STRONGLY S-CLOSED SPACES AND FIRMLY CONTRA-CONTINUOUS FUNCTIONS

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Abstract. In the present paper, we offer a new form of firm continuity, called firm contra-continuity, by which we characterize strongly S-closed spaces. Moreover, we investigate the basic properties of firmly contra-continuous functions. We also introduce and investigate the notion of locally contra-closed graphs.

1. Introduction

Kupka [8] has used firm continuity to investigate compactness. Recently Caldas, et al. have used firm semi-continuity to study semi-compactness. In this note we continue this line of investigation by introducing a form of firm continuity, which we call firm contra-continuity, and using it to study strongly S-closed spaces. Dontchev [6] introduced strongly S-closed spaces and showed that contra-continuous images of strongly S-closed spaces are compact. Baker [2] extended this result by showing that subcontra-continuous images of strongly S-closed spaces are compact. Quite recently, Ganster et al. [7] further investigated, among others, the notion of strongly S-closedness. Our purpose in this note is to characterize strongly S-closed spaces in terms of firm contra-continuity and subcontra-continuity. In particular, we show that a space X is strongly S-closed if and only if for every space Y every subcontra-continuous function $f : X \to Y$ is firmly contra-continuous. Moreover, some of the basic properties of firmly contra-continuous functions are investigated. For example, we show that firm contra-continuity implies slight continuity. Finally, we introduce the notion of locally contra-closed graphs and present some of its fundamental properties.

2. Preliminaries

The symbols X and Y represent topological spaces with no separation properties assumed unless explicitly stated. All sets are considered to be subsets of topological spaces. The closure and interior of a set A are signified by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A set A is regular open if $A = \mathrm{Int}(\mathrm{Cl}(A))$. A set A is semiopen [9] (respectively, preopen [10], β-open [1]) provided that $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$ (respectively, $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$, $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$). A set A is regular closed (respectively, semiclosed, preclosed, β-closed) if the complement of A is regular open (respectively, semiopen, preopen, β-open). We denote the intersection of all semiclosed sets containing A by $sCl(A)$. Recall that a set $A \subset X$ is called a semi-generalized closed set (briefly sg-closed set) [3] if $sCl(A) \subset U$ whenever $A \subset U$ and U is semi-open. The complement of an sg-closed set is called sg-open.

Definition 1. *A space X is said to be S-closed* [12] *(respectively, almost compact* [6]*) if every semiopen cover (respectively, open cover) of X has a finite subfamily, the closures of whose members cover X.*

Definition 2. *1) A space X is said to be strongly S-closed* [6] *if every closed cover of X has a finite subcover.*
2) Let A be a subset of X. We say that A is strongly S-closed relative to X if every cover of A by closed sets of X has a finite subcover.

Observe that if X is regular and strongly S-closed then the weight of X does not exceed $2^{|A|}$, where A is the finite dense subset of X. Recall that the least cardinal of a base for the space X is called the weight of X.

Remark 2.1. *Dontchev* [6] *showed that strongly S-closedness and compactness are independent of each other. For example the Hilbert cube is compact but not strongly S-closed. But the real line with a topology in which non-empty open sets are the ones containing the origin is an example of a strongly S-closed space which is not compact (see* [6]*, Remark 3.1). He also noticed that a set is regular closed if and only if it is both closed and sg-open. It follows that a topological space X is S-closed if and only if it is strongly S-closed and sg-compact. Recall that a topological space X is called sg-compact* [4] *if every cover of X by sg-open sets has a finite subcover.*

Definition 3. *A function $f : X \to Y$ is said to be contra-continuous* [6] *if $f^{-1}(V)$ is closed for every open subset V of Y.*

Definition 4. *A function $f : X \to Y$ is said to be subcontra-continuous* [2] *provided there is an open base $\mathcal{B}$ for Y such that $f^{-1}(V)$ is closed for every $V \in \mathcal{B}$.*

Definition 5. *A function $f : X \to Y$ is said to be β-continuous* [1] *(respectively, precontinuous* [10]*) if $f^{-1}(V)$ is β-open (respectively, preopen) for every open subset V of Y.*

Definition 6. *A function $f : X \to Y$ is said to be firmly continuous* [8] *if for every open cover Λ of Y there exists a finite open cover Γ of X such that for every $U \in \Gamma$ there exists $V \in \Lambda$ such that $f(U) \subseteq V$.*

3. Characterization of strongly S-closed spaces

Definition 7. *A function $f : X \to Y$ is said to have property φ [8] provided that for every open cover Λ of Y there exists a finite cover (the members of which need not be open) $\{A_1, A_2, \ldots, A_n\}$ of X such that for each $i \in \{1, 2, \ldots, n\}$ there exists $V \in \Lambda$ for which $f(A_i) \subseteq V$.*

Definition 8. *A function $f : X \to Y$ is said to be firmly contra-continuous if for every open cover Λ of Y there exists a finite closed cover $\mathcal{F}$ of X such that for every $F \in \mathcal{F}$ there exists $V \in \Lambda$ such that $f(F) \subseteq V$.*

The following examples show that firm contra-continuity is independent of firm continuity.

Example 3.1. *Let $X = \{a, b, c\}$ have the topology $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$ and let $f : X \to X$ be the identity mapping. Since f is continuous and X is finite, f is obviously firmly continuous. Since any closed cover of X must contain X, we see that f is not firmly contra-continuous.*

Example 3.2. *Let $X = \{a, b, c\}$ have the topologies $\tau = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}$ and $\sigma = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$, and let $f : (X, \tau) \to (X, \sigma)$ be the identity mapping. Since any τ-open cover of X must contain X, it follows that f is not firmly continuous. However, since f is contra-continuous and X is finite, f is firmly contra-continuous.*

If (X, τ) is a topological space, then the topology on X with a base consisting of the τ-closed sets will be denoted by τ_c.

Theorem 3.3. *For a space (X, τ) the following properties are equivalent:*

(a) *(X, τ) is strongly S-closed;*
(b) *For every space Y, every subcontra-continuous function $f : X \to Y$ is firmly contra-continuous;*
(c) *The identity function $f : (X, \tau) \to (X, \tau_c)$ is firmly contra-continuous;*
(d) *The identity function $f : (X, \tau) \to (X, \tau_c)$ has property φ;*
(e) *For every space Y, every subcontra-continuous function $f : X \to Y$ has property φ.*

Proof. (a) $\Rightarrow$ (b) Assume X is strongly S-closed and that $f : X \to Y$, where Y is an arbitrary space, is subcontra-continuous with respect to the base $\mathcal{B}$ for Y. Let Λ be an open cover of Y. Therefore for each $y \in f(X)$ there exists $V_y \in \Lambda$ such that $y \in V_y$ and there exists $B_y \in \mathcal{B}$ such that $y \in B_y \subseteq V_y$. Then $\{B_y : y \in f(X)\}$ is an open cover of $f(X)$. Since the subcontra-continuous image of a strongly S-closed space is compact [2], we have that $f(X)$ is compact. Therefore there is a finite subcover $\{B_{y_i} : i = 1, 2, \ldots, n\}$ which covers $f(X)$. If we let $F_i = f^{-1}(B_{y_i})$ for every $i \in \{1, 2, \ldots, n\}$, then $\{F_i : i = 1, 2, \ldots, n\}$ is a finite closed cover of X

for which $f(F_i) \subseteq B_{y_i} \subseteq V_{y_i}$ for every $i \in \{1, 2, \ldots, n\}$. Hence, f is firmly contra-continuous.

(b) $\Rightarrow$ (c) The proof is clear since the identity function $f : (X, \tau) \to (X, \tau_c)$ is subcontra-continuous with respect to the base consisting of the τ-closed sets.

(c) $\Rightarrow$ (d) The proof is clear since firm contra-continuity obviously implies property φ.

(d) $\Rightarrow$ (a) Assume the identity function $f : (X, \tau) \to (X, \tau_c)$ has property φ. Let $\mathcal{F}$ be a closed cover of (X, τ). Then $\mathcal{F}$ is an open cover of (X, τ_c). Since the identity function $f : (X, \tau) \to (X, \tau_c)$ has property φ, there exists a finite cover $\{A_1, A_2, \ldots, A_n\}$ of (X, τ) such that for each $i \in \{1, 2, \ldots, n\}$ there exists $F_i \in \mathcal{F}$ for which $A_i = f(A_i) \subseteq F_i$. Obviously $\{F_i : i = 1, 2, \ldots, n\}$ is a finite subcover of $\mathcal{F}$, which proves that (X, τ) is strongly S-closed.

(b) $\Rightarrow$ (e) The proof is clear since firm contra-continuity implies property φ.

(e) $\Rightarrow$ (d) The proof is clear since the identity function $f : (X, \tau) \to (X, \tau_c)$ is subcontra-continuous with respect to the base $\mathcal{B}$ consisting of the τ-closed sets. $\square$

Since a subcontra-continuous, β-continuous image of an S-closed space is compact [2], we have the following version of the implication (a) $\Rightarrow$ (b) in Theorem 3.3.

Theorem 3.4. *If X is an S-closed space, then for every space Y, every subcontra-continuous, β-continuous function $f : X \to Y$ is firmly contra-continuous.*

Similarly, since a subcontra-continuous, precontinuous image of an almost compact space is compact [2], we have the following result.

Theorem 3.5. *If X is almost compact, then for every space Y, every subcontra-continuous, precontinuous function $f : X \to Y$ is firmly contra-continuous.*

4. Additional properties of firmly contra-continuous functions

All of the results in this section are special cases of the following theorem.

Theorem 4.1. *Let $f : X \to Y$ be firmly contra-continuous. If V is an open subset of Y and A is a closed subset of Y such that $A \subseteq V$, then $Cl(f^{-1}(A)) \subseteq f^{-1}(V)$.*

Proof. Let $x \in f^{-1}(A)$. Since $\{V, Y - A\}$ is an open cover of Y, there exists a finite closed cover $\mathcal{F}$ of X such that for every $F \in \mathcal{F}$, we have

$f(F) \subseteq V$ or $f(F) \subseteq Y - A$. Let $F_x \in \mathcal{F}$ such that $x \in F_x$. Then $f(F_x) \subseteq V$ and hence $f^{-1}(A) \subseteq \cup_{x \in f^{-1}(A)} F_x \subseteq f^{-1}(V)$. Since $\mathcal{F}$ is finite, $\cup_{x \in f^{-1}(A)} F_x$ is a finite union of closed sets and hence closed. Therefore $\mathrm{Cl}(f^{-1}(A)) \subseteq \cup_{x \in f^{-1}(A)} F_x \subseteq f^{-1}(V)$. □

Kupka [8] observed that a firmly continuous function need not be continuous. The following example shows that a firmly contra-continuous function need not be subcontra-continuous, even when the domain is strongly S-closed. In particular, the requirements that f be subcontra-continuous and firmly contra-continuous in Theorem 3.3(b) cannot be interchanged.

Example 4.2. *Let $X = [0, 3]$ have the topology $\sigma = \{U \subseteq X : 3 \in U\} \cup \{\emptyset\}$ and let Y be the real numbers with the topology $\tau = \{(a, +\infty) : a \in Y\} \cup \{Y, \emptyset\}$. Finally, let $f : (X, \sigma) \to (Y, \tau)$ be the inclusion mapping. To see that f is firmly contra-continuous, note that every open cover of Y must contain either Y or a set of the form $(a, +\infty)$, where $a < 0$, and that both of these sets contain $f(X)$. To see that f is not subcontra-continuous, let $\mathcal{B}$ be an open base for Y. Then there exists $B \in \mathcal{B}$ such that $3 \in B \subsetneq (2, +\infty)$. Then $B = (a, +\infty)$ where $2 \leq a < 3$ and hence $f^{-1}(B) = (a, 3]$, which is not closed in X. Finally note that X is strongly S-closed.*

Recall that a space is called zero dimensional provided it has a clopen base.

Corollary 4.3. *If $f : X \to Y$ is firmly contra-continuous and Y is zero dimensional, then f is subcontra-continuous.*

Proof. Assume $\mathcal{B}$ is a clopen base for Y and let $B \in \mathcal{B}$. Then by Theorem 4.1, if we let $A = V = B$, we have $\mathrm{Cl}(f^{-1}(B)) \subseteq f^{-1}(B)$, which proves that $f^{-1}(B)$ is closed. Therefore f is subcontra-continuous with respect to the base $\mathcal{B}$. □

Definition 9. *A function $f : X \to Y$ is said to be slightly continuous* [11] *provided that for every $x \in X$ and for every clopen subset V of Y containing $f(x)$, there exists an open subset U of X containing x such that $f(U) \subseteq V$.*

The following characterizations of slight continuity will be useful.

Theorem 4.4. *For a function $f : X \to Y$ the following properties are equivalent:*

(a) *f is slightly continuous;*
(b) [11] *The inverse image of every clopen subset of Y is an open subset of X;*
(c) *The inverse image of every clopen subset of Y is a closed subset of X;*

(d) [11] *The inverse image of every clopen subset of Y is a clopen subset of X.*

The proof of the following corollary is analogous to that of Corollary 4.3.

Corollary 4.5. *If $f : X \to Y$ is firmly contra-continuous, then f is slightly continuous.*

If the codomain of a function is either T_1 or regular, then Theorem 4.1 can be used to prove that firm contra-continuity implies a local version of contra-continuity.

Definition 10. *A function $f : X \to Y$ is said to be locally contra-continuous provided that for every $x \in X$ and for every open subset V of Y containing $f(x)$, there exists a closed subset F of X containing x such that $f(F) \subseteq V$.*

Example 4.6. *The identity mapping on the real numbers with the usual topology is locally contra-continuous, but not contra-continuous. Actually the identity function on any regular or T_1 space with an open nonclosed set has this property.*

Corollary 4.7. *If $f : X \to Y$ is firmly contra-continuous and Y is either regular or T_1, then f is locally contra-continuous.*

Proof. Assume Y is regular. Let $x \in X$ and let V be an open subset of Y containing $f(x)$. Then there exists an open subset U of Y such that $f(x) \in U \subseteq \mathrm{Cl}(U) \subseteq V$. By Theorem 4.1 $x \in \mathrm{Cl}(f^{-1}(\mathrm{Cl}(U))) \subseteq f^{-1}(V)$. Thus, if $F = \mathrm{Cl}(f^{-1}(\mathrm{Cl}(U)))$, then F is a closed set containing x for which $f(F) \subseteq V$ and therefore f is locally contra-continuous.

The proof for the case where Y is T_1 is analogous if $\{f(x)\}$ is used in place of U. $\square$

5. Locally contra-closed graphs

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) \mid x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 11. *A function $f : X \to Y$ has a locally contra-closed graph if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist a closed subset D of X containing x and an open subset V of Y containing y such that $(D \times V) \cap G(f) = \emptyset$.*

Lemma 5.1. *A function $f : X \to Y$ has a locally contra-closed graph if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist a closed subset D of X containing x and an open subset V of Y containing y such that $f(D) \cap V = \emptyset$.*

Proof. It is an immediate consequence of Definition 11 and the fact that for any subsets $D \subset X$ and $V \subset Y$, $(D \times V) \cap G(f) = \emptyset$ if and only if $f(D) \cap V = \emptyset$. $\square$

Theorem 5.2. *If $f : X \to Y$ is locally contra-continuous and Y is Hausdorff, then $G(f)$ is locally contra-closed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$. Since Y is Hausdorff, there exist open subsets V_1 and V_2 of Y containing y and $f(x)$, respectively, such that $V_1 \cap V_2 = \emptyset$. Since f is locally contra-continuous, there exists a closed set D of X containing x such that $f(D) \subset V_2$. This means that $f(D) \cap V_1 = \emptyset$. It follows that $G(f)$ is locally contra-closed in $X \times Y$. $\square$

Corollary 5.3. *If $f : X \to Y$ is firmly contra-continuous and Y is Hausdorff, then $G(f)$ is locally contra-closed in $X \times Y$.*

Theorem 5.4. *If $f : X \to Y$ has a locally contra-closed graph, $f(K)$ is closed in Y for each subset K strongly S-closed relative to X.*

Proof. Suppose that y is a point in $Y \setminus f(K)$. We have $(x, y) \notin G(f)$ for each $x \in K$. Since $G(f)$ is locally contra-closed, there exists a closed subset D_x of X containing x and an open set V_x of Y containing y such that $f(D_x) \cap V_x = \emptyset$. The family $\{D_x \mid x \in K\}$ is a cover of K by closed sets of X. Then, there exists a finite subset K_0 of K such that $K \subset \bigcup\{D_x \mid x \in K_0\}$. Set $V = \bigcap\{V_x \mid x \in K_0\}$. Now we have

$$f(K) \cap V \subset \textstyle\bigcup_{x \in K_0}(f(D_x) \cap V) \subset \bigcup_{x \in K_0}(f(D_x) \cap V_x) = \emptyset.$$

This shows that $y \notin Cl(f(K))$ and hence $f(K)$ is closed in Y. $\square$

Corollary 5.5. *If $f : X \to Y$ is a surjection with a locally contra-closed graph, then Y is T_1.*

Proof. Suppose that q is a point of Y. Since f is surjective, there exists a point $d \in X$ such that $f(d) = q$. The singleton $\{d\}$ is strongly S-closed relative to X. By Theorem 5.4, $\{q\}$ is closed in Y. Since the singleton sets in Y are closed, Y is T_1. $\square$

Theorem 5.6. *If $f : X \to Y$ is a injection with a locally contra-closed graph, then X is T_1.*

Proof. Let x and y be two distinct points of X. Then $f(x) \neq f(y)$. Since f has a locally contra-closed graph, there exist a closed set D in X containing x and an open set V in Y containing $f(y)$ such that $f(D) \cap V = \emptyset$. This means that $y \notin D$ and therefore X is T_1. $\square$

Corollary 5.7. *If $f : X \to Y$ is a bijection with a locally contra-closed graph, then both X and Y are T_1.*

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