A NOTE ON CLASS $Q(N)$ OPERATORS

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Abstract. Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. In this paper we introduce two new classes of operators: class $Q(N)$ and class $Q^*(N)$. An operator $T \in \mathcal{L}(\mathcal{H})$ is of class $Q(N)$ for a fixed real number $N \geq 1$, if T satisfies $N\|Tx\|^2 \leq \|T^2x\|^2 + \|x\|^2$ for all $x \in \mathcal{H}$. And an operator $T \in \mathcal{L}(\mathcal{H})$ is of class $Q^*(N)$ for a fixed real number $N \geq 1$, if T satisfies $N\|T^*x\|^2 \leq \|T^2x\|^2 + \|x\|^2$ for all $x \in \mathcal{H}$. We prove the basic properties of these classes of operators.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $\mathcal{L}(\mathcal{H})$ denote the C^* algebra of all bounded operators on $\mathcal{H}$. The null operator and the identity on $\mathcal{H}$ will be denoted by O and I, respectively. If T is an operator, then T^* is its adjoint, and $\|T\| = \|T^*\|$. The operator T is an isometry, if $\|Tx\| = \|x\|$ for all $x \in \mathcal{H}$. The operator T is called unitary operator if $T^*T = TT^* = I$.

Aluthge in [1] defined a transformation $\tilde{T}$ of operator T by $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, where $T = U|T|$ is the polar decomposition of operator T. $\tilde{T}$ is called Aluthge transformation.

Yamazaki in [7] defined the *-Aluthge transformation of operator T. The $*$-Aluthge transformation is defined by $\tilde{T}^{(*)} \overset{def}{=} (\tilde{T}^*)^* = |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}$.

It is proved that $U^*|T^*|^{\frac{1}{2}} = |T|^{\frac{1}{2}}U^*$, $U^*|T^*| = |T|U^*$, $U|T|^{\frac{1}{2}} = |T^*|^{\frac{1}{2}}U$, $U|T| = |T^*|U$.

Duggal, Kubrusly, Levan in [3] introduced a new class of operators, the class Q. An operator $T \in \mathcal{L}(\mathcal{H})$ belongs to class Q if $T^{*2}T^2 - 2T^*T + I \geq O$. It is proved that an operator $T \in \mathcal{L}(\mathcal{H})$ is of class Q if $\|Tx\|^2 \leq \frac{1}{2}(\|T^2x\|^2 + \|x\|^2)$. They showed that this class of operators properly induces the paranormal operators and studied various properties of class Q operators.

D. Senthilkumar and T. Prasad in [6] defined the new class of operators, the M-class Q. An operator $T \in \mathcal{L}(\mathcal{H})$ is of M-class Q for a fixed real number $M \geq 1$, if it satisfies $M^2T^{*2}T^2 - 2T^*T + I \geq 0$. They proved that an operator $T \in \mathcal{L}(\mathcal{H})$ is of M-class Q, if $\|Tx\|^2 \leq \dfrac{1}{2}(M^2\|T^2x\|^2 + \|x\|^2)$, for all $x \in \mathcal{H}$ and for a fixed real number $M \geq 1$.

V. R. Hamiti in [5] defined the new class of operators, the M-class Q^*. An operator $T \in \mathcal{L}(\mathcal{H})$ is of M-class Q^* for a fixed real number $M \geq 1$, if it satisfies $M^2T^{*2}T^2 - 2TT^* + I \geq 0$. It is proved that an operator $T \in \mathcal{L}(\mathcal{H})$ is of M-class Q^*, if $\|T^*x\|^2 \leq \frac{1}{2}(M^2\|T^2x\|^2 + \|x\|^2)$, for all $x \in \mathcal{H}$ and for a fixed real number $M \geq 1$.

Analyzing the good qualities of class Q, class Q^*, M-class Q and M-class Q^* of operators, we introduced general classes of operators which include these classes of operators and some of their properties. Based on this, we introduced two new classes of operators, class $Q(N)$ and class $Q^*(N)$.

Definition 1.1. An operator $T \in \mathcal{L}(\mathcal{H})$ is of class $Q(N)$ for a fixed real number $N \geq 1$, if T satisfies

$$N\|Tx\|^2 \leq \|T^2x\|^2 + \|x\|^2$$

for all $x \in \mathcal{H}$.

Definition 1.2. An operator $T \in \mathcal{L}(\mathcal{H})$ is of class $Q^*(N)$ for a fixed real number $N \geq 1$, if T satisfies

$$N\|T^*x\|^2 \leq \|T^2x\|^2 + \|x\|^2$$

for all $x \in \mathcal{H}$.

The definition of these classes of operators was not a big problem. But the challenge was to prove which properties were valid for these classes of operators, how classes were related, and did any operators in these classes exist, or were they empty classes of operators.

While we studied these classes of operators, we observed that the classes $Q(N)$ and $Q^*(N)$ of operators were interesting for the following reasons.

(1) They could be compared to several classes of operators. From the definitions we see that
  class $Q =$ class $Q(2)$; class $Q^* =$ class $Q^*(2)$;
  M-class $Q =$ class $Q(N)$ for $M = 1$ and $N = 2$,
  i.e., 1-class $Q =$ class $Q(2)$;
  M-class $Q^* =$ class $Q^*(N)$ for $M = 1$ and $N = 2$,
  i.e. 1-class $Q^* =$ class $Q^*(2)$.
(2) They are invariant to unitary equivalence.
(3) The inverse of their elements is also in those classes whenever it exists.
(4) The following inclusions are valid.
  class $Q(1) \supseteq$ class $Q(2) \supseteq$ class $Q(3) \supseteq \cdots \supseteq$ class $Q(N)$;
  class $Q^*(1) \supseteq$ class $Q^*(2) \supseteq$ class $Q^*(3) \supseteq \cdots \supseteq$ class $Q^*(N)$.
(5) And there are examples of operators from these classes of operators.

These were in fact the motivation for studying these two new classes of operators.

During the proof of the properties of these operators, we used almost the same techniques as those used previously for other classes of operators. What was more challenging was that the properties had to be proved for every fixed number $N \geq 1$. For example, for classes Q and Q^* of operators, all of their properties were proved for $N = 2$, where for classes $Q(N)$ and $Q^*(N)$, their properties should be proved for every fixed number $N \geq 1$. This changed the working technique of this paper.

During the study, we noticed that we proved some very important properties for these two new classes of operators. These results are stated in the following section.

2. Main Results

First, we state a proposition which gives necessary and sufficient conditions for an operator T to be of class $Q(N)$.

Proposition 2.1. *An operator $T \in \mathcal{L}(\mathcal{H})$ is of class $Q(N)$ if and only if*

$$T^{*2}T^2 - NT^*T + I \geq 0$$

for a fixed real number $N \geq 1$.

Proof. Since T is of class $Q(N)$ for a fixed real number $N \geq 1$, then

$$N\|Tx\|^2 \leq \|T^2x\|^2 + \|x\|^2$$

for all $x \in \mathcal{H}$. Then,

$$\begin{aligned}
&(T^2x|T^2x) - N(Tx|Tx) + (x|x) \geq 0 \\
\Leftrightarrow &(T^{*2}T^2x|x) - N(T^*Tx|x) + (x|x) \geq 0 \\
\Leftrightarrow &((T^{*2}T^2 - NT^*T + I)x|x) \geq 0 \\
\Leftrightarrow &T^{*2}T^2 - NT^*T + I \geq 0.
\end{aligned}$$

$\square$

Similarly, we state a proposition giving necessary and sufficient conditions for an operator T to be of class $Q^*(N)$.

Proposition 2.2. *An operator $T \in \mathcal{L}(\mathcal{H})$ is of class $Q^*(N)$ if and only if*

$$T^{*2}T^2 - NTT^* + I \geq 0$$

for a fixed real number $N \geq 1$.

Proof. Since T is of class $Q^*(N)$, for a fixed real number $N \geq 1$, then

$$N\|T^*x\|^2 \leq \|T^2x\|^2 + \|x\|^2$$

for all $x \in \mathcal{H}$. Then,

$$\begin{aligned}
&(T^2x|T^2x) - N(T^*x|T^*x) + (x|x) \geq 0\\
\Leftrightarrow&(T^{*2}T^2x|x) - N(TT^*x|x) + (x|x) \geq 0\\
\Leftrightarrow&((T^{*2}T^2 - NTT^* + I)x|x) \geq 0\\
\Leftrightarrow&T^{*2}T^2 - NTT^* + I \geq 0.
\end{aligned}$$

□

In the following we give the inclusion of these classes of operators.

Proposition 2.3. *Class $Q^*(N) \subseteq$ class $Q^*(N-1)$ for $N \geq 2$.*

Proof. Since T is of class $Q^*(N)$ for a fixed real number $N \geq 2$, then

$$N\|T^*x\|^2 \leq \|T^2x\|^2 + \|x\|^2$$

for all $x \in \mathcal{H}$. Then,

$$(N-1)\|T^*x\|^2 \leq N\|T^*x\|^2 \leq \|T^2x\|^2 + \|x\|^2.$$

Hence, T is an operator of class $Q^*(N-1)$.

It follows that class $Q^*(1) \supseteq$ class $Q^*(2) \supseteq$ class $Q^*(3) \supseteq \cdots \supseteq$ class $Q^*(N)$. □

A similar inclusion can also be proved.

Class $Q(1) \supseteq$ class $Q(2) \supseteq$ class $Q(3) \supseteq \cdots \supseteq$ class $Q(N)$.

Now we will prove some basic properties of class $Q(N)$ and class $Q^*(N)$ operators.

Proposition 2.4. *Let T be an operator of class $Q(N)$.*

a) *If T double commutes with an isometric operator S, then TS is an operator of the class $Q(N)$.*
b) *If S is unitarily equivalent to operator T, then S is an operator of the class $Q(N)$.*
c) *If T is an invertible operator, then T^{-1} is operator of class $Q(N)$.*

Proof. a) Let $B = TS$, $TS = ST$, $S^*T = TS^*$, and $S^*S = I$.

$$\begin{aligned}
&B^{*2}B^2 - NB^*B + I\\
=&(TS)^{*2}(TS)^2 - N(TS)^*(TS) + I\\
=&T^{*2}T^2 - NT^*T + I \geq 0,
\end{aligned}$$

so TS is an operator of class $Q(N)$.

b) Since operator S is unitarily equivalent to operator T, there exists an unitary operator U such that $S = U^*TU$. Since T is an operator of class $Q(N)$, then

$$T^{*2}T^2 - NT^*T + I \geq 0.$$

Hence,

$$
\begin{aligned}
&S^{*2}S^2 - NS^*S + I \\
&\qquad = (U^*TU)^{*2}(U^*TU)^2 - N(U^*TU)^*(U^*TU) + I \\
&\qquad = U^*(T^{*2}T^2 - NT^*T + I)U \geq 0
\end{aligned}
$$

so S is an operator of class $Q(N)$.

c) If T is invertible and T is an operator of class $Q(N)$, then

$$N\|Tx\|^2 \leq \|T^2x\|^2 + \|x\|^2$$

and

$$N\|x\|^2 = N\|TT^{-1}x\|^2 \leq \|T^2(T^{-1}x)\|^2 + \|T^{-1}x\|^2$$

for $x \in \mathcal{H}$ and $N \geq 1$. For any y in $\mathcal{H} = ran(T)$ so that $y = Tx$, $x = T^{-1}y$, and $T^{-1}x = T^{-1}(T^{-1}y) = T^{-2}y$, we have

$$N\|T^{-1}y\|^2 \leq \|y\| + \|T^{-2}y\|^2$$

by the above inequality, and so T^{-1} is an operator of class $Q(N)$. □

Proposition 2.5. *Let T be an operator of class $Q^*(N)$. If S is unitarily equivalent to operator T, then S is an operator of the class $Q^*(N)$.*

Proposition 2.6. *Let $T \in L(H)$.*

a) *If $\|T\| \leq \dfrac{1}{\sqrt{N}}$, then T is an operator of class $Q(N)$.*

b) *If $T^2 = 0$, then T is an operator of class $Q(N)$ if and only if $\|T\| \leq \dfrac{1}{\sqrt{N}}$.*

Proof. a) From $\|T\| \leq \dfrac{1}{\sqrt{N}}$, we have $\|T\|^2 \leq \dfrac{1}{N}$. Then,

$$
\begin{aligned}
&\|Tx\|^2 \leq \frac{1}{N}\|x\|^2, \text{ for all } x \in H \\
&\langle Tx, Tx\rangle - \frac{1}{N}\langle x, x\rangle \leq 0, \text{ for all } x \in H \\
&\langle (I - NT^*T)x, x\rangle \geq 0, \text{ for all } x \in H \\
&I - NT^*T \geq 0 \\
&T^{*2}T^2 - NT^*T + I \geq 0,
\end{aligned}
$$

so T is an operator of class $Q(N)$.

b) Let T be an operator of class $Q(N)$. Then,

$$T^{*2}T^2 - NT^*T + I \geq 0.$$

Since $T^2 = 0$, we have

$$\begin{aligned}
& I - NT^*T \geq 0 \\
\Leftrightarrow & NT^*T - I \leq 0 \\
\Leftrightarrow & \langle (NT^*T - I)x, x\rangle \leq 0, \text{ for all } x \in H \\
\Leftrightarrow & \langle NT^*Tx, x\rangle - I\langle x, x\rangle \leq 0, \text{ for all } x \in H \\
\Leftrightarrow & N\langle Tx, Tx\rangle - I\langle x, x\rangle \leq 0, \text{ for all } x \in H \\
\Leftrightarrow & N\|Tx\|^2 - I\|x, x\|^2 \leq 0, \text{ for all } x \in H \\
\Leftrightarrow & \|T\| \leq \frac{1}{\sqrt{N}}.
\end{aligned}$$

□

Proposition 2.7. *Let $T \in L(H)$.*

a) *If $\|T^*\| \leq \frac{1}{\sqrt{N}}$, then T is an operator of class $Q^*(N)$.*
b) *If $T^2 = 0$, then T is an operator of class $Q^*(N)$ if and only if $\|T^*\| \leq \frac{1}{\sqrt{N}}$.*

Proposition 2.8. *Let M be a closed T invariant subset of $\mathcal{H}$. Then, the restriction $T_{|M}$ of a class $Q(N)$ operator T to M is of class $Q(N)$.*

Proof. Let $u \in M$. Then,

$$\begin{aligned}
\|T\,|_M\, u\|^2 = \|Tu\|^2 &\leq \frac{1}{N}\left(\|T^2u\|^2 + \|u\|^2\right) \\
&= \frac{1}{N}\left(\|(T\,|_M)^2u\|^2 + \|u\|^2\right).
\end{aligned}$$

This implies that $T\,|_M$ is an operator of class $Q(N)$. □

Proposition 2.9. *Let M be a closed T invariant subset of $\mathcal{H}$. Then, the restriction $T_{|M}$ of a class $Q^*(N)$ operator T to M is of class $Q^*(N)$.*

Theorem 2.10. *Let $T \in \mathcal{L}(\mathcal{H})$ be an invertible operator and S be an operator such that S commutes with operator T^*T. Then, S is of class $Q(N)$ if and only if TST^{-1} is of class $Q(N)$.*

Proof. Let S be an operator of class $Q(N)$. Then,

$$S^{*2}S^2 - NS^*S + I \geq 0.$$

From this we have

$$T[S^{*2}S^2 - NS^*S + I]T^* \geq 0.$$

Now, we prove that operator TT^* commutes with operator $T[S^{*2}S^2 - NS^*S + I]T^*$. Since operator S commutes with operator T^*T, operator S^* also commutes with operator T^*T. From this we have

$$\begin{aligned}
&T[S^{*2}S^2 - NS^*S + I]T^*[TT^*] \\
=&T[S^{*2}S^2 - NS^*S + I][T^*T]T^* \\
=&T[T^*T][S^{*2}S^2 - NS^*S + I]T^* \\
=&[TT^*]T[S^{*2}S^2 - NS^*S + I]T^*.
\end{aligned}$$

Thus, operator TT^* commutes with operator $T[S^{*2}S^2 - NS^*S + I]T^*$. Then, operator $[TT^*]^{-1}$ also commutes with operator $T[S^{*2}S^2 - NS^*S + I]T^*$. Since the operators $[TT^*]^{-1}$ and $T[S^{*2}S^2 - NS^*S + I]T^*$ are positive, then

$$T[S^{*2}S^2 - NS^*S + I]T^*[TT^*]^{-1} \geq 0.$$

Since operator S commutes with operator T^*T, we get

$$(TST^{-1})^*(TST^{-1}) = T^{*-1}S^*T^*TST^{-1} = TS^*ST^{-1}, \tag{2.1}$$

$$\begin{aligned}
(TST^{-1})^{*2}(TST^{-1})^2 &= (TST^{-1})^*(TST^{-1})^*(TST^{-1})(TST^{-1}) \\
&= TS^{*2}S^2T^{-1}. \qquad (2.2)
\end{aligned}$$

To prove that TNT^{-1} is an operator of class $Q(N)$, we substitute equations 2.1 and 2.2 in the above expression and obtain

$$(TST^{-1})^{*2}(TST^{-1})^2 - N(TST^{-1})^*(TST^{-1}) + I$$

and we have

$$\begin{aligned}
&(TST^{-1})^{*2}(TST^{-1})^2 - N(TST^{-1})^*(TST^{-1}) + I \\
=&TS^{*2}S^2T^{-1} - NTS^*ST^{-1} + I \\
=&T^{*-1}T^*[T[S^{*2}S^2 - NS^*S + I]T^{-1}]TT^{-1} \\
=&T^{*-1}T^*T[S^{*2}S^2 - NS^*S + I]T^{-1}TT^{-1} \\
=&T[S^{*2}S^2 - NS^*S + I]T^{-1}.
\end{aligned}$$

Now we prove that the last expression is positive. Since

$$T[S^{*2}S^2 - NS^*S + I]T^*[TT^*]^{-1} \geq 0,$$

we have

$$\begin{aligned}
&T[S^{*2}S^2 - NS^*S + I]T^*[TT^*]^{-1} \geq 0, \\
&T[S^{*2}S^2 - NS^*S + I]T^*T^{*-1}T^{-1} \geq 0, \\
&T[S^{*2}S^2 - NS^*S + I]T^{-1} \geq 0.
\end{aligned}$$

Hence, TST^{-1} is an operator of class $Q(N)$.

Conversely, let TST^{-1} be an operator of class $Q(N)$. Then

$$(TST^{-1})^{*2}(TST^{-1})^2 - N(TST^{-1})^*(TST^{-1}) + I \geq 0.$$

Similarly, after substituting equations 2.1 and 2.2, we have

$$T[S^{*2}S^2 - NS^*S + I]T^{-1} \geq 0,$$
$$T^*T[S^{*2}S^2 - NS^*S + I]T^{-1}T \geq 0,$$
$$[T^*T][S^{*2}S^2 - NS^*S + I] \geq 0.$$

Operator $[T^*T]$ commutes with operator S and hence, with operator $[T^*T][S^{*2}S^2 - NS^*S + I]$. Therefore, operator $[T^*T]^{-1}$ commutes with operator $[T^*T][S^{*2}S^2 - NS^*S + I]$. Since operators $[T^*T]^{-1}$ and $[T^*T][S^{*2}S^2 - NS^*S + I]$ are positive we have

$$[T^*T]^{-1}[T^*T][S^{*2}S^2 - NS^*S + I] \geq 0.$$

Therefore,

$$S^{*2}S^2 - NS^*S + I \geq 0.$$

Hence, S is an operator of class $Q(N)$. □

Theorem 2.11. *Let $T \in \mathcal{L}(\mathcal{H})$ be an invertible operator and S be an operator such that S commutes with operator T^*T. Then, S is of class $Q^*(N)$ if and only if TST^{-1} is of class $Q^*(N)$.*

The next two propositions give necessary and sufficient conditions for a weighted shift operator T with decreasing weighted sequence (α_n) to be an operator of these classes of operators.

Proposition 2.12. *A weighted shift operator T with decreasing weighted sequence (α_n) is an operator of class $Q(N)$ if and only if*

$$|\alpha_n|^2|\alpha_{n+1}|^2 - N|\alpha_n|^2 + 1 \geq 0$$

for every n.

Proof. Since T is a weighted shift, its adjoint T^* is also a weighted shift and defined by $T(e_n) = |\alpha_n|e_{n+1}$. Thus, we have

$$T^*(e_n) = |\alpha_{n-1}|e_{n-1},$$
$$(T^*T)(e_n) = |\alpha_n|^2 e_n,$$
$$(T^{*2}T^2)(e_n) = |\alpha_n|^2|\alpha_{n+1}|^2 e_n.$$

Now, since T is an operator of class $Q(N)$,

$$T^{*2}T^2 - NT^*T + I \geq 0$$
$$\Leftrightarrow |\alpha_n|^2|\alpha_{n+1}|^2 - N|\alpha_n|^2 + 1 \geq 0.$$

□

Example 1. A weighted shift operator T with decreasing weighted sequence $\alpha_n = \dfrac{1}{\sqrt{N}}, N > 0$ is an operator of class $Q(N)$ for every fixed real number $N \geq 1$ (it is clear from Proposition 2.12).

Proposition 2.13. *A weighted shift operator T with decreasing weighted sequence (α_n) is an operator of class $Q^*(N)$ if and only if*

$$|\alpha_n|^2|\alpha_{n+1}|^2 - N|\alpha_{n-1}|^2 + 1 \geq 0$$

for every n.

Proof. Since T is an operator of class $Q^*(N)$, it follows that

$$T^{*2}T^2 - NTT^* + I \geq 0$$
$$\Leftrightarrow |\alpha_n|^2|\alpha_{n+1}|^2 - N|\alpha_{n-1}|^2 + 1 \geq 0.$$

□

In the following we give one example where it is shown that there exists an operator from the class $Q^*(3)$, which is not a $Q(3)$ operator. This shows that the classes of operators $Q(N)$ and $Q^*(N)$ are independent.

Example 2. Let T be a weighted shift operator with a decreasing weighted sequence as follows.

$$\alpha_n = \begin{cases} 0, \ n \leq 0 \\ 1, \ n = 1 \\ \dfrac{1}{2}, \ n = 2 \\ 4, \ n \geq 3 \end{cases}.$$

After some calculations from Proposition 2.13, it follows that T is an operator of class $Q^*(3)$ for every n.

But from Proposition 2.12, it follows that T is not an operator of class $Q(3)$. For example for $n = 1$ we have

$$\alpha_1^2 \cdot \alpha_2^2 - 3 \cdot \alpha_1^2 + 1 = 1 \cdot \frac{1}{4} - 3 \cdot 1 + 1 < 0.$$

In the following, we will give the equivalence between Aluthge transformation and $*$-Aluthge transformation of class $Q(N)$ and class $Q^*(N)$ of operators.

Theorem 2.14. *Let $T \in L(H)$. Then, $\tilde{T}$ is an operator of class $Q(N)$ if and only if $\tilde{T}^{(*)}$ is an operator of class $Q(N)$.*

Proof. Assume that $\tilde{T}$ is an operator of class $Q(N)$. Then

$$\tilde{T}^{*2}\tilde{T}^2 - N\tilde{T}^*\tilde{T} + I \geq 0.$$

We need to prove that $\tilde{T}^{(*)}$ is an operator of class $Q(N)$.

$$\begin{aligned}
&\tilde{T}^{(*)*2}\tilde{T}^{(*)2} - N\tilde{T}^{(*)*}\tilde{T}^{(*)} + I \\
=&(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^{*2}(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^2 - N(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}) + I \\
=&(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}) \\
&- N(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}) + I \\
=&(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}) \\
&- N(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}) + I \\
=&UU^*(|T^*|^{\frac{1}{2}}U^*|T^*|U^*|T^*U|T^*|U|T^*|^{\frac{1}{2}} - N|T^*|^{\frac{1}{2}}U^*|T^*|U|T^*|^{\frac{1}{2}} + I)UU^* \\
=&U(U^*|T^*|^{\frac{1}{2}}U^*|T^*|U^*|T^*U|T^*|U|T^*|^{\frac{1}{2}}U \\
&- NU^*|T^*|^{\frac{1}{2}}U^*|T^*|U|T^*|^{\frac{1}{2}}U + I)U^* \\
=&U(|T|^{\frac{1}{2}}U^*|T|U^*|T|U|T|U|T|^{\frac{1}{2}} - N|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}} + I)U^* \\
=&U(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) \\
&- N(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) + I)U^* \\
=&U(\tilde{T}^{*2}\tilde{T}^2 - N\tilde{T}^*\tilde{T} + I)U^* \geq 0.
\end{aligned}$$

Therefore,

$$\tilde{T}^{(*)*2}\tilde{T}^{(*)2} - N\tilde{T}^{(*)*}\tilde{T}^{(*)} + I \geq 0.$$

Hence, $\tilde{T}^{(*)}$ is an operator of class $Q(N)$.

Conversely, assume that $\tilde{T}^{(*)}$ is an operator of class $Q(N)$. Then

$$\tilde{T}^{(*)*2}\tilde{T}^{(*)2} - N\tilde{T}^{(*)*}\tilde{T}^{(*)} + I \geq 0.$$

We need to prove that $\tilde{T}$ is an operator of class $Q(N)$.

Consider

$$\begin{aligned}
&\tilde{T}^{*2}\tilde{T}^{2} - N\tilde{T}^{*}\tilde{T} + I \\
=&(|T|^{\frac{1}{2}}U^{*}|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^{*}|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) - N(|T|^{\frac{1}{2}}U^{*}|T|^{\frac{1}{2}}) \\
&(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) + I \\
=&U^{*}U[(|T|^{\frac{1}{2}}U^{*}|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^{*}|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) - N(|T|^{\frac{1}{2}}U^{*}|T|^{\frac{1}{2}}) \\
&(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) + I]U^{*}U \\
=&U^{*}[U|T|^{\frac{1}{2}}U^{*}|T|U^{*}|T|U|T|U|T|^{\frac{1}{2}}U^{*} - NU|T|^{\frac{1}{2}}U^{*}|T|U|T|^{\frac{1}{2}}U^{*} + I]U \\
=&U^{*}[|T^{*}|^{\frac{1}{2}}U^{*}|T^{*}|U^{*}|T^{*}|U|T^{*}|U|T^{*}|^{\frac{1}{2}} - N|T^{*}|^{\frac{1}{2}}U^{*}|T^{*}|U|T^{*}|^{\frac{1}{2}} + I]U \\
=&U^{*}[(|T^{*}|^{\frac{1}{2}}U^{*}|T^{*}|^{\frac{1}{2}})(|T^{*}|^{\frac{1}{2}}U^{*}|T^{*}|^{\frac{1}{2}})(|T^{*}|^{\frac{1}{2}}U|T^{*}|^{\frac{1}{2}})(|T^{*}|^{\frac{1}{2}}U|T^{*}|^{\frac{1}{2}}) \\
&- N(|T^{*}|^{\frac{1}{2}}U^{*}|T^{*}|^{\frac{1}{2}})(|T^{*}|^{\frac{1}{2}}U|T^{*}|^{\frac{1}{2}}) + I]U \\
=&U^{*}[(|T^{*}|^{\frac{1}{2}}U|T^{*}|^{\frac{1}{2}})^{*2}(|T^{*}|^{\frac{1}{2}}U|T^{*}|^{\frac{1}{2}})^{2} \\
&- N(|T^{*}|^{\frac{1}{2}}U|T^{*}|^{\frac{1}{2}})^{*}(|T^{*}|^{\frac{1}{2}}U|T^{*}|^{\frac{1}{2}}) + I]U \\
=&U^{*}[\tilde{T}^{(*)*2}\tilde{T}^{(*)2} - N\tilde{T}^{(*)*}\tilde{T}^{(*)} + I]U \geq 0.
\end{aligned}$$

Therefore,

$$\tilde{T}^{*2}\tilde{T}^{2} - N\tilde{T}^{*}\tilde{T} + I \geq 0.$$

Hence, $\tilde{T}$ is an operator of class $Q(N)$. □

Theorem 2.15. *Let $T \in L(H)$. Then $\tilde{T}$ is an operator of class $Q^{*}(N)$ if and only if $\tilde{T}^{(*)}$ is an operator of class $Q^{*}(N)$.*

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