# A NEW CLOSURE OPERATOR IN BITOPOLOGICAL SPACES AND ASSOCIATED SEPARATION AXIOMS

R. Raja Rajeswari and M. Lellis Thivagar

**Abstract.** The aim of this paper is to introduce a new closure operator and an associated new topology in bitopological spaces. We also define some new separation axioms and a comparative study is done.

**1. Introduction and Preliminaries.** In 1986, Maki introduced new sets called $\Lambda$-sets and $\vee$-sets of a given set $B$. He obtained $B^{\Lambda}$ and $B^{\vee}$ as the intersection of all open sets containing $B$ and the union of all closed sets contained in $B$, respectively. By using the concept of $\Lambda$-sets and $\vee$-sets, Maki, Umehara, and Yamamura [4, 5, 6] defined and investigated different new classes of sets. In this paper, we define a new class of closure operator, which is also a Kuratowski closure operator with respect to the generalized $\Lambda_u$-sets, an analogue of Maki's work [4]. Thus, a new topology $\tau^{\vee_u}$ is formed. We also characterize the class of ultra-$T_{1/2}$ spaces using the newly defined spaces $T_U^L$ and $T_{\vee_u}^R$.

Let us now recall some definitions which are useful to read this paper. Throughout this paper, $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ (or simply $X$ and $Y$) denote the bitopological spaces on which no separation axioms are assumed unless explicitly stated.

Definition 1.1. [3] A subset $A$ of $X$ is called
(i) $\tau_1\tau_2$-open if $A \in \tau_1 \cup \tau_2$;
(ii) $\tau_1\tau_2$-closed if $A^c \in \tau_1 \cup \tau_2$.

Definition 1.2. [3] Let $A$ be a subset of $X$. Then the $\tau_1\tau_2$-closure of $A$ is denoted as $\tau_1\tau_2$-$Cl(A)$ and defined as $\tau_1\tau_2$-$Cl(A) = \cap\{F \mid A \subset F$ and $F$ is $\tau_1\tau_2$-closed$\}$.

Definition 1.3. [3] A subset $A$ of $X$ is called $(1,2)\alpha$-open if A $\subset \tau_1$-$Int(\tau_1\tau_2$-$Cl(\tau_1$-$Int(A)))$. The complement of a $(1,2)\alpha$-open set is known as a $(1,2)\alpha$-closed set. The family of all $(1,2)\alpha$-open and $(1,2)\alpha$-closed sets are denoted as $(1,2)\alpha O(X)$ and $(1,2)\alpha C(X)$ (or $(1,2)\alpha Cl(X)$), respectively.

Definition 1.4. [3] A subset $A$ of $X$ is called a $(1,2)\alpha$g-closed set if $(1,2)\alpha Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \alpha O(X)$, where $(1,2)\alpha Cl(A) = \cap\{F \mid F \subset A$ and $F \in (1,2)\alpha C(X)\}$.

Definition 1.5. A space $X$ is said to be
(i) [1] ultra-$R_0$ space if and only if for every $x \in G$, where $G$ is a $(1,2)\alpha$-open set, $(1,2)\alpha Cl(\{x\}) \subset G$;
(ii) [3] ultra-$T_0$ space if and only if for any two distinct points $x$ and $y$ in $X$, there exists a $(1,2)\alpha$-open set $G$, containing $x$ but not $y$.

Definition 1.6. [3] A bitopological space $X$ is called ultra-$T_{1/2}$ if every $(1,2)\alpha$g-closed set is $(1,2)\alpha$-closed.

Remark 1.7. [3] In an ultra-$T_{1/2}$ space, every singleton set is either $(1,2)\alpha$-open or $(1,2)\alpha$-closed.

Definition 1.8. [9] In a bitopological space $X$, a subset $B$ of $X$ is an ultra-$\Lambda$-set (briefly, $\Lambda_u$-set) if $B = B^{\Lambda_u}$, where $B^{\Lambda_u} = \cap\{G \mid G \supseteq B \text{ and } G \in (1,2)\alpha O(X)\}$.

Definition 1.9. [9] In a bitopological space $X$, a subset $B$ of $X$ is an ultra-$\vee$-set (briefly $\vee_u$-set) if $B = B^{\vee_u}$, where $B^{\vee_u} = \cup\{F \mid F \subseteq B \text{ and } F \in (1,2)\alpha C(X)\}$. The family of all $\Lambda_u$-sets (resp. family of $\vee_u$-sets) is denoted by $\Lambda_u O(X)$ (resp. $\vee_u O(X)$).

Proposition 1.10. [9] For any bitopological space X, the following hold:
(i) The sets $\emptyset$ and $X$ are both $\vee_u$ sets and $\Lambda_u$-sets.
(ii) Every union of $\Lambda_u$-sets is a $\Lambda_u$-set.
(iii) Every intersection of $\vee_u$-set is a $\vee_u$-set.
(iv) $B^{\vee_u} \subseteq B$.
(v) $B \subseteq B^{\Lambda_u}$.
(vi) $(B^c)^{\Lambda_u} = (B^{\vee_u})^c$, that is $(X - B)^{\Lambda_u} = X - B^{\vee_u}$.

Proposition 1.11. [9] Let $\{B_i : i \in I\}$ be subsets of a bitopological space $(X, \tau_1, \tau_2)$. Then the following are valid:
(i) $(\bigcup\limits_{i\in I} B_i)^{\Lambda_u} = \bigcup\limits_{i\in I} B_i^{\Lambda_u}$;
(ii) $(\bigcap\limits_{i\in I} B_i)^{\Lambda_u} \subseteq \bigcap\limits_{i\in I} B_i^{\Lambda_u}$;
(iii) $(\bigcup\limits_{i\in I} B_i)^{\vee_u} \supseteq \bigcup\limits_{i\in I} B_i^{\vee_u}$ for any index set $I$.

Definition 1.12. [9] In a space $X$, a subset $B$ is called
(i) Generalized-$\Lambda_u$-set (briefly g.$\Lambda_u$-set) of $X$ if $B^{\Lambda_u} \subseteq F$ whenever $B \subseteq F$ and $F \in (1,2)\alpha CL(X)$. $D^{\Lambda_u}(X)$ denotes the family of all g.$\Lambda_u$-sets of $X$;
(ii) Generalized $\vee_u$-set (briefly g.$\vee_u$-set) of $X$ if $B^c$ is a g.$\Lambda_u$-set. $D^{\vee_u}(X)$ denotes the family of all g.$\vee_u$-sets of $X$.

Theorem 1.13. [9] A subset $A$ of $(X, \tau_1, \tau_2)$ is a g.$\vee_u$-set if and only if $U \subseteq B^{\vee_u}$ whenever $U \subseteq B$ and $U$ is a $(1,2)\alpha$-open set.

**2. A New Closure Operator $C^{\Lambda_u}$.** By using the family of $\Lambda_u$-sets of a bitopological space $X$ we define a closure operator $C^{\Lambda_u}$ and the associated topology $\tau^{\Lambda_u}$.

Definition 2.1. For any subset $B$ of a bitopological space $X$ we define $C^{\Lambda_u}(B) = \cap\{G : B \subseteq G \text{ and } G \in D^{\Lambda_u}\}$ and $Int^{\vee_u}(B) = \cup\{F : F \subseteq B \text{ and } F \in D^{\vee_u}\}$.

Proposition 2.2. Let $A$ and $B$ be subsets of a bitopological space $X$. Then
(i) $C^{\Lambda_u}(B^c) = (Int^{\vee_u}(B))^c$;
(ii) If $A \subseteq B$, then $C^{\Lambda_u}(A) \subseteq C^{\Lambda_u}(B)$;
(iii) If $B$ is a g.$\Lambda_u$-set, then $C^{\Lambda_u}(B) = B$;
(iv) If $B$ is a g.$\vee_u$-set, then $Int^{\vee_u}(B) = B$.

Theorem 2.3. $C^{\Lambda_u}$ is a Kuratowski closure operator.

Proof. (i) $C^{\Lambda_u}(\emptyset) = \emptyset$ is obvious.
(ii) $A \subset C^{\Lambda_u}(A)$ is true from the definition.
(iii) We now prove that $C^{\Lambda_u}(A \cup B) = C^{\Lambda_u}(A) \cup C^{\Lambda_u}(B)$.
Suppose there exists a point $x \in X$ such that $x \notin C^{\Lambda_u}(A \cup B)$. Then there exists a subset $G \in D^{\Lambda_u}$ such that $A \cup B \subseteq G$ and $x \notin G$. Then $A \subseteq G$, $B \subseteq G$, and $x \notin G$ which implies $x \notin C^{\Lambda_u}(A)$ and $x \notin C^{\Lambda_u}(B)$. So $C^{\Lambda_u}(A) \cup C^{\Lambda_u}(B) \subseteq C^{\Lambda_u}(A \cup B)$.
Suppose that there exists a point $x \in X$ such that $x \notin (C^{\Lambda_u}(A) \cup C^{\Lambda_u}(B))$. Then there exists two sets $G_1$ and $G_2$ in $D^{\Lambda_u}$ such that $A \subseteq G_1$ and $B \subseteq G_2$ but $x \notin G_1$ and $x \notin G_2$. Now let $G = G_1 \cup G_2$. By Proposition 2.4 of [9], $G \in D^{\Lambda_u}$. Then $A \cup B \subseteq G$ and $x \notin G$ and so $x \notin C^{\Lambda_u}(A \cup B)$, which implies $C^{\Lambda_u}(A \cup B) \subseteq C^{\Lambda_u}(A) \cup C^{\Lambda_u}(B)$.
(iv) We now prove $C^{\Lambda_u}(C^{\Lambda_u}(B)) = C^{\Lambda_u}(B)$. Suppose there exists a point $x \in X$ such that $x \notin C^{\Lambda_u}(B)$. Then there exists a $U \in D^{\Lambda_u}$ such that $x \notin U$ and $B \subseteq U$. By Proposition 2.2, $C^{\Lambda_u}(B) \subseteq C^{\Lambda_u}(U) = U$. Thus, we have $x \notin C^{\Lambda_u}(C^{\Lambda_u}(B))$. Hence, $C^{\Lambda_u}(C^{\Lambda_u}(B)) \subseteq C^{\Lambda_u}(B)$. Also by (ii), $C^{\Lambda_u}(B) \subseteq C^{\Lambda_u}(C^{\Lambda_u}(B))$. Therefore, $C^{\Lambda_u}(C^{\Lambda_u}(B)) = C^{\Lambda_u}(B)$.

Definition 2.4. Let $\tau^{\Lambda_u}$ be a bitopological space generated by $C^{\Lambda_u}$ in the usual manner.
$\tau^{\Lambda_u} = \{B : B \subseteq X, C^{\Lambda_u}(B^c) = B^c\}$.
Here, we also define another family of subsets.
$\rho^{\Lambda_u} = \{B : C^{\Lambda_u}(B) = B\}$. Then we can also say that
$\rho^{\Lambda_u} = \{B : B^c \in \tau^{\Lambda_u}\}$.

Theorem 2.5. For a space $X$, the following hold:
(i) $\tau^{\Lambda_u} = \{B : B \subseteq X.\ Int^{\vee_u}(B) = B\}$.
(ii) $(1,2)\alpha O(X) \subseteq D^{\Lambda_u} \subseteq \rho^{\Lambda_u}$.
(iii) $(1,2)\alpha CL(X) \subseteq D^{\vee_u} \subseteq \tau^{\Lambda_u}$.

Proof. (i) Let $A \subseteq X$. Then $A \in \tau^{\Lambda_u}$ if and only if $C^{\Lambda_u}(A^c) = A^c$. By Proposition 2.2, $C^{\Lambda_u}(A^c) = [Int^{\vee_u}(A)]^c = A^c$, which implies $Int^{\vee_u}(A) = A$ and so $A \in \tau^{\Lambda_u}$.
(ii) Let $B \in (1,2)\alpha O(X)$. Then $B$ is a $\Lambda_u$-set and, by the definition of $\Lambda_u$-set and g.$\Lambda_u$-set, $B$ is a g.$\Lambda_u$-set. So $B \in D^{\Lambda_u}$. Then $C^{\Lambda_u}(B) = B$ which implies $B \in \rho^{\Lambda_u}$. So $(1,2)\alpha O(X) \subseteq D^{\Lambda_u} \subseteq \rho^{\Lambda_u}$.
(iii) Let $B \in (1,2)\alpha C(X)$. Then $B$ is a g.$\vee_u$-set. So $B \in D^{\vee_u}$ and so $Int^{\vee_u}(B) = B$, which implies $C^{\Lambda_u}(B^c) = B^c$. So $B \in \tau^{\Lambda_u}$. Hence, $(1,2)\alpha CL(X) \subseteq D^{\vee_u} \subseteq \tau^{\Lambda_u}$.

Proposition 2.6. Let $X$ be a bitopological space. Then
(i) for each $x \in X$, $\{x\}$ is either $(1,2)\alpha$-open or $\{x\}^c$ is a g.$\Lambda$-set;
(ii) for each $x \in X$, $\{x\}$ is $(1,2)\alpha$-open or $\{x\}$ is a g.$\vee_u$-set.

Proof. Assume $\{x\}$ is not $(1,2)\alpha$-open. Then $X - \{x\}$ is not a $(1,2)\alpha$-closed set. So the only $(1,2)\alpha$-closed set containing $\{x\}^c$ is $X$ and so $\{x\}^c$ is a g.$\Lambda_u$-set. Hence, $\{x\}$ is a g.$\vee_u$-set.

Proposition 2.7. If $(1,2)\alpha O(X) = \tau^{\Lambda_u}$, then every singleton set $\{x\}$ of $X$ is $\tau^{\Lambda_u}$-open.

Proof. Suppose $\{x\}$ is not $(1,2)\alpha$-open. By Proposition 2.6, $\{x\}^c$ is a g.$\Lambda_u$-set. Then $x \in \tau^{\Lambda_u}$. If $\{x\}$ is $(1,2)\alpha$-open, then by assumption $\{x\} \in \tau^{\Lambda_u}$.

Proposition 2.8. Let $X$ be a bitopological space. Then
(i) if $(1,2)\alpha CL(X) = \tau^{\Lambda_u}$, then every g.$\Lambda_u$-set of $X$ is $(1,2)\alpha$-open;
(ii) if every g.$\Lambda_u$-set of $X$ is $(1,2)\alpha$-open, then $\tau^{\Lambda_u} = \{B : B \subseteq X, B = B^{\Lambda_u}\}$.

Proof. Let $B$ be a g.$\Lambda_u$-set of $X$. That is, $B \in D^{\Lambda_u}$ and, by Theorem 2.5, $B \in \rho^{\Lambda_u}$ and so $B^c \in \tau^{\Lambda_u}$. By the assumption $B^c \in (1,2)\alpha CL(X)$, we have $B \in (1,2)\alpha O(X)$.
(ii) Let $A \subseteq X$ and $A \in \tau^{\Lambda_u}$. Then $C^{\Lambda_u}(A^c) = A^c = \cap\{G : A^c \subseteq G \text{ and } G \in D^{\Lambda_u}\} = \cap\{G : A^c \subseteq G \text{ and } G \in (1,2)\alpha O(X)\} = (A^c)^{\Lambda_u}$. Then, by Proposition 1.10, $A^c = (A^c)^{\Lambda_u} = X - A^{\vee_u}$. So we get $A = A^{\vee_u}$. That is, $A \in \tau^{\Lambda_u} = \{B : B \subseteq X \text{ and } B = B^{\vee_u}\}$.

Remark 2.9. From Definition 1.8, 2.1, and by Theorem 2.5, we can say that $(1,2)\alpha CL(X) \subseteq \vee_u O(X) \subseteq D^{\vee_u} \subseteq \tau^{\Lambda_u}$.

## 3. New Separation Axioms.

Definition 3.1. A bitopological space $(X, \tau_1, \tau_2)$ is called a
(i) $T_u^L$-space if and only if every g.$\vee_u$-set is a $\vee_u$-set;
(ii) $T_u^{LT}$ if and only if every g.$\vee_u$-set is a $(1,2)\alpha$-closed set.

Remark 3.2. Every $T_u^{LT}$ space is a $T_u^L$ space, but the converse is not always true as can be seen from the following example.

Example 3.3. The space $X$ defined in this example is a $T_u^L$ space, but it is not a $T_u^{LT}$ space. Let $X = \{a,b,c,d\}$, $\tau_1 = \{\phi, X, \{a\}, \{c\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}\}$ and $\tau_2 = \{\phi, X, \{a,b\}\}$. Then $(1,2)\alpha O(X) = \{\phi, X, \{a\}, \{c\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}\}$ and $D^{\Lambda_u}(X) = \{\phi, X, \{a\}, \{c\}, \{a,c\}, \{b,c\}, \{c,d\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}\}$.

Definition 3.4. A space $X$ is said to be a
(i) $T_{\vee_u}$ space if every $\tau^{\Lambda_u}$-open set is a g.$\vee_u$-set;
(ii) $T_{\vee_u}^R$ space if every $\tau^{\Lambda_u}$-open set is a $\vee_u$-set.

Example 3.5. $X = \{a,b,c,d\}$, $\tau_1 = \{\emptyset, X, \{d\}\}$, and $\tau_2 = \{\emptyset, X, \{c,d\}\}$. Then
$(1,2)\alpha O(X) = \{\emptyset, X, \{d\}, \{a,d\}, \{b,d\}, \{c,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$,
$(1,2)\alpha CL(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$, $D^{\Lambda_u} = \{\emptyset, X, \{d\}, \{a,d\}, \{b,d\}, \{c,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$, and $D^{\vee_u} = (1,2)\alpha CL(X) = \tau^{\Lambda_u}$. Here, $X$ is a $T^R_{\vee_u}$ space.

Remark 3.6. Every $T^R_{\vee_u}$-space is a $T_{\vee_u}$-space. But the converse is not always true as seen from the following example.

Example 3.7. Let $X = \{a,b,c\}$, $\tau_1 = \{\phi, X, \{a\}\}$, and $\tau_2 = \{\phi, X, \{b,c\}\}$. Then $D^{\wedge_u} = \{\phi, X, \{a\}, \{a,b\}, \{a,c\}\} = \tau^{\Lambda_u}$. This space $X$ is a $T_{\vee_u}$ space but not a $T^R_{\vee_u}$-space.

Lemma 3.8. For a g.$\vee_u$-set $B$ of a space $X$, if $x \in X$ is a point such that $x \in B$ and $X \notin B^{\vee_u}$, then $\{x\}$ is neither $(1,2)\alpha$-closed nor $(1,2)\alpha$-open.

Proof. By the definition of $B^{\vee_u}$, the set $\{x\}$ is not $(1,2)\alpha$-closed. By Theorem 1.13, $\{x\}$ is not $(1,2)\alpha$-open.

Lemma 3.9. For a bitopological space $X$, every singleton set $\{x\}$ is either $(1,2)\alpha$-closed or $\{x\}^c$ is $(1,2)\alpha$g-closed.

Proof. If $\{x\}$ is not $(1,2)\alpha$-closed, then the only $(1,2)\alpha$-open set containing $X - \{x\}$ is $X$. Hence, $\{x\}^c$ is $(1,2)\alpha$g-closed.

Theorem 3.10. The following statements in $(X, \tau_1, \tau_2)$ are equivalent.
(i) $X$ is an ultra-$T_{1/2}$ space;
(ii) $X$ is a $T^L_u$-space;
(iii) $X$ is a $T^R_{\vee_u}$-space.

Proof. (i)$\Rightarrow$(ii) Suppose $X$ is not an $T^L_u$-space. Then there exists a g.$\vee_u$-set which is not a $\vee_u$-set. Let $B^{\vee_u} \subset B$ but $B^{\vee_u}$ is not equal to $B$. Then there exists an $x \in B$ but $x \notin B^{\vee_u}$. Hence, $\{x\}$ is not a $(1,2)\alpha$-closed set. By Lemma 3.9, $X - \{x\}$ is a $(1,2)\alpha$g-closed set. On the other hand, $\{x\}$ is not $(1,2)\alpha$-open (by Lemma 3.8). Therefore, $X - \{x\}$ is not $(1,2)\alpha$-closed but it is $(1,2)\alpha$g-closed. This is a contradiction to the assumption that $X$ is an ultra-$T_{1/2}$ space.
(ii)$\Rightarrow$(i) Suppose $X$ is not an ultra-$T_{1/2}$ space. Then there exists a $B \subset X$ such that $B$ is a $(1,2)\alpha$g-closed set but not $(1,2)\alpha$-closed. Since $B$ is not $(1,2)\alpha$-closed, there exists a point $x \in X$ such that $x \in \alpha Cl(B)$ but $x \notin B$. By Proposition 2.6, the set $\{x\}$ is either $(1,2)\alpha$-open or a g.$\vee_u$-set.

Case (i). $\{x\}$ is $(1,2)\alpha$-open. Then since $x \in (1,2)\alpha Cl(B)$, $\{x\} \cap B = \phi$. This is a contradiction.

Case (ii). If $\{x\}$ is g.$\vee_u$-set and $\{x\}$ is not $(1,2)\alpha$-closed, $\{x\}^{\vee_u} = \phi$. Hence, $\{x\}$ is not a g.$\vee_u$-set. This is a contradiction.

Case (iii). If $\{x\}$ is a g.$\vee_u$-set and $\{x\}$ is $(1,2)\alpha$-closed, then $X-\{x\}$ is a $(1,2)\alpha$-open set containing $B$. As $B$ is a $(1,2)\alpha$g-closed set, $(1,2)\alpha Cl(B) \subseteq X-\{x\}$, again this is a contradiction that $x \in (1,2)\alpha Cl(B)$.

(ii)$\Rightarrow$(iii) Let $B$ be a $\tau^{\Lambda_u}$-set. That is, $B = Int^{\vee_u}(B)$. By assumption, $D^{\vee_u}(X) = \vee_u O(X)$. Also, $(Int^{\vee_u}(B))^{\vee_u} = (\{\bigcup\{F : F \subseteq B, F \in D^{\vee_u}\}^{\vee_u} \supset \bigcup\{F : F^{\vee_u} \subseteq B, F \in D^{\vee_u}\} = Int^{\vee_u}(B)$. Again by Proposition 1.11, $(Int^{\vee_u}(B))^{\vee_u} \subseteq Int^{\vee_u}(B)$. Hence, $Int^{\vee_u}(B)$ is a $\vee_u$-set.
(iii)$\Rightarrow$(ii) Let $B$ be a g.$\vee_u$-set. Then $Int^{\vee_u}(B) = B$ (by Definition 2.1) and, by assumption, it is a $\vee_u$-set.

Theorem 3.11. If $X$ is an ultra-$T_{1/2}$ space, then $X$ is a $T_{\vee_u}$ space.

Proof. By Remark 2.9, we have $(1,2)\alpha CL(X) \subseteq \vee_u O(X) \subset D^{\vee_u} \subseteq \tau^{\Lambda_u}$. Again by Theorem 3.10, $\vee_u O(X) = \tau^{\vee_u}$. Therefore, $D^{\vee_u} = \tau^{\vee_u}$. Hence, $X$ is a $T_{\vee_u}$ space.

Remark 3.12. The converse of Theorem 3.11 need not always be true. This is shown by the following example.

Example 3.13. $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, X, \{a\}, \{b,c\}\}$, and $\tau_2 = \{\emptyset, X, \{a\}\}$. Then $(1,2)\alpha O(X) = \{\emptyset, X, \{a,\}, \{b,c\}\}$, $(1,2)\alpha CL(X) = \{\emptyset, X, \{a\}, \{b,c\}\}$, $(1,2)\alpha GCL(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$, $D^{\Lambda_u}(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\} = D_u^{\vee}(X) = \tau^{\Lambda_u}$. Hence, the space $X$ is $T_{\vee_u}$ but not ultra-$T_{1/2}$.

Lemma 3.14. For a space $X$, every singleton set is a g.$\Lambda_u$-set if and only if $G = G^{\vee_u}$ for every $(1,2)\alpha$-open set $G$.

Proof. Let $G$ be a $(1,2)\alpha$-open set and let $y \in X-G$. Then $\{y\}$ is a g.$\Lambda_u$-set and $X-G$ is a $(1,2)\alpha$-closed set. $\{y\}^{\Lambda_u} \subseteq X-G$. Again, $U\{y\}^{\Lambda_u} \subseteq X-G$ for $y \in X-G$. By Proposition 1.11, $(U\{y\})^{\Lambda_u} = \cup\{\{y\}^{\Lambda_u}\}$ for $y \in X-G$ and hence, $\{U\{y\}^{\Lambda_u} : y \in X-G\} = (X-G)^{\Lambda_u} = \{(U\{y\}))^{\Lambda_u} : y \in X-G\} \subseteq X-G$. Again by Proposition 1.11, $X-G \subseteq (X-G)^{\Lambda_u}$. Therefore, $(X-G)^{\Lambda_u} = X-G = X-G^{\vee_u}$ and so $G = G^{\vee_u}$.

Lemma 3.15. The bitopological space $X$ is an ultra-$R_0$ space if and only if $G = G^{\vee_u}$, where $G$ is a $(1,2)\alpha$-open set.

Proof. Let $X$ be ultra-$R_0$. Let $x \in G$. Then $(1,2)\alpha Cl(\{x\}) \subseteq G$. So we have $\{x\} \subseteq (1,2)\alpha Cl(\{x\}) \subseteq G$ for each $x \in G$. Then $\{\cup\{x\} : x \in G\} \subseteq \{\cup(1,2)\alpha Cl(\{x\}); x \in G\} \subset G$. Now let $F = (1,2)\alpha Cl(\{x\})$. Then we have $G = \cup\{F : F \subseteq G$ and $F \in (1,2)\alpha Cl(\{x\})\} = G^{\vee_u}$.
Conversely, let $G = U\{F_i : F_i \subseteq G$ and $F \in (1,2)\alpha C(X)\}$ and also, let $x \in G$. Then $x \in F_i$ for some $i$ and $F_i$ is $(1,2)\alpha$-closed. Then $(1,2)\alpha Cl(\{x\}) \subseteq (1,2)\alpha Cl(F_i) = F_i \subseteq G$. Hence, $X$ is ultra-$R_0$.

Theorem 3.16. If $X$ is an ultra-$R_0$ space, then $X$ is $T_{\vee_u}$.

Proof. By Lemma 3.15, if $X$ is ultra-$R_0$ and $G$ is any $(1,2)\alpha$-open set, then $G = G^{\Lambda_u}$. By Lemma 3.14, every singleton set $\{b\}$ is a g.$\Lambda_u$-set.

Now let $B$ be any subset of $X$. Then $B = \cup\{\{b\} = b \in B\}$ is also a g.$\Lambda_u$-set. Thus, every subset of $X$ is a g.$\Lambda_u$-set. Hence, $D^{\Lambda_u} = P(X)$ and so $D^{\vee_u} = \tau^{\Lambda_u}$. That is, $X$ is $T_{\vee_u}$.

Remark 3.17. Every $T_{\vee_u}$-space need not always be an ultra-$R_0$-space. This can be seen by the following example.

Example 3.18. Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$, and $\tau_2 = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$. Then $(1,2)\alpha O(X) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$, $(1,2)\alpha CL(X) = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$, $D^{\Lambda_u} = \{\emptyset, X, \{a\}, \{b, a\}, \{a, c\}\}$. Here, $D_u^{\vee} = (1,2)\alpha CL(X) = \tau^{\vee}$. The space is not ultra-$R_0$ but $T_{\vee_u}$.

Remark 3.19. The concepts of ultra-$T_0$ and $T_{\vee_u}$ are independent. Example 3.13 and the following example justify this claim.

Example 3.20. Let $X$ be the set of all real numbers and $\tau_1 = \{\{\emptyset, X\} \bigcap \{(a, \infty) : a \in X\}$, $\tau_2 = \{\emptyset, X\}$. Now $(1,2)\alpha O(X) = \tau_1$. Here, the space $X$ is ultra-$T_0$ but not $T_{\vee_u}$.

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R. Raja Rajeswari
Department of Mathematics
Sri Parasakthi College
Courtallam - 627802
Tirunelveli Dt., TamilNadu, India
email: raji_arul2000@yahoo.co.in

M. Lellis Thivagar
Department of Mathematics
Arul Anndar College
Karumathur - 625514
Madurai Dt., TamilNadu, India
email: mlthivagar@yahoo.co.in