplication be given by:*
$a.b = (0, 0, 3\ a_1 b_1 - a_1 b_2 + a_2 b_2, a_1 b_2)$ *for* $a = (a_1, a_2, a_3, a_4)$ *and* $b = (b_1, b_2, b_3, b_4)$.
It is not hard to see that A is a non-commutative positive square $\mathcal{O}'$-algebra. It follows that A cannot be an almost f-algebra. On the other hand, A is not a d-algebra because $(e_1 + e_2) \wedge (e_1 - e_2) = 0$ and $e_1 (e_1 + e_2) \wedge e_1 (e_1 - e_2) = 2e_3 + e_4$.

Examples 1 and 2 above show that the class of $\mathcal{O}'$-algebras is larger then the class of almost f-algebras.

It is shown by Render in [13, Theorem 1] that in every real Banach ℓ-algebra with closed cone C containing all squares (or *positive square* Banach ℓ-algebra) we have actually more, $rad\,(A) = N(A) = \{x \in A, x^3 = 0\}$. Moreover, Diem proved in [6, Theorem 3.9.(ii)] that the index of a positive nilpotent element in an ℓ-algebra with positive squares does not exceed 3. Recently, Lavric showed in [9] that for any ℓ-ring A with positive squares, $N\,(A) = \{x \in A;\ x^4 = 0\} = \{x \in A;\ 2x^3 = 0\}$.

Remark 1. *In general, for an arbitrary positive square algebra A, $N(A)$ need not be an o-ideal of A, as it is shown in the following example which is due to Bernau and Huijsmans.*

Example 3. *([2], Example 1.6) Take A be the plane $\mathbb{R}^2$ with the coordinatewise vector space operations and cone*
$A^+ = \{(a_1, a_2) \in \mathbb{R}^2;\ a_1 \geq 0 \text{ and } |a_2| \leq a_1\}$,
provided with the following multiplication:
$ab = (a_1, a_2)(b_1, b_2) = (a_1 b_1, 0)$ *for* $a = (a_1, a_2)$ *and* $b = (b_1, b_2)$.

A straightforward calculation shows that A is a positive square algebra. Let $e_1 = (1,0)$ and $e_2 = (0,1)$. Since $(e_1+e_2)\wedge(e_1-e_2)=0$, then $((e_1+e_2)(e_1-e_2))^2 = e_1$. Put $a = (e_1+e_2)-(e_1-e_2)$, then $a \in N(A)$, but $a^+ = (e_1+e_2) \notin N(A)$ and $a^- = (e_1-e_2) \notin N(A)$. Then $N(A)$ is not an o-ideal of A.

It is proven by Lavric in [9, Theorem 1.5] that if A is a *positive square* ring, then $N(A)$ is order convex (that means that if $0 \le x \le y$, $y \in N(A)$ then $x \in N(A)$). If in addition A is a *positive square* $\mathcal{O}'$-algebra, then the result can be improved. More precisely, $N(A)$ becomes an ℓ-ideal.

We need the following result.

Proposition 1. *Let A be an algebra ordered by a multiplicative cone A_+. If A_+ contains all squares then $x^3 = 0$ for $x \in A_+$ implies that $cxd = cdx = 0$ for all $c, d \in A_+$.*

Proof. The assumption implies that $0 \le (nx^2 - c)^2$ for all natural numbers n and for all $c \in A_+$. Hence $0 \le nx^2c + ncx^2 \le c^2$. It follows that $x^2c = cx^2 = 0$. Similarly $0 \le (nx-d)^2$ implies that $0 \le nxd + ndx \le n^2x^2 + d^2$. Multiplication of the last inequality with $c \ge 0$ yields $0 \le ncxd + ncdx \le cd^2$. The Archimedean property leads to the result $cxd = cdx = 0$. The proof is complete. ∎

Now we state the main result of this section.

Theorem 1. *Let A be a positive square $\mathcal{O}'$-algebra. Then $N(A) = \{x \in A;\ x^3 = 0\}$ is an ℓ-ideal of A.*

Proof. First $N(A) = \{x \in A;\ x^3 = 0\}$ by a result of Lavric (see [9], Theorem 1.5). Let $y \in N(A)$. We claim that $y^+, y^- \in N(A)$. Then $y^2 = (y^+ - y^-)^2 = (y^+)^2 + (y^-)^2 - y^+y^- - y^-y^+ \in N(A)$. It follows from the fact that A is an $\mathcal{O}'$-algebra that $y^+y^-, y^-y^+ \in N(A)$. In particular, according to [6, Theorem 3.9.(i)], $y^+y^-a = ay^+y^- = 0$ and $by^-y^+ = y^-y^+b = 0$, for all $a, b \in A$. Hence

$$y^4 = \left(y^+ - y^-\right)^4 = \left(y^+\right)^4 + \left(y^-\right)^4.$$

Since $y^4 = 0$, it follows $(y^+)^4 = (y^-)^4 = 0$. This implies that $y^+, y^- \in N(A)$. According to the previous proposition, we deduce that $|y| = y^+ + y^- \in N(A)$.
Let $x, y \in A$ such as $|x| \le |y|$ and $y \in N(A)$. As $|x| \le |y|$, we have that $|x| \in N(A)$ and $x \in N(A)$.

Let us prove, that $N(A)$ is a subspace of A. Let $x, y \in N(A)$ then $|x|, |y|, x^+, x^-, y^+, y^- \in N(A)$. A simple combination between [6, Theorem 3.9.(i)] and the previous proposition yields

$$(x+y)^3 = xyx + yxy = xy^+x - xy^-x + yx^+y + yx^-y = 0$$

Hence $N(A)$ is an o-ideal of A.
Now let $0 \le x \in N(A)$, and $0 \le y \in A$. We want to show that xy and $yx \in N(A)$. It follows from [6, Theorem3.9. (i)] that $x^2y = yx^2 = 0$. Since $0 \le (nx-y)^2$, we deduce that

$$n(xy + yx) \le y^2 + nx^2 \ \ (n = 1, 2....).$$

Multiplying in both sides the previous inequality by y, we get

$$n\,(yxy + yyx) \leq y^3 \;\; (n = 1, 2....)\,,$$

and

$$n\,(xyy + yxy) \leq y^3 \;\; (n = 1, 2....)\,.$$

Since A is an Archimedean vector lattice, this implies that $yxy = yyx = xyy = yxy = 0$, and hence $(xy)^2 = (yx)^2 = 0$. From the fact that A is a vector lattice, it follows that $(xy)^2 = (yx)^2 = 0$ for all $(x, y) \in N\,(A) \times A$, which shows that xy and $yx \in N(A)$. ∎

Remark 2. *In general, for an arbitrary $\mathcal{O}'$-algebra A, $N(A)$ need not be equal to $\{x \in A;\; x^3 = 0\}$, as it is shown in the following example.*

Example 4. *Let A be the set of all $(n \times n)$-matrices of the form*

$$M = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ 0 & a_{2,2} & \cdots & a_{2,n} \\ . & & & \\ . & & & \\ 0 & & 0 & a_{n,n} \end{pmatrix}$$

with the usual addition, scalar multiplication and partial ordering. Moreover, it is not hard to prove that, under composition, A is an $\mathcal{O}$-algebra. In addition, it is obvious that $N\,(A) \supsetneq \{M \in A;\; M^3 = 0\}$, whenever $n > 3$.

The next corollary turns out to be useful later.

Corollary 1. *Let A be a positive square $\mathcal{O}'$-algebra. Then $(ab)^2 = 0$ as soon as $|a| \wedge |b| = 0$.*

Proof. Let $a, b \in A$ such that $|a| \wedge |b| = 0$. As A is an $\mathcal{O}'$-algebra, it follows that $|a|\,|b| \in N(A)$. By Proposition 1 and Theorem 1,

$$c\,|a|\,|b|\,d = cd\,|a|\,|b| = 0$$

for all $0 \leq c, d \in A$. In particular, if $c = |a|$ and $d = |b|$, then

$$(|a|\,|b|)^2 = 0.$$

Now since

$$0 \leq (ab)^2 \leq (|a|\,|b|)^2 = 0,$$

one can easily deduced that $(ab)^2 = 0$. This completes the proof. ∎

Remark 3. *It is obvious that every positive square algebra A is a positive square $\mathcal{O}'$-algebra if and only if A satisfies the following condition:*

$$a \wedge b = 0 \text{ then } (ab)^2 = 0 \text{ for all } a, b \in A$$

We next present two other corollaries.

Corollary 2. *Let A be a positive square $\mathcal{O}'$-algebra, then*
(i) $A/N(A)$ is a semiprime f-algebra.
(ii) If either $a \wedge b \in N(A)$ or $a \vee b \in N(A)$, then $ab \in N(A)$.

Proof. *(i)* Suppose that $\overline{a} \wedge \overline{b} = \overline{0}$ in the $\ell-$algebra $A/N(A)$ (here $\overline{a}$ denotes the coset $a + N(A)$). It follows from $a \wedge b \in N(A)$ that $\overline{a} = \overline{a} - \overline{a} \wedge \overline{b}$ and $\overline{b} = \overline{b} - \overline{a} \wedge \overline{b}$. Hence , $\overline{a}\overline{b} = \left(\overline{a} - \overline{a} \wedge \overline{b}\right)\left(\overline{b} - \overline{a} \wedge \overline{b}\right) = \overline{0}$, as $A/N(A)$ is an almost f-algebra. Beside this suppose that $a \in A$; $\overline{a}^n = \overline{0}$ for $n \in \mathbb{N}^*$, so $a^n \in N(A)$ this yields $a \in N(A)$. Then $A/N(A)$ is a semiprime f-algebra.

(ii) This is a direct consequence of *(i)*. ■

Corollary 3. *Let A be a positive square $\mathcal{O}'$-algebra. Then $cab = abc = 0$ whenever $a \wedge b = 0$ and $c \in A$.*

Proof. Let $a, b \in A$ such as $a \wedge b = 0$. In view of Corollary 1, we have $(ab)^2 = 0$ Let $c \in A^+$. It follows from the fact that A is a *positive square* algebra that

$$0 \leq (nab - c)^2$$

and

$$n\,(abc + cab) \leq c^2.$$

By the Archimedean property, $abc + cab = 0$. Hence

$$abc = cab = 0 \text{ for all } 0 \leq c \in A.$$

And consequently, $abc = cab = 0$ for all $c \in A$, since every element of A is the difference of two positive elements. ■

It is well-known that almost f-algebras are commutative. But it was shown in the previous examples that arbitrary *positive square* $\mathcal{O}'$-algebras need not to be commutative. Moreover, Lavric proved in [9, Theorem 3.2.] that in every partially ordered ring A which is *f-decomposable* (that is: every element $a \in A$ can be expressed as $a = a_1 - a_2$ with $a_1, a_2 \in A^+$ and $a_1a_2 = a_2a_1 = 0$) then all triples of elements of A commute. In what follows, we improve the preceding result in an elementary way.

Theorem 2. *Let A be a positive square $\mathcal{O}'$-algebra. Then all triples elements of A, commute, that is*

$$a_1a_2a_3 = a_{\sigma(1)}a_{\sigma(2)}a_{\sigma(3)} \textit{ for any permutation } \sigma \textit{ of } \{1, 2, 3\}.$$

Proof. Let $0 \leq a, b, c \in A$. Take $B = A \times A$ with the coordinatewise vector space operations and partial ordering inherited from A and with the following product :

$$X * Y = \begin{pmatrix} x \\ y \end{pmatrix} * \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ xx'a \end{pmatrix}$$

for all $x, x', y, y' \in A$.
According to Corollary 3, we conclude that $(B, *)$ is an almost f- algebra. Then $(B, *)$ is a commutative algebra and so

$$xx'a = x'xa.$$

In particular, if $x = b$ and $x' = c$ we have $bca = cba$ for all $0 \leq a, b, c \in A$. Also, if B is equipped with the following multiplication

$$X \widehat{*} Y = \begin{pmatrix} x \\ y \end{pmatrix} \widehat{*} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ axx' \end{pmatrix},$$

then we deduce that

$$abc = acb$$

for all $0 \leq a, b, c \in A$. Analogously, it is not hard to prove the desired assertion, since every element of A is the difference of two positive elements. ■

As a consequence, we obtain the following.

Corollary 4. *Let A be a positive square $\mathcal{O}$-algebra. Then*
(i) $zxy = z(x \wedge y)(x \vee y)$ for all $x, y, z \in A$,
(ii) $(xy)^2 = (x \wedge y)^2 (x \vee y)^2$ for all $x, y, \in A$.

Proof. (i) Let $x, y, z \in A$. It is known that $(x - x \wedge y) \wedge (y - x \wedge y) = 0$. From the previous Corollary $z(x - x \wedge y)(y - x \wedge y) = 0$.
So $z\left[xy - x(x \wedge y) - (x \wedge y)y + (x \wedge y)^2\right] = 0$. Now by Theorem 2

$$zxy = z((x \wedge y)(x + y - (x \wedge y))) = z(x \wedge y)(x \vee y)$$

This gives the desired result .
The second assertion is obtained in a similar way. ■

3 The main results

Let us recall that some of the relevant notions in this section. Let A be a vector lattice and let $u \in A^+$. The sequence $\{x_n\}_{n=1}^{\infty}$ in A is said *to converge u-uniformly to $x \in A$*, whenever, for every $\varepsilon > 0$, there exists a natural number $N_\varepsilon > 0$ such that $|x_n - x| < \varepsilon u$ for all $n > N_\varepsilon$. This is denoted by $x_n \to x\,(u)$. The element u is called *the regulator of convergence.* The sequence $\{x_n\}_{n=1}^{\infty}$ is said *to converge relatively uniformly to* $x \in A$, if $x_n \to x\,(u)$ for some $u \in A^+$. We shall write $x_n \to x\,(r.u.)$.

The notion of *relatively uniform Cauchy sequence* is defined in the obvious way. The vector lattice A is called *relatively uniformly complete* whenever every relatively uniform Cauchy sequence has a unique limit. For any non empty subset D of A we define the pseudoclosure D' of D to be the set of all $a \in A$ for which there exists $a_n \in D$ $(n = 1, 2...)$ such that $a_n \to a\,(r.u.)$. Furthermore, D is called *relatively uniformly closed* whenever $D = D'$. If $\widehat{A}$ is the Dedekind completion of the vector lattice A, then $\overline{A}$, the closure of A in $\widehat{A}$ with respect to the relatively uniform topology, is a *relatively uniformly completion* of A.

It is proved by Beukers and Huijsmans [4, Theorem 5] that if A is a relatively uniformly complete semiprime f- algebra and $F \in \mathbb{R}^+ [X_1, ..., X_n]$ a homogeneous polynomial of degree $p \in \mathbb{N}$, then for all $0 \leq a_1, ..., a_n \in A$, there exists (unique) $0 \leq a \in A$ such that

$$a^p = F(a_1, ..., a_n).$$

In this section we show that this result subsists in the case of a *positive square* $\mathcal{O}'$-algebra (evidently, we lose uniqueness). In order to reach this aim, we need the following proposition.

Proposition 2. *Let A be a relatively uniformly complete positive square $\mathcal{O}'$-algebra and $p > 2$ a natural number. Then*
(i) for every $0 \leq a_1, ..., a_p \in A$, there exists $0 \leq a \in A$ such that

$$a^p = a_1 ... a_p,$$

(ii) for every $0 \leq a, b \in A$, there exists $0 \leq c \in A$ such that

$$c^p = a^p + b^p.$$

Proof. (i) Let $0 \leq a_1, ..., a_p \in A$ and put $e = a_1 + ... + a_p$. It is well known that the principal order ideal A_e, generated by e in A, can be equipped by a multiplication denoted by $(\times)$ in such a manner that $(A_e, \times)$ becomes a relatively uniformly complete f-algebra with e as unit (see [12], Remark 19.5).
Consider $B = A_e \times A_{e^p}$ with the coordinatewise vector space operations and partial ordering inherited from $A \times A$ and with the following multiplication: for fixed $0 \leq b \in A_e$

$$X * Y = \begin{pmatrix} x \\ y \end{pmatrix} * \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ (b \times x) . x' . a_3 ... a_p \end{pmatrix}$$

for all $x, x' \in A_e$ and $y, y' \in A_{e^p}$. By Corollary 3, we deduce that $(B, *)$ is an almost f- algebra. Then $(B, *)$ is a commutative ℓ-algebra. So

$$(b \times x) . x' . a_3 ... a_p = (b \times x') . x . a_3 ... a_p.$$

In particular, if $b = a_1$, $x = e$ and $x' = a_2$, we have

$$a_1 a_2 . a_3 ... a_p = (a_1 \times a_2)\, e a_3 ... a_p.$$

Analogously, we conclude that

$$a_1a_2a_3...a_p = (a_1 \times a_2 \times a_3 \times .. \times a_{p)} \underbrace{e...e}_{(p\text{-}1)\text{times}}$$

Now, applying [4, Theorem 5] to A_e which is a relatively uniformly complete f-algebra with unit e, there exists a positive element a in A_e such that

$$a_1 \times a_2 \times a_3 \times .. \times a_p = a^{\times p},$$

therefore

$$a_1a_2a_3...a_p = (a^{\times p})e...e = a^p.$$

This gives the desired result.
(ii) Let $0 \leq a, b \in A$ and take $e = a + b$. Using the same argument and notation of (i), we deduce that

$$a^p + b^p = (a^{\times p})e...e + (b^{\times p})e...e = \left(a^{\times p} + b^{\times p}\right) e...e.$$

Since the property *(ii)* is valid for relatively uniformly complete f-algebra (see [4, Theorem 5]), there exists $0 \leq c \in A_e$ such that

$$\left(a^{\times p} + b^{\times p}\right) = c^{\times p},$$

and finally

$$a^p + b^p = \left(c^{\times p}\right) e...e = c^p$$

and we are done. ■

Now, the following main result of this section is a simple inference of the previous proposition.

Theorem 3. *Let A be a relatively uniformly complete positive square $\mathcal{O}'$–algebra and let F be a homogeneous polynomial of degree $p \in \mathbb{N}^*$, $p \neq 2$ in $\mathbb{R}^+ [X_1, ..., X_n]$. Then for all $0 \leq a_1, .., a_n \in A$, there exists $0 \leq a \in A$ such that*

$$a^p = F(a_1, ..., a_n).$$

Since any $\mathcal{O}$–algebra in which squares are positive is a relatively uniformly complete positive square $\mathcal{O}'$–algebra, we deduce the following.

Corollary 5. *Let A be an $\mathcal{O}$–algebra in which squares are positive and let F be a homogeneous polynomial of degree $p \in \mathbb{N}^*$, $p \neq 2$ in $\mathbb{R}^+ [X_1, ..., X_n]$. Then for all $0 \leq a_1, .., a_n \in A$, there exists $0 \leq a \in A$ such that*

$$a^p = F(a_1, ..., a_n).$$

The next example shows that the preceding theorem (and evidently the previous corollary) need not to be true in the case $p = 2$.

Example 5. *Take* $A = \mathbb{R}^4$ *with the coordinatewise vector space operations and the cone*
$A^+ = \{a = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4;\ a_1 \geq 0, |a_2| \leq a_1,\ a_3 \geq 0$ *and* $|a_4| \leq a_3\}$,
equipped with the following multiplication: $a.b = (0, 0, 2a_1b_1 + a_2b_2, a_2b_2)$ *for* $a = (a_1, a_2, a_3, a_4)$ *and* $b = (b_1, b_2, b_3, b_4)$.
A straightforward calculation shows that A *is a positive square* $\mathcal{O}'$*–algebra (also an* $\mathcal{O}$*–algebra). However Theorem 3 and Corollary 5 fail to hold if* $p = 2$*. Indeed, let* $a = (2, -2, 0, 0)$ *and* $b = (1, 1, 0, 0)$.
Assume now that there exists $0 \leq c \in A$ *such that* $a.b = c^2$ *then we obtain the following:*
$(0, 0, 2, -2) = (0, 0, 2\ c_1^2 + c_2^2, c_2^2)$, *then* $c_2^2 = -2$, *a contradiction.*

Remark 4. *The previous theorem and corollary do not hold if the* $\mathcal{O}'$*-algebra (or the* $\mathcal{O}$*-algebra)* A *is not a positive square algebra. We illustrate this with the following counterexample.*

Example 6. *Take* $A = \{f : [-1, 1] \to \mathbb{R}\}$ *with the usual operations and order, and define* $\alpha \in A$ *by*

$$\alpha(x) = \begin{cases} 0 & if \quad x \in \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\} \\ 1 & if \quad x \notin \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\} \end{cases}.$$

For all $f, g \in A$ *define*

$$f * g(x) = \begin{cases} \alpha(x) f(x)\ g(x) & if \quad -1 \leq x \leq -\frac{1}{2} \\ \alpha(x)\ f(x)\ g\left(x - \frac{1}{2}\right) & if \quad -\frac{1}{2} \leq x \leq 0 \\ \alpha(x)\ f(x)\ g(x) & if \quad 0 \leq x \leq \frac{1}{2} \\ \alpha(x)\ f\left(x - \frac{3}{2}\right)\ g(x) & if \quad \frac{1}{2} \leq x \leq 1 \end{cases}.$$

It not hard to prove that A *is a positive square* $\mathcal{O}'$*-algebra (and* $\mathcal{O}$*-algebra) under the multiplication* $(*)$.
Let f_1, f_2, f_3, *defined by*

$$f_1(x) = \begin{cases} 0 & if \quad -1 \leq x < -\frac{1}{2} \\ 1 & if \quad -\frac{1}{2} \leq x < 0 \\ 2 & if \quad 0 \leq x < \frac{1}{2} \\ 0 & if \quad \frac{1}{2} \leq x \leq 1 \end{cases},\quad f_2(x) = \begin{cases} 1 & if \quad -1 \leq x < -\frac{1}{2} \\ 1 & if \quad -\frac{1}{2} \leq x < 0 \\ 0 & if \quad 0 \leq x < \frac{1}{2} \\ 0 & if \quad \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$\text{and } f_3(x) = \begin{cases} 2 & if \quad -1 \leq x < -\frac{1}{2} \\ 0 & if \quad -\frac{1}{2} \leq x < 0 \\ 1 & if \quad 0 \leq x < \frac{1}{2} \\ 0 & if \quad \frac{1}{2} \leq x \leq 1 \end{cases}.$$

Assume now that there exists $f \in A$ *such that* $f_1 * f_2 * f_3 = f^{*3}$*. We obtain the following system:*

$$\begin{cases} f^3(x) & = & 0 & if \quad -1 < x < -\frac{1}{2} \\ f(x)\ f^2\left(x - \frac{1}{2}\right) & = & 2 & if \quad -\frac{1}{2} < x < 0 \\ f^3(x) & = & 0 & if \quad 0 < x < \frac{1}{2} \\ f^2\left(x - \frac{3}{2}\right)\ f(x) & = & 0 & if \quad \frac{1}{2} < x < 1 \end{cases}.$$

The first equation implies that $f(x) = 0 \left(-1 < x < -\frac{1}{2}\right)$ *and the second equation becomes* $f(x)\ f^2\left(x - \frac{1}{2}\right) = 0 \left(-\frac{1}{2} < x < 0\right)$, *a contradiction.*

Recall that a linear mapping T defined on the vector lattice A with values in the vector lattice B is called *positive* if $T(A^+) \subset B^+$ (notation $T \in \mathcal{L}^+(A, B)$ or $T \in \mathcal{L}^+(A)$ if $A = B$). The linear mapping $T \in \mathcal{L}^+(A, B)$ is called *lattice* (or *Riesz*) *homomorphism* (notation $T \in Hom(A, B)$ or $T \in Hom(A)$ if $A = B$) whenever $a \wedge b = 0$ implies $T(a) \wedge T(b) = 0$. The linear mapping T defined on the vector lattice A with values in the vector lattice B is called *order bounded* whenever for each order interval $[x, y]$ of A, the set $T([x, y])$ is included in an order interval in B.

In the sequel, we intend to generalize Theorem 4.1 in [14] by Render. Our proof is identical in concept with the proof of [8, Theorem 5.1].

Theorem 4. *Let A be a relatively uniformly complete positive square $\mathcal{O}'$-algebra and B be a semiprime f-algebra. If $T : A \to B$ is an algebra homomorphism then T is positive, hence T is a lattice homomorphism.*

Proof. Let $0 \leq u \in A$. Since A is a relatively uniformly complete $\mathcal{O}'$-algebra, it follows from the previous theorem that there exists $v \in A^+$ which satisfies $u^5 = v^4$. Consequently, $(T(u))^5 = (T(v))^4 \geq 0$. On the other hand,

$$(T(u))^5 = \left(T(u)^+ - T(u)^-\right)^5 = (T(u)^+)^5 - (T(u)^-)^5,$$

because $T(u)^+ T(u)^- = 0$. Moreover, since $(T(u)^+)^5 \wedge (T(u)^-)^5 = 0$, we deduce that $(T(u)^-)^5 = ((T(u))^5)^-$. Hence $(T(u)^-)^5 = 0$.
Since B is a semiprime f-algebra, it follows that $T(u)^- = 0$, i.e., $T(u) \geq 0$.
Let now $a, b \in A$ such that $a \wedge b = 0$. Then

$$0 \leq (T(a) \wedge T(b))^2 \leq T(a)\, T(b) = T(ab) = 0$$

as $ab \in N(A)$. Since B is a semiprime f-algebra, it follows that

$$T(a) \wedge T(b) = 0.$$

The proof is complete. ∎

Since $\mathbb{R}$ is a semiprime f-algebra, the following corollary is straightforwardly deduced from the previous theorem, also the result of Render [14, Theorem 4.1] follows.

Corollary 6. *Let A be an $\mathcal{O}$-algebra. If the positive cone contains all squares then every multiplicative functional $T : A \to \mathbb{R}$ is a positive lattice homomorphism.*

Remark 5. *We note that, in the result of Render [14, Theorem 4.1], the assumption that T is order bounded is superfluous as it is shown in the previous corollary.*

The assumption that A is relatively uniformly complete in Theorem 4 is not redundant (see [8, Example 5.2]). But there exists a variant of the theorem in question. In fact, one may assume that T is order bounded instead of imposing relatively uniform completeness on A. To reach this we need the following.
Triki advocated in [18] that for any ℓ-algebra A, the multiplication in A can be extended in a unique way into an ℓ-algebra multiplication on $\overline{A}$ (the closure of A in $\widehat{A}$ with respect to the relatively uniform topology) in such a manner that A becomes a subalgebra of $\overline{A}$. Moreover, $\overline{A}$ is an f-algebra (respect almost f-algebra, d-algebra) whenever A is an f-algebra (respect almost f-algebra, d-algebra). Our following result shows that it is true for every *positive square* $\mathcal{O}'$-algebra. The proof is left to the reader, because it is identical in concept with the proof of [18, Theorem 4].

Theorem 5. *Let A be a positive square $\mathcal{O}'$-algebra. Then the multiplication in A can be extended in a unique way into a positive square $\mathcal{O}'$-algebra multiplication on $\overline{A}$ in such a manner that A becomes a subalgebra of $\overline{A}$.*

Also we need the following result, which is a direct consequence of ([10], Theorem 3.3).

Lemma 1. *Let A be a vector lattice. If $\alpha = 0$, we set $A_0 = A$, and for every countable ordinal α, we set $A_\alpha = (A_\beta)'$, the pseudo closure of A_β in A, whenever $\beta + 1 = \alpha$ and we set $A_\alpha = \cup \{A_\beta; \beta < \alpha\}$ otherwise. Then $\overline{A} = \cup \{A_\alpha; \alpha < \mathcal{N}_1\}$, the relatively uniform closure of A in $\widehat{A}$.*

As a consequence we deduce a result about the extension of order bounded algebra homomorphism.

Proposition 3. *Let A and B be two ℓ-algebras and T be an order bounded algebra homomorphism from A into B. Then T has a unique extension $\overline{T}$ defined from $\overline{A}$ into $\overline{B}$ and $\overline{T}$ is an algebra homomorphism.*

Proof. It is well known that T has a unique order bounded extension $\overline{T}$ defined from $\overline{A}$ into $\overline{B}$. This is a classical result due to $L.V.Kantorovi\widetilde{c}$ (see [1], Theorem 2.8). It remains to prove that $\overline{T}$ is an algebra homomorphism.

On account of the transfinite induction principle it suffices to prove that $\overline{T}/A'$ is multiplicative. To this end, let $a, b \in A'$ arbitrary and let (a_n) and (b_n) be sequences in A with a, b as $(r.u)$ limits respectively. Then we have $\overline{T}(a_n b_n) = T(a_n)$ $T(b_n) = \overline{T}(a_n)\ \overline{T}(b_n)$. But since $\overline{T}$ is order bounded, it follows that $\overline{T}$ is $(r.u)$ continuous and hence
$\lim \overline{T}(a_n b_n) = \overline{T}(ab) = \lim \overline{T}(a_n) \lim \overline{T}(b_n) = \overline{T}(a)\ \overline{T}(b)$ and we are done. ∎

Thus now, we have gathered all ingredients for our aim.

Theorem 6. *Let A be a positive square $\mathcal{O}'$-algebra and B be a semiprime f-algebra. If $T : A \to B$ is an order bounded algebra homomorphism then T is positive, and hence T is a lattice homomorphism.*

Proof. Since T has an extension $\overline{T}$ defined from $\overline{A}$ into $\overline{B}$, which is an algebra homomorphism, we deduce that, according to Theorem 4, that $\overline{T}$ is positive. Hence $\overline{T}$ is a lattice homomorphism and so is T. This completes the proof. ∎

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