Hiroshima Math. J.
49 (2019), 35–46

Projective classification of jets of surfaces in 4-space

Dedicated to Professor Takashi Nishimura on the occasion of his 60th birthday.

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(Received July 25, 2017)
(Revised July 18, 2018)

Abstract. We classify jets of Monge forms of generic surfaces in 4-space via projective transformations, which is an extension of Platonova's result for surfaces in 3-space.

1. Introduction

We are concerned with the local projective geometry of smooth surfaces in the real projective space $\mathbb{P}^4$. In the present paper, we classify jets of generic surfaces via projective transformations, which is called *the projective classification*.

For surfaces in $\mathbb{P}^3$, Platonova [21, 22] completed the projective classification of jets of generic surfaces, and various results concerning projective differential geometry of surfaces have been given using normal forms of this classification (cf. [1, 2, 11, 18, 21, 24]). An extension of Platonova's result for generic two parameter families of surfaces was obtained in [23] (see also [8, 10, 11, 20]). The same study for surfaces in 4-space was proposed in [2, page 61]; however, there have been no results. Theorem 1 in the present paper is the answer to the proposal in [2].

Let M be a smooth surface embedded in $\mathbb{R}^4 \subset \mathbb{P}^4$ containing the origin of $\mathbb{R}^4$, where $\mathbb{R}^4$ is identified with an open chart $\{[x : y : z : w : 1]\} \subset \mathbb{P}^4$. We write M in the Monge form as $(z, w) = f(x, y) = (f_1(x, y), f_2(x, y))$ at the origin where $f_i(0, 0) = df_i(0, 0) = 0$ for $i = 1, 2$. Two jets of surfaces at some points are said to be *projectively equivalent* if there is a projective transformation sending one to the other. Our result is the following.

Theorem 1. *There is an open everywhere dense subset $\mathcal{O}$ of the space of compact smooth surfaces M in $\mathbb{P}^4$ such that the germ at each point on*

2010 *Mathematics Subject Classification*. Primary 53A20; Secondary 58K05.
Key words and phrases. Projective differential geometry, surfaces in 4-space, projection of surfaces, singularities of smooth maps.

Type	Normal form	Condition	cod.
Π_{E_1}	$(x^2 - y^2 + xy^2 + \alpha y^3 + y^2\phi_2, xy + \psi_4)$	$\alpha \neq 0$	0
Π_{E_2}	$(x^2 - y^2 + y^3 + y^2\phi_2, xy + \psi_4)$	–	1
Π_{E_3}	$(x^2 - y^2 + y^2\phi_2, xy + \psi_4)$	–	2
Π_S	$(x^2 + y^3 + y\phi_3, y^2 + \alpha x^3 + x\psi_3)$	$\alpha \neq 0$	0
Π_B	$(x^2 + y^3 + y\phi_3, y^2 + x\psi_3)$	–	1
Π_{2B}	$(x^2 + y\phi_3, y^2 + x\psi_3)$	–	2
Π_H	$(x^2 + \alpha xy^2 + y^3 + y\phi_3, xy + x\psi_3)$	–	1
Π_P	$(x^2 + xy^2 + \alpha y^4, xy + \beta y^3 + \psi_4)$	$\beta, \Lambda \neq 0$	2
Π_I^+	$(x^2 + y^2 + \alpha x^2 y + y\phi_3, \psi_3 + \psi_4)$	$b_{30} - b_{12} \neq 0$	2
Π_I^-	$(xy + \alpha x^3 + \phi_4, \psi_3 + \psi_4)$	$b_{03} \neq 0,\ a_{22} = 0$	2

Table 1. Strata of codimension ≤ 2 in the space of 4-jets of Monge forms. Here $\phi_s = \sum_{i+j=s} a_{ij}x^i y^j$, $\psi_s = \sum_{i+j=s} b_{ij}x^i y^j$, $\alpha, \beta, a_{ij}, b_{ij} \in \mathbb{R}$ are moduli parameters and $\Lambda = 6\beta^2 + 4\alpha - 15\beta + 5$. The naming of each type comes from types of central projections of the surfaces from view points on asymptotic lines (see Section 3).

$M \in \mathcal{O}$ is projectively equivalent to a germ with the 4*-jet of the Monge form in Table* 1.

The study of local geometric aspects of surfaces in 4-space is a relatively new subject (see [4, 5, 6, 9, 12, 13, 17, 19]). In general, calculations on local differential geometry of surfaces tend to be very complicated. Our classification in Theorem 1 simplifies such calculations, since a lot of terms in 4-jets of Monge forms are eliminated (see Remark 2). In addition, our normal forms in Table 1 contain a lot of moduli parameters including coefficients of higher order terms of degree grater than 4. They must be interpreted as some projective differential invariants. For example, when we look at the $\mathcal{A}$-type of the central projection of the Π_P-type surface germ, it is observed that the central projection from a view point on the asymptotic line is $\mathcal{A}$-equivalent to $P_3(c) : (x, xy^2 + cy^4, xy + y^3)$ where $c = \alpha/\beta$ is a moduli parameter (see Section 3). The first author found also that α and β are expressed by combinations of some cross-ratio invariants and they determine the topological type of BDE (binary differential equations) of asymptotic curves in [6].

In Section 2, we consider a stratification of the 4-jet space of Monge forms via projective transformations, and obtain simple normal forms of jets of Monge forms. This gives the proof of Theorem 1. Section 3 is an appendix, showing the complete stratification of the jet space of Monge forms induced from the $\mathcal{A}$-classification of central projections. Although the result of Section 3 was implicitly given in the Ph.D. thesis [14] of Mond, we believe that this explicit style will help the readers.

2. The classification of Monge forms by projective transformations and proof of Theorem 1

In this section we consider the classification of jets of Monge forms of generic surfaces by projective transformations. The projective linear group $PGL(5)$ is defined as the quotient space $GL(5)/\sim$, where $A \sim A'$ if there exists a non-zero constant λ such that $A = \lambda A'$. For a natural number ℓ, let V_ℓ denote the space of polynomials in x, y consisting of monomials of degree greater than 1 and less than or equal to ℓ, then $V_\ell \times V_\ell$ means the ℓ-jet space of Monge forms of surfaces in $\mathbb{P}^4$. We define the following subgroup

$$G(5) := \{\Psi \in PGL(5) \mid \Psi(0) = 0, \Psi(W) = W\}$$

of $PGL(5)$, where $0 = [0;0;0;0;1]$ is the origin and W is the xy-plane in $\mathbb{R}^4$. $G(5)$ forms a 16-dimensional subgroup of $PGL(5)$ and acts on $V_\ell \times V_\ell$. Thus $V_\ell \times V_\ell$ can be stratified into strata of $G(5)$-orbits. In the following, we use the word "codimesion" to mean the codimension of a stratum in $V_\ell \times V_\ell$.

Let $f = (f_1, f_2)$ and $g = (g_1, g_2)$ be Monge forms of surfaces. We say that the k-jets of these Monge forms are *projectively equivalent* and write $j^k f \sim j^k g$ if there exists a projective transformation $\Psi \in G(5)$ which transforms one to the other. In this paper we check the equivalence of jets of Monge forms in the following way. With the coordinate (x, y, z, w) of $\mathbb{R}^4$, a projective transformation $\Psi \in G(5)$ is regarded locally as a diffeomorphism germ $\mathbb{R}^4, 0 \to \mathbb{R}^4, 0$ given by

$$\Psi(x, y, z, w) = \left(\frac{q_1(x, y, z, w)}{p(x, y, z, w)}, \frac{q_2(x, y, z, w)}{p(x, y, z, w)}, \frac{q_3(x, y, z, w)}{p(x, y, z, w)}, \frac{q_4(x, y, z, w)}{p(x, y, z, w)}\right),$$

where $q_i = q_{i1}x + q_{i2}y + q_{i3}z + q_{i4}w$, for $i = 1, 2$, $q_j = q_{j3}z + q_{j4}w$, for $j = 3, 4$ and $p = 1 + p_1 x + p_2 y + p_3 z + p_4 w$. Define

$$F_1(x, y, z, w) = \frac{q_3}{p} - f_1\left(\frac{q_1}{p}, \frac{q_2}{p}\right)$$

$$F_2(x, y, z, w) = \frac{q_4}{p} - f_2\left(\frac{q_1}{p}, \frac{q_2}{p}\right).$$

Then

$$F_1(x, y, g_1, g_2) = F_2(x, y, g_1, g_2) = o(k)$$

Type	Normal form	cod.
elliptic	$(x^2 - y^2, xy)$	0
hyperbolic	(x^2, y^2)	0
parabolic	(x^2, xy)	1
inflection	$(x^2 + y^2, 0)$ or $(xy, 0)$	2
degenerate inflection	$(x^2, 0)$	3
degenerate inflection	$(0, 0)$	4

Table 2. The classification of $J^2(2,2)$ (which is equal to the 2-jet space of Monge forms $f = (f_1, f_2)$) by $GL(2) \times GL(2)$-actions given by Gibson in [2].

implies $j^k f \sim j^k g$ (o is Landau's symbol). Hence, to check the equivalence, we have to solve algebraic equations $F_1 = F_2 = o(k)$ in terms of $q_i s$ and $p_i s$ for a given Monge form $f = (f_1, f_2)$ and some simplified normal form $g = (g_1, g_2)$.

We begin with simplifying 2-jets of Monge forms, then deal with higher jets. However we stop this process with the 4-jets. This is because, the dimension of $G(5)$ acting on the jet space of Monge forms is just 16, and it does not give so good normal forms for higher jets. In addition, we only have to treat strata with codimension ≤ 2 when considering generic surfaces, which is verified by the natural extension of a transversality theorem by J. Bruce in [3] (see also [5]).

2.1. 2-jet. We first deal with the classification of 2-jets of Monge forms. In the 2-jet space, the condition $F_1 = F_2 = o(2)$ for any $j^2 f, j^2 g \in V_2 \times V_2$ gives equations of just q_{i1}, q_{i2}, q_{j3}, q_{j4} with $i = 1, 2$ and $j = 3, 4$, and the classification by projective transformations is reduced to the classification of $V_2 \times V_2 \subset J^2(2,2)$ by the natural action of $\mathscr{G} = GL(2, \mathbb{R}) \times GL(2, \mathbb{R})$. The $\mathscr{G}$-orbits are classified in [2] described as in Table 2. We classify now the higher jets of germs with a 2-jet in Table 2.

2.2. Elliptic case. Suppose that $j^2 f = (x^2 - y^2, xy)$ and write

$$j^3(f_1, f_2) = \left(x^2 - y^2 + \sum_{i+j=3} a_{ij} x^i y^j, xy + \sum_{i+j=3} b_{ij} x^i y^j\right)$$

where $a_{ij}, b_{ij} \in \mathbb{R}$. The following equivalence

$$j^3(f_1, f_2) \sim (x^2 - y^2 + y^2 \phi_1, xy),$$

is given by the projective transformation Ψ with

$$q_1 = x + b_{03}z + (-a_{21} + b_{12} - b_{30})w, \qquad q_2 = y - b_{30}z + (-b_{21} + b_{03} + a_{30})w,$$

$$q_3 = z, \qquad q_4 = w, \qquad p = 1 + (a_{30} + 2b_{03})x + (2b_{12} - a_{21})y.$$

Here ϕ_k means a homogeneous polynomial of degree k. Consider

$$j^4(f_1, f_2) = \left(x^2 - y^2 + y^2\phi_1 + \sum_{i+j=4} c_{ij}x^iy^j, xy + \sum_{i+j=4} d_{ij}x^iy^j\right)$$

where $c_{ij}, d_{ij} \in \mathbb{R}$, then

$$j^4(f_1, f_2) \sim (x^2 - y^2 + y^2(\phi_1 + \phi_2), xy + \phi_4),$$

by Ψ with $q_1 = x$, $q_2 = y$, $q_3 = z$, $q_4 = w$, $p = 1 + c_{40}z + c_{31}w$.

Write $\phi_1 = \bar{c}_{12}x + \bar{c}_{03}y$. Then

$$j^4(f_1, f_2) \sim \begin{cases} (x^2 - y^2 + xy^2 + \alpha y^3 + y^2\phi_2, xy + \phi_4) & \text{if } \bar{c}_{12}, \bar{c}_{03} \neq 0; \\ (x^2 - y^2 + y^3 + y^2\phi_2, xy + \phi_4) & \text{if } \bar{c}_{03} \neq 0, \bar{c}_{12} = 0; \\ (x^2 - y^2 + y^2\phi_2, xy + \phi_4) & \text{if } \bar{c}_{12} = \bar{c}_{03} = 0. \end{cases}$$

where $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

2.3. Hyperbolic case. Suppose that $j^2f = (x^2, y^2)$ and write

$$j^3(f_1, f_2) = \left(x^2 + \sum_{i+j=3} a_{ij}x^iy^j, y^2 + \sum_{i+j=3} b_{ij}x^iy^j\right)$$

where $a_{ij}, b_{ij} \in \mathbb{R}$. The following equivalence

$$j^3(f_1, f_2) \sim (x^2 + a_{03}y^3, y^2 + b_{30}x^3)$$

is given by the projective transformation Ψ with

$$q_1 = x + \frac{1}{2}(-a_{30} + b_{12})z - \frac{1}{2}a_{12}w, \qquad q_2 = y - \frac{1}{2}b_{21}z + \frac{1}{2}(a_{21} - b_{03})w,$$

$$q_3 = z, \qquad q_4 = w, \qquad p = 1 + b_{12}x + a_{21}y.$$

We can eliminate two more coefficients in the 4-jet. Put

$$j^4(f_1, f_2) = \left(x^2 + a_{03}y^3 + \sum_{i+j=4} c_{ij}x^iy^j, y^2 + b_{30}x^3 + \sum_{i+j=4} d_{ij}x^iy^j\right)$$

where $c_{ij}, d_{ij} \in \mathbb{R}$, then

$$j^4(f_1, f_2) \sim (x^2 + a_{03}y^3 + y\phi_3, y^2 + b_{30}x^3 + x\psi_3),$$

by Ψ with $q_1 = x$, $q_2 = y$, $q_3 = z$, $q_4 = w$, $p = 1 - c_{40}z - d_{04}w$. Here ϕ_3 and ψ_3 mean homogeneous polynomials of degree 3.

Then

$$j^4(f_1, f_2) \sim \begin{cases} (x^2 + y^3 + y\phi_3, y^2 + \alpha x^3 + x\psi_3) & \text{if } a_{03}, b_{30} \neq 0; \\ (x^2 + y^3 + y\phi_3, y^2 + x\psi_3) & \text{if } a_{03} \neq 0 \text{ and } b_{30} = 0; \\ (x^2 + y\phi_3, y^2 + x\psi_3) & \text{if } a_{03} = b_{30} = 0 \end{cases}$$

where $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

2.4. Parabolic case. Suppose that $j^2 f = (x^2, xy)$ and write

$$j^3(f_1, f_2) = \left(x^2 + \sum_{i+j=3} a_{ij} x^i y^j, xy + \sum_{i+j=3} b_{ij} x^i y^j \right)$$

where $a_{ij}, b_{ij} \in \mathbb{R}$. It is easy to show that

$$j^3(f_1, f_2) \sim (x^2 + a_{12}xy^2 + a_{03}y^3, xy + \bar{b}_{12}xy^2 + b_{03}y^3)$$

where $\bar{b}_{12} = b_{12} - \frac{1}{2}a_{21}$. If $a_{03} \neq 0$, then

$$j^3(f_1, f_2) \sim (x^2 + (a_{12} + 3b_{03})xy^2 + a_{03}y^3, xy)$$

with the equivalence given by Ψ with

$$q_1 = x - \frac{(-\bar{b}_{12}a_{03} + 3a_{12}b_{03} + 3b_{03}^2)}{a_{03}} w,$$

$$q_2 = \frac{b_{03}}{a_{03}} x + y + \frac{b_{03}^2(a_{12}b_{03} - a_{03}\bar{b}_{12})}{a_{03}^3} z - \frac{b_{03}(2b_{03}^2 + \bar{b}_{12}a_{03})}{a_{03}^2} w, \qquad q_3 = z,$$

$$q_4 = \frac{b_{03}}{a_{03}} z + w, \qquad p = 1 + \frac{b_{03}^2(a_{12} + b_{03})}{a_{03}^2} x - \frac{(-2\bar{b}_{12}a_{03} + 4a_{12}b_{03} + 3b_{03}^2)}{a_{03}} y.$$

Then the 4-jet can be written in the form

$$j^4(f_1, f_2) \sim (x^2 + \alpha xy^2 + y^3 + y\phi_3, xy + x\psi_3),$$

where $\alpha = \frac{(a_{12}+3b_{03})}{a_{03}^{3/2}}$, ϕ_3 and ψ_3 mean homogeneous polynomials of degree 3.

If $a_{03} = 0$ but $a_{12} \neq 0$, we obtain

$$j^3(f_1, f_2) \sim (x^2 + xy^2, xy + \beta y^3)$$

with the projective transformation Ψ given by

$$q_1 = a_{12}x + a_{12}b_{12}w, \qquad q_2 = y, \qquad q_3 = a_{12}^2 z,$$

$$q_4 = a_{12}w, \qquad p = 1 + 2b_{12}y,$$

where $\beta = \frac{b_{03}}{a_{12}}$. If we put

$$j^4(f_1, f_2) = \left(x^2 + xy^2 + \sum_{i+j=4} a_{ij} x^i y^j, xy + \beta y^3 + \sum_{i+j=4} b_{ij} x^i y^j \right),$$

then $\beta \neq 0$ leads to

$$j^4(f_1, f_2) \sim (x^2 + xy^2 + \tilde{\alpha} y^4, xy + \beta y^3 + \phi_4)$$

by a projective transformation Ψ with

$$q_1 = x + \frac{1}{2}(-q_{21}^2 + p_1)z + (3\beta q_{21} - 3q_{21})w,$$

$$q_2 = y + q_{21}x + \frac{1}{2}(-2\beta q_{21}^3 + q_{21}^3 + p_1 q_{21})z + \frac{1}{2}(-q_{21}^2 + p_1)w,$$

$$q_3 = z, \qquad q_4 = q_{21}z + w, \qquad p = 1 + p_1 x - (-6\beta q_{21} - 4q_{21})y + p_3 z + p_4 w,$$

where $p_1 = \frac{1}{\Lambda^2}\xi_1$, $p_3 = \frac{1}{\Lambda^4}\xi_2$, $p_4 = \frac{1}{\Lambda^3}\xi_3$, $q_{21} = -\frac{a_{13}}{\Lambda}$, ξ_i are combinations of the coefficients of the 4-jet, $\Lambda = 6\beta^2 + 4\tilde{\alpha} - 15\beta + 5 \neq 0$ and $\tilde{\alpha} = a_{44}$. Note that the terms with degree 4 of $j^4(f^1, f^2)$ can not be removed if $\Lambda = 0$. The ϕ_4 is a homogeneous polynomial of degree 4.

2.5. Inflection case. Suppose that $j^2 f = (x^2 + y^2, 0)$ and write

$$j^3(f_1, f_2) = \left(x^2 + y^2 + \sum_{i+j=3} a_{ij} x^i y^j, \sum_{i+j=3} b_{ij} x^i y^j \right).$$

Let $b_{30} - b_{12} \neq 0$. It follows that

$$j^3(f_1, f_2) \sim \left(x^2 + y^2 + \alpha x^2 y, \sum_{i+j=3} b_{ij} x^i y^j \right)$$

by Ψ with

$$q_1 = x, \qquad q_2 = y, \qquad q_3 = z + \frac{a_{30} - a_{12}}{b_{30} - b_{12}} w, \qquad q_4 = w,$$

$$p = 1 - \frac{a_{30}b_{12} - a_{12}b_{30}}{b_{30} - b_{12}} x - \frac{a_{30}b_{03} - a_{12}b_{03} - a_{03}b_{30} + a_{03}b_{12}}{b_{30} - b_{12}} y$$

where α is a scalar constant. Now, we take

$$j^4(f_1, f_2) = \left(x^2 + y^2 + \alpha x^2 y + \sum_{i+j=4} c_{ij} x^i y^j, \phi_3 + \sum_{i+j=4} d_{ij} x^i y^j \right),$$

where $c_{ij}, d_{ij} \in \mathbb{R}$, then it follows that

$$j^4(f_1, f_2) \sim (x^2 + y^2 + \alpha x^2 y + y\psi_3, \phi_3 + \phi_4)$$

by Ψ with $q_1 = x$, $q_2 = y$, $q_3 = z$, $q_4 = w$, $p = 1 + c_{40}z$. Here ϕ_k and ψ_k mean homogeneous polynomials of degree k.

Next, suppose that $j^2 f = (xy, 0)$ and write

$$j^3(f_1, f_2) = \left(xy + \sum_{i+j=3} a_{ij} x^i y^j, \sum_{i+j=3} b_{ij} x^i y^j \right).$$

If $b_{03} \neq 0$, then

$$j^3(f_1, f_2) \sim \left(xy + \alpha x^3, \sum_{i+j=3} b_{ij} x^i y^j \right)$$

by Ψ with

$$q_1 = x, \qquad q_2 = y, \qquad q_3 = z + \frac{a_{03}}{b_{03}} w, \qquad q_4 = w,$$

$$p = 1 + \frac{a_{21}b_{03} - a_{03}b_{21}}{b_{03}} x + \frac{a_{12}b_{03} - a_{03}b_{12}}{b_{03}} y$$

where α is a scalar constant. Finally, we consider

$$j^4(f_1, f_2) = \left(xy + \alpha x^3 + \sum_{i+j=4} c_{ij} x^i y^j, \phi_3 + \sum_{i+j=4} d_{ij} x^i y^j \right),$$

where $c_{ij}, d_{ij} \in \mathbb{R}$. Thus, it follows that

$$j^4(f_1, f_2) \sim (xy + \alpha x^3 + \bar{\xi}_4, \phi_3 + \phi_4)$$

by Ψ with $q_1 = x$, $q_2 = y$, $q_3 = z$, $q_4 = w$, $p = 1 + c_{22}z$. The ϕ_k means a homogeneous polynomial of degree k and $\bar{\xi}_4$ is a homogeneous polynomial of degree 4 without the term $x^2 y^2$.

3. Appendix

Consider a point $p \in \mathbb{P}^4 - M$, which is sometimes called *a view point*, and define $\pi_p : \mathbb{P}^4 - \{p\} \to \mathbb{P}^3$ as the canonical projection which maps $x \in \mathbb{P}^4 - \{p\}$ to the line generated by $x - p$. *The central projection of the surface M from $p \in \mathbb{P}^4 - M$* is given by the composite map

$$\varphi_{p,M} := \pi_p \circ \imath : M \to \mathbb{P}^3$$

(see also [23]).

Name	Normal form	$\mathscr{A}_e$-cod.
immersion	$(x, y, 0)$	0
cross-cap	(x, y^2, xy)	0
$S_k^{\pm}$	$(x, y^2, y^3 \pm x^{k+1}y)$	$k = 1, 2, 3, 4$
$B_k^{\pm}$	$(x, y^2, x^2y \pm y^{2k+1})$	$k = 2, 3, 4$
$C_k^{\pm}$	$(x, y^2, xy^3 \pm x^k y)$	$k = 3, 4$
$H_k^{\pm}$	$(x, xy \pm y^{3k-1}, y^3)$	$k = 2, 3, 4$
$F_4^{\pm}$	$(x, y^2, x^3y + y^5)$	4
$P_3(c)$	$(x, xy + y^3, xy^2 + cy^4),\ c \neq 0, \frac{1}{2}, 1, \frac{3}{2}$	3
$P_4(0)$	$(x, xy + y^3, xy^2 + y^7)$	4
$P_4(\frac{1}{2})$	$\left(x, xy + y^3, xy^2 + \frac{1}{2}y^4 + y^5\right)$	4
$P_4(1)$	$(x, xy + y^3, xy^2 + y^4 + y^6)$	4
$P_4(\frac{3}{2})$	$\left(x, xy + y^3, xy^2 + \frac{3}{2}y^4 + y^5\right)$	4
R_4	$(x, xy + y^6 + by^7, xy^2 + y^4 + cy^6)$	4
T_4	$(x, xy + y^3, y^4)$	4
X_4	$(x, y^3, x^2y + xy^2 + y^4)$	4
Y_4	$(x, y^3 - x^2y, xy^2 + y^4)$	4

Table 3. The $\mathscr{A}$-classification of germs of $\mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ of corank one with $\mathscr{A}_e$-codimension ≤ 4 [14, 15].

Strata in Tables 1 can be divided into finer ones when we consider $\mathscr{A}$-types of germs of central projections. Here two map germs $g, h : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ are said to be $\mathscr{A}$-equivalent if and only if there exist diffeomorphism germs σ, τ of the source and the target at the origins such that $h = \tau \circ g \circ \sigma^{-1}$. In Mond's Ph.D. thesis [14], he first obtained the $\mathscr{A}$-classification of map germs $\mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ of corank one as in Table 3 (see also [15, 16]). Then he calculated the condition for a surface germ to allow each $\mathscr{A}$-type as the singularity of the central projection. Thus we can obtain the finer stratification of the jet space of Monge forms from his calculations. Table 4 shows the list of the above finer strata with codimension ≤ 2.

Finally we briefly explain the notations in Table 4. We say that a line on the tangent plane at a point $\boldsymbol{x} \in M$ is an *asymptotic line of M* if the germ of $\varphi_{p,M}$ at $\boldsymbol{x}$ is equivalent to one of singularities worse than the crosscap (S_0-type) for all view points p on the line. The fourth column in Table 4 shows types of central projections from view points on asymptotic lines. Since the $\mathscr{A}$-type of the central projection depends on the position of the view point $p \in \mathbb{P}^4 - M$, the table is a little bit complicated. In general, for almost all view points on the asymptotic lines, central projections of surfaces give the less degenerated $\mathscr{A}$-types of singularities such as S_1, B_2, H_2, H_3 and $P_3(c)$. On the other hand, some degenerated singularities appear for view points at some discrete points on asymptotic lines. The types written inside brackets in Table 4 mean the

Type	Name	cod.	Projection
Π_{E_i}	Π_E	0	—
Π_S	$\Pi_{S(2)}$	0	$S_1(S_2) \mid S_1(S_2)$
	$\Pi_{S(3)}$	1	$S_1(S_3) \mid S_1(S_2)$
	$\Pi^*_{S(3)}$	2	$S_1(S_3) \mid S_1(S_3)$
	$\Pi_{S(4)}$	2	$S_1(S_4) \mid S_1(S_2)$
Π_B	$\Pi_{BC(3,3)}$	1	$B_2(B_3, C_3) \mid S_1(S_2)$
	$\Pi_{BC(3,4)}$	2	$B_2(B_3, C_4) \mid S_1(S_2)$
	$\Pi_{BF(3,4)}$	2	$B_2(B_3, F_4) \mid S_1(S_2)$
	$\Pi^*_{BC(3,3)}$	2	$B_2(B_3, C_3) \mid S_1(S_3)$
	$\Pi_{BC(4,3)}$	2	$B_2(B_4, C_3) \mid S_1(S_2)$
Π_{2B}	$\Pi_{2BC(3,3)}$	2	$B_2(B_3, C_3) \mid B_2(B_3, C_3)$
Π_H	$\Pi_{H(2)}$	1	H_2
	$\Pi_{H(4)}$	2	$H_3(H_4)$
Π_P	$\Pi_{P(3)}$	2	$P_3(c)$
Π_I^+	Π_I^+	2	$S_1, S_2(S_3), B_2(B_3)$
Π_I^-	Π_I^-	2	$S_1, S_2(S_3), B_2(B_3), H_2$

Table 4. Strata of codimension ≤ 2 induced from the $\mathscr{A}$-orbits. $i = 1, 2$ or 3.

latter types which appear for special view points. For instance, a surface germ of the $\Pi_{BF(3,4)}$-type has two asymptotic lines. The central projection gives the B_2-type for almost all view points on one side of the asymptotic lines, and the B_3 and F_4-type for some discrete view points on the same line; it gives the S_1-type for almost all view points on the other side of the lines, and the S_2-type for some discrete view points on the line.

Remark 1. (1) Surface germs of the elliptic type (Π_{E_1}, Π_{E_2} and Π_{E_3}) have no asymptotic lines.

(2) Surface germs of the hyperbolic type (Π_S, Π_B and Π_{2B}) have two asymptotic lines, and the vertical side line in the column distinguishes types of singularities for the above different asymptotic lines.

(3) Surface germs of the Π_H or Π_P-type have only one asymptotic line. It is interesting that there are no special positions on the asymptotic line for the $\Pi_{H_{(2)}}$ or $\Pi_{P_{(3)}}$-type. That is, the central projection gives the same types of singularities for all view points on the asymptotic lines.

(4) For surface germs of the inflection types (Π_I^+ and Π_I^-), central projections give the S_1-type for almost all view points on the tangent planes.

There are some special lines on the tangent plane, and the central projection gives more degenerate singularities noted as $S_2(S_3)$, $B_2(B_3)$ or H_2 (the use of the bracket follows the previous convention). The configurations of these singularities are given in [4].

Remark 2. In [14], we can see the explicit conditions defining strata in Table 3. Since Mond's calculation begins with the general Monge forms $f = (\sum_{i+j\geq 2} a_{ij}x^iy^j, \sum_{i+j\geq 2} b_{ij}x^iy^j)$ with $a_{ij}, b_{ij} \in \mathbb{R}$, the list of conditions are very big. By using normal forms in Table 1, the conditions can be made relatively small. For instance, take the Monge form of the Π_S-type, and write $f = (x^2 + y^3 + \sum_{i+j\geq 4} a_{ij}x^iy^j, y^2 + \alpha x^3 + \sum_{i+j\geq 4} b_{ij}x^iy^j)$ with $\alpha, a_{ij}, b_{ij} \in \mathbb{R}$, $\alpha \neq 0$ and $a_{40} = b_{04} = 0$. Then the conditions $a_{41} \cdot b_{14} \neq 0$ and $a_{31} = b_{13} = 0$ determine the proper stratum of the $\Pi^*_{S(3)}$-type.

Acknowledgement

We would like to thank Takashi Nishimura and Farid Tari for organizing the JSPS-CAPES no.002/14 bilateral project in 2014–2015. The authors are supported by the project for their stays in ICMC-USP and Hokkaido University, respectively. The first author thanks also the FAPESP no.2012/00066-9 for supporting part of this work. The second author thanks also JSPS KAKENHI Grant Numbers 16J02200 for supporting part of this work. We are also very grateful to Farid Tari and Toru Ohmoto for their supervisions, and the referee for the valuable comments.

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