SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

161. [2006, 147] *Proposed by José Luis Díaz-Barrero, Universidad Politècnica de Cataluña, Barcelona, Spain.*

Show that

$$\int_0^1 \sqrt[3]{1+\ln(1+x)}\,dx \int_0^1 \sqrt[3]{(1+\ln(1+x))^2}\,dx < 2\ln 2.$$

Solution by Joe Howard, Portales, New Mexico. We use an inequality due to Chebyshev found on page 135 (Problem 75) of G. Klambauer, *Problems and Propositions in Analysis*, (1979), Marcel Dekker. With $p(x)=1$ and f and g monotonically increasing on $[0,1]$, we have

$$\int_0^1 f(x)dx \cdot \int_0^1 g(x)dx \le \int_0^1 f(x)\cdot g(x)dx.$$

Assuming

$$f(x) = \big(1+\ln(1+x)\big)^{\frac{1}{3}} \qquad (f'(x)>0 \text{ on } [0,1])$$

and

$$g(x) = \big(1+\ln(1+x)\big)^{\frac{2}{3}} \qquad (g'(x)>0 \text{ on } [0,1])$$

the conditions of the inequality are met. Now

$$\int_0^1 \big(1+\ln(1+x)\big)dx = 1 + \int_1^2 \ln t dt = 1 + \big[t\ln t - t\big]_1^2 = 2\ln 2.$$

Also, from the proof of Problem 75 (above), f or g must be constant to have equality. Hence, the inequality is strict.

Also solved by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri (jointly); Kenneth B. Davenport, *Dallas, Pennsylvania; Paolo Perfetti, Universitá degli studi "Tor Vergata" Roma, Italy; Huizeng Qin, Shadong University of Technology, Zibo, Shadong, People's Republic of China and Youmin Lu, Bloomsburg University, Bloomsburg, Pennsylvania (jointly) ; and the proposer.*

162. [2006; 147] *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, Massachusetts.*

Find all positive integers a and n (with $n > 1$ and $a < n$) such that

$$\sin \frac{\pi}{n} + \sin \frac{a\pi}{n} = \sin \frac{(a+2)\pi}{n}.$$

Solution by Huizeng Qin, Shadong University of Technology, Zibo, Shadong, People's Republic of China and Youmin Lu, Bloomsburg University, Bloomsburg, Pennsylvania (jointly). Moving the second term from the left side to the right side, one obtains

$$\sin \frac{\pi}{n} = \sin \frac{(a+2)\pi}{n} - \sin \frac{a\pi}{n} = 2\cos \frac{(a+1)\pi}{n} \sin \frac{\pi}{n}.$$

Thus,

$$\cos \frac{(a+1)\pi}{n} = \frac{1}{2}.$$

Since $0 \leq \frac{a+1}{n} \leq 1$, we have

$$\frac{(a+1)\pi}{n} = \frac{\pi}{3}.$$

Solving the equation, we get

$$a = \frac{n}{3} - 1.$$

Therefore,

$$a = \frac{n}{3} - 1, \quad \text{if} \quad n = 3k \quad \text{and} \quad k = 1, 2, 3, \ldots .$$

Also solved by Joe Howard, Portales, New Mexico; Joe Flowers, St. Mary's University, San Antonio, Texas; and the proposer. A partial solution by Kenneth B. Davenport, Dallas, Pennsylvania was also received.

163. [2006; 147] *Proposed by José Luis Díaz-Barrero, Universidad Politècnica de Cataluña, Barcelona, Spain.*

Let u, v, z, w be complex numbers. Prove that

$$2\mathrm{Re}\big(uz+vw\big) \leq 3\big(|u|^2+|v|^2\big)+\frac{1}{3}\big(|z|^2+|w|^2\big).$$

Solution by Joe Flowers, St. Mary's University, San Antonio, Texas. From the inequality

$$\big(3|u|-|z|\big)^2+\big(3|v|-|w|\big)^2 \geq 0,$$

we obtain

$$9|u|^2-6|u|\cdot|z|+|z|^2+9|v|^2-6|v|\cdot|w|+|w|^2 \geq 0,$$

or equivalently,

$$6\big(|u|\cdot|z|+|v|\cdot|w|\big) \leq 9\big(|u|^2+|v|^2\big)+|z|^2+|w|^2.$$

Therefore,

$$\begin{aligned} 6\mathrm{Re}\big(uz+vw\big) &\leq 6|uz+vw| \\ &\leq 6\big(|u|\cdot|z|+|v|\cdot|w|\big) \\ &\leq 9\big(|u|^2+|v|^2\big)+|z|^2+|w|^2, \end{aligned}$$

and dividing through by 3 yields the desired inequality.

Also solved by Huizeng Qin, Shadong University of Technology, Zibo, Shadong, People's Republic of China and Youmin Lu, Bloomsburg University, Bloomsburg, Pennsylvania (jointly) and the proposer.

164. [2006; 148] *Proposed by Ovidiu Furdui (student), Western Michigan University, Kalamazoo, Michigan.*

Evaluate

$$\int_0^1 \left\{\frac{1}{x}\right\} \ln x \, dx,$$

where $\{x\}$ is the fractional part of x.

Solution by the proposer. The integral equals $\gamma + \gamma_1 - 1$ where

$$\gamma = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \ln n \approx 0.5772156$$

is the Euler-Mascheroni constant and

$$\gamma_1 = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{\ln k}{k} - \frac{\ln^2 n}{2} \right] \approx -0.07278$$

is the Stieltjes constant.

If we make the substitution $\frac{1}{x} = t$, we obtain

$$I = \int_0^1 \left\{\frac{1}{x}\right\} \ln x dx = -\int_1^{\infty} \ln t \frac{\{t\}}{t^2} dt = -\sum_{k=1}^{\infty} \int_k^{k+1} \frac{\ln t}{t^2}(t-k)dt,$$

and two integrations by parts yield

$$\int_k^{k+1} \frac{\ln t}{t} dt = \frac{\ln^2(k+1)}{2} - \frac{\ln^2(k)}{2}$$

and

$$\int_k^{k+1} \frac{\ln t}{t^2} dt = \frac{\ln k}{k} - \frac{\ln(k+1)}{k+1} + \frac{1}{k(k+1)}.$$

Hence, we have

$$I = -\sum_{k=1}^{\infty}\left(\frac{\ln^2(k+1)-\ln^2(k)}{2} + \ln(k+1) - \ln(k) - \frac{\ln(k+1)}{k+1} - \frac{1}{k+1}\right).$$

Let

$$S_n = \sum_{k=1}^{n}\left(\frac{\ln^2(k+1)-\ln^2(k)}{2} + \ln(k+1) - \ln(k) - \frac{\ln(k+1)}{k+1} - \frac{1}{k+1}\right)$$

be the n^{th} partial sum of the preceding series. This series is a telescoping series, so we obtain

$$S_n = \left(\ln(n+1) - \sum_{k=2}^{n}\frac{1}{k}\right) + \left(\frac{\ln^2(n+1)}{2} - \sum_{k=2}^{n+1}\frac{\ln k}{k}\right),$$

and, in the limit,

$$\lim_{n\to\infty} S_n = -(\gamma - 1) - \gamma_1 = -\gamma_1 - \gamma + 1.$$

Thus,

$$I = -\lim_{n\to\infty} S_n = \gamma_1 + \gamma - 1 \approx -0.4955.$$

Remark 1. To prove that γ_1 i.e., the Stieltjes constant exists and is finite, the following inequality will be used.

$$\frac{\ln k}{k} \geq \frac{\ln^2(k+1)}{2} - \frac{\ln^2(k)}{2} \geq \frac{\ln(k+1)}{k+1}, \tag{0.1}$$

for all $k \geq 3$. For the proof of (0.1) one can apply the Mean Value Theorem to the function $f(x) = \ln^2 x/2$ on the interval $(k, k+1)$, and then use the fact that the function $x \to \ln x/x$ decreases on $[3, \infty)$.

Let

$$x_n = \sum_{k=1}^{n}\frac{\ln k}{k} - \frac{\ln^2 n}{2}.$$

Then in view of (0.1) we obtain that

$$x_{n+1} - x_n = \frac{\ln(n+1)}{n+1} - \frac{\ln^2(n+1)}{2} + \frac{\ln^2 n}{2} \leq 0.$$

Thus, the sequence decreases. The boundedness of the sequence $(x_n)_{n\in\mathbb{N}}$ can be obtained by iterating (0.1). These show that the sequence (x_n) converges.

Remark 2. The constant γ_1 is known in the literature as the Stieltjes constant. For more information about this constant, see page 118 of [1]. The sequence x_n is very slowly convergent. Symbolic calculations using Maple show that $x_{220,000} = -0.07278819$ which gives only 4 significant figures of the Stieltjes constant.

Reference

1. Julian Havil, *Gamma, Exploring Euler's Constant,* Princeton University Press, Princeton, NJ, 2003.

Also solved by Huizeng Qin, Shandong University of Technology, Zibo, Shadong, People's Republic of China and Youmin Lu, Bloomsburg University, Bloomsburg, Pennsylvania (jointly) and Paolo Perfetti, Universitá degli studi "Tor Vergata" Roma, Italy.