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# The skew growth functions for the monoid of type $\mathrm{B_{ii}}$ and others

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**Abstract.** For a class of positive homogeneously presented cancellative monoids whose heights are greater than or equal to 2, we will present several explicit calculations of the skew growth functions for them. By the inversion formula, the spherical growth functions for them can be determined. For most of them, the direct calculations are not known. The datum of certain lemmas for proving the cancellativity of the monoids are indispensable to the calculations of the skew growth functions. By improving the technique to show the lemmas, we succeed in the calculations.

## 1. Introduction

Let $M$ be a positive homogeneously finitely presented monoid $\langle L|R\rangle_{mo}$ that satisfies the cancellation condition (i.e. $axb = ayb$ implies $x = y$). Due to the homogeneity of the defining relations in the monoid $M$, we naturally define a map $\deg : M \to \mathbb{Z}_{\geq 0}$ defined by assigning to each equivalence class of words the length of the words. In [S1], by considering the set $\mathrm{Tmcm}(M)$ of all towers $T = (I_0, J_1, J_2, \dots, J_n)$ in $M$, the author defined the *skew growth function* (see § 3 for details) as

$$N_{M,\deg}(t) := 1 + \sum_{T \in \mathrm{Tmcm}(M)} (-1)^{\#J_1 + \cdots + \#J_n - n + 1} \sum_{\Delta \in \mathrm{mcm}(J_n)} t^{\deg(\Delta)}.$$

In this article, for four kinds of positive homogeneously presented cancellative monoids $G^+_{\mathrm{B_{ii}}}$, $G^+_m$, $H^+_m$ and $M_{\mathrm{abel},m}$, we will present several explicit calculations of the skew growth functions for them. The monoid $G^+_{\mathrm{B_{ii}}}$ is studied in [I1]. The presentation of it is associated with a Zariski-van Kampen presentation of the fundamental group of the complement of a certain divisor in $\mathbb{C}^3$. The difining equation of the divisor is $z(-2y^3 + 4x^3z + 18xyz + 27z^2)$. The monoids $G^+_m$, $H^+_m$ and $M_{\mathrm{abel},m}$ are constructed artificially, for which the towers of them do not stop on the first stage $J_1$. The presentations of the monoids $G^+_{\mathrm{B_{ii}}}$, $G^+_m$, $H^+_m$ and $M_{\mathrm{abel},m}$ are the following

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$$G^+_{\mathrm{B_{ii}}} := \left\langle a, b, c \,\middle|\, \begin{array}{l} cbb = bba, \\ ab = bc, \\ ac = ca \end{array} \right\rangle_{mo},$$

$$G^+_m := \left\langle a, b, c \,\middle|\, \begin{array}{l} cb^m = b^m a, \\ ab = bc, \\ ac = ca \end{array} \right\rangle_{mo} \qquad (m = 3, 4, \dots),$$

$$H^+_m := \left\langle a, b, c \,\middle|\, \begin{array}{l} b(ab)^m ba = cb(ab)^m b, \\ ab = bc, \\ ac = ca \end{array} \right\rangle_{mo} \qquad (m = 1, 2, \dots),$$

$$M_{\mathrm{abel}, m} := \left\langle a, b \,\middle|\, \begin{array}{l} a^m = b^m, \\ ab = ba \end{array} \right\rangle_{mo} \qquad (m = 2, 3, \dots).$$

For a class of positive homogeneously presented cancellative monoids whose heights are greater than or equal to 2, calculations of the skew growth functions have not been known yet. For the calculations, the datum of certain lemmas for proving the cancellativity of the monoids are indispensable. The results of calculations of the skew growth functions are the following

$$N_{G^+_{\mathrm{B_{ii}}}, \deg}(t) = \frac{(1-t)^4}{1 - t + t^2},$$

$$N_{G^+_m, \deg}(t) = (1-t)(t^{m+2} + t^{m+1} - 2t + 1),$$

$$N_{H^+_m, \deg}(t) = (1-t)(t^{2m+5} + t^{2m+4} + t^{2m+3} - 2t + 1),$$

$$N_{M_{\mathrm{abel}, m}, \deg}(t) = \frac{(1-t)^2}{1 - t^m}.$$

The *spherical growth function* for a monoid $M$ is defined as

$$P_{M, \deg}(t) := \sum_{u \in M} t^{\deg(u)}.$$

In [S1], K. Saito has shown the inversion formula for $M$ with respect to the map $\deg : M \to \mathbb{Z}_{\geq 0}$

$$P_{M, \deg}(t) \cdot N_{M, \deg}(t) = 1.$$

Hence, by the inversion formula, we can calculate the spherical growth function $P_{M, \deg}(t)$ for the monoids $G^+_{\mathrm{B_{ii}}}$, $G^+_m$, $H^+_m$ and $M_{\mathrm{abel}, m}$.

Let us explain more details of the contents. In analogy with the spherical growth function for a finitely generated group, the spherical growth function for a monoid is defined. That has been studied by several authors ([A-N] [B]

[Bro] [Del] [I1] [S2, S3, S4, S5] [Xu]). If $M = \langle L|R \rangle_{mo}$ satisfies the **condition** $\mathscr{L}$ that any subset $J$ of $I_0$ (:= the image of the set $L$ in $M$) admits either the least right common multiple $\Delta_J$ or no common multiple in $M$, then the inversion function $P_{M,\deg}(t)^{-1}$ is given in a form of polynomial. Since a positive homogeneously presented cancellative monoid $M = \langle L|R \rangle_{mo}$ does not always satisfy the **condition** $\mathscr{L}$, if we try to generalize the formula, the consideration to obtain the above formula is invalid. To resolve this obstraction, for a subset $J$ of $I_0$ we will examine the set $\mathrm{mcm}(J)$ of *minimal common right multiples* of elements of $J$. However, the datum $\{\mathrm{mcm}(J)\}_{J \subset I_0}$ is still not sufficient to recover the inversion formula, since a subset $J'$ of $\mathrm{mcm}(J)$ in general may have common right multiples. Thus we need to consider the set $\mathrm{mcm}(J')$ for a subset $J'$ of $\mathrm{mcm}(J)$. Then, we may again need to consider $\mathrm{mcm}(J'')$ for a subset $J''$ of $\mathrm{mcm}(J')$, and so on. Repeating this process, we are naturally led to consider a notion of *tower*: a finite sequence $J_1, J_2, \dots, J_n$ of subsets of $M$ such that $J_1 \subset I_0, J_2 \subset \mathrm{mcm}(J_1), \dots, J_n \subset \mathrm{mcm}(J_{n-1})$. In [S1], K. Saito has succeeded in generalizing the inversion formula for a rather wider class of monoids (in this article, we explain it in a restricted form).

For the set $\mathrm{Tmcm}(M)$ of all towers $T = (I_0, J_1, J_2, \dots, J_n)$ in $M$, we put

$$h(M, \deg) := \max\{n \mid T = (I_0, J_1, J_2, \dots, J_n) \in \mathrm{Tmcm}(M)\}$$

and call it the *height* of the monoid $M$. The inversion formula covers all the cases $0 \le h(M, \deg) \le \infty$. For a non-abelian monoid $M = \langle L|R \rangle_{mo}$ whose $h(M, \deg)$ is equal to $\infty$, one may think that calculations of the skew growth functions are not practicable. However, in § 5, we will carry out the non-trivial calculation for the monoid $G^+_{\mathrm{B_{ii}}}$, partially because for any tower $T = (I_0, J_1, J_2, \dots, J_n)$ the set $\mathrm{mcm}(J_i)$ can be calculated explicitly for each $J_i$ due to Lemma 3. For the same reason, for the monoids $G^+_m$ and $H^+_m$ whose $h(M, \deg)$ is equal to 2 and the abelian monoid $M_{\mathrm{abel},m}$ whose $h(M, \deg)$ is equal to $\infty$, we can calculate the skew growth functions in § 5.

As far as we know, for non-abelian monoids that do not satisfy the **condition** $\mathscr{L}$, there are few examples for which the cancellativity of them has been shown, since the pre-existing technique to show the cancellativity has only limited applicability ([G] [B-S] [Deh1] [Deh2]). For calculations of the skew growth functions, improvement of the technique to show the cancellativity is expected. In [Deh1], [Deh2], if presentation of a positive homogeneously presented monoid satisfies some condition, called *completeness*, the cancellativity of it can be trivially checked. However, in general, the presentation of a monoid is not complete. When the presentation is not complete, to obtain a complete presentaion, some procedure, called *completion*, is carried out. From our experience, for most of non-abelian monoids that do not satisfy the

**condition** $\mathscr{L}$, these procedures do not finish in finite steps. For monoids of this kind, nothing is discussed in [Deh1], [Deh2]. Thus we attempt improving the technique for this class of monoids. On the other hand, the presentations of the examples $G^+_{\mathrm{B_{ii}}}$, $G^+_m$ and $H^+_m$ are not complete and the procedures do not finish in finite steps (Remark 5). Nevertheless, in §4, we show the cancellativity of them successfully by improving the technique. Other successful examples are contained in [I1], [I2], [S-I].

In §6 we will deal with two monoids $M_4$ and $G^+(4_1)$ whose towers do not stop on the first stage $J_1$. The skew growth functions for them can be calculated with comparative ease.

## 2. Positive homogeneous presentation

In this section, we first recall from [S-I], [B-S] some basic definitions and notations. Secondly, for a positive homogeneously finitely presented group

$$G = \langle L|R\rangle,$$

we associate a monoid defined by it. We give some basic definitions in a positive homogeneously presented monoid. Lastly, we define two operations on the set of subsets of a monoid.

First, we recall from [S-I] basic definitions on a monoid $M$.

Definition 1. 1. *A monoid $M$ is called* cancellative, *if a relation $AXB = AYB$ for $A, B, X, Y \in M$ implies $X = Y$.*

2. *For two elements $u$, $v$ in $M$, we denote*

$$u|_l v$$

*if there exists an element $x$ in $M$ such that $v = ux$. We say that $u$ divides $v$ from the left, or, $v$ is a right-multiple of $u$.*

3. *We say that $M$ is* conical, *if* 1 *is the only invertible element in $M$.*

Next, we recall from [B-S] some terminologies and concepts. Let $L$ be a finite set. We denote by $F(L)$ the free group generated by $L$, and by $L^*$ the free monoid generated by $L$ inside $F(L)$. We call the elements of $F(L)$ *words* and the elements of $L^*$ *positive words*. The empty word $\varepsilon$ is the identity element of $L^*$. Let $G = \langle L|R\rangle$ be a positive homogeneously presented group (i.e. the set $R$ of relations consists of those of the form $R_i = S_i$ where $R_i$ and $S_i$ are positive words of the same length), where $R$ is the set of relations. We often use the same symbols for the images of the letters and words under the quotient homomorphism $F(L) \to G$ and the equivalence relation on elements $A$ and $B$ in $G$ is denoted by $A = B$.

Next, we recall from [S-I], [I1] some basic concepts on positive homogeneously presented monoid.

Definition 2. *Let $G = \langle L|R\rangle$ be a positive homogeneously finitely presented group, where $L$ is the set of generators (called alphabet) and $R$ is the set of relations. Then we associate a monoid $G^+ = \langle L|R\rangle_{mo}$ defined as the quotient of the free monoid $L^*$ generated by $L$ by the equivalence relation defined as follows:*

i) *two words $U$ and $V$ in $L^*$ are called* elementarily equivalent *if either $U = V$ in $L^*$ or $V$ is obtained from $U$ by substituting a substring $R_i$ of $U$ by $S_i$ where $R_i = S_i$ is a relation of $R$ ($S_i = R_i$ is also a relation if $R_i = S_i$ is a relation),*

ii) *two words $U$ and $V$ in $L^*$ are called* equivalent, *denoted by $U \doteq V$, if there exists a sequence $W_0, W_1, \dots, W_n$ of words in $L^*$ for $n \in \mathbb{Z}_{\geq 0}$ such that $U = W_0$, $V = W_n$ and $W_i$ is elementarily equivalent to $W_{i-1}$ for $i = 1, \dots, n$.*

*Due to the homogeneity of the relations, we define a homomorphism:*

$$\deg : G^+ \to \mathbb{Z}_{\geq 0}$$

*by assigning to each equivalence class of words the length of the words.*

Remark 1. *For a positive homogeneously presented group $G = \langle L|R\rangle$, the associated monoid $G^+ = \langle L|R\rangle_{mo}$ is conical.*

Remark 2. *In* [S1], *for a monoid $M$, the quotient set $M/{\sim}$ is considered, where the equivalence relation $\sim$ on $M$ is defined by putting $u \sim v \Leftrightarrow_{\mathrm{def.}} u|_l v \,\&\, v|_l u$. Due to the conicity, if $M$ is a positive homogeneously presented monoid, then we see that $M/{\sim} = M$.*

Lastly, we consider two operations on the set of subsets of a monoid $M$. For a subset $J$ of $M$, we put

$$\mathrm{cm}_r(J) := \{u \in M \mid j|_l u, \text{ for any } j \in J\},$$

$$\min_r(J) := \{u \in J \mid \exists v \in J \text{ such that } v|_l u \Rightarrow v = u\},$$

and their composition: the set of *minimal common multiples* of the set $J$ by

$$\mathrm{mcm}(J) := \min_r(\mathrm{cm}_r(J)).$$

## 3. Generating functions $P_{M,\deg}$ and $N_{M,\deg}$

In this section, for a positive homogeneous presented cancellative monoid

$$M = \langle L|R\rangle_{mo},$$

we define a spherical growth function $P_{M,\deg}$ and a skew growth function $N_{M,\deg}$. Next, we recall from [S1] the inversion formula for the spherical growth function of $M$.

First, we introduce a concept of towers of minimal common multiples in $M$.

Definition 3. *A* tower *of* $M$ *of height* $n \in \mathbb{Z}_{\geq 0}$ *is a sequence*

$$T := (I_0, J_1, J_2, \dots, J_n)$$

*of subsets of* $M$ *satisfying the followings.*

i) $I_0 :=$ *the image of the set* $L$ *in* $M$.

ii) $\mathrm{mcm}(J_k) \neq \emptyset$ *and we put* $I_k := \mathrm{mcm}(J_k)$ *for* $k = 1, \dots, n$.

iii) $J_k \subset I_{k-1}$ *such that* $1 < \# J_k < \infty$ *for* $k = 1, \dots, n$.

*Here, we call* $I_0$, $J_k$ *and* $I_k$, *the* ground, *the* $k$th stage *and the* set of minimal common multiples on the $k$th stage *of the tower* $T$, *respectively. In particular, the set of minimal common multiples on the top stage is denoted by* $|T| := I_n$.

*The set of all towers of* $M$ *shall be denoted by* $\mathrm{Tmcm}(M)$. *We put*

$$h(M, \deg) := \max\{\textit{the height of } T \in \mathrm{Tmcm}(M)\}$$

*and call it the* height *of the monoid* $M$.

Remark 3. i) *It is clear that* $M$ *is a free monoid if and only if* $h(M, \deg) = 0$.

ii) *All of the monoids discussed in* [A-N], [B-S], [S2], [S3] *have* $h(M, \deg) \leq 1$.

iii) *For the following cancellative monoid* $G^+_{\mathrm{B_{ii}}}$, *we have* $h(G^+_{\mathrm{B_{ii}}}, \deg) = \infty$ *(see Proposition* 6 *in* §5*).*

iv) *For the two cancellative monoids* $G^+_m$ *and* $H^+_m$ $(m = 1, 2, \dots)$, *we have* $h(G^+_m, \deg) = 2$ *(see Proposition* 8 *in* §5*) and* $h(H^+_m, \deg) = 2$ *(see Proposition* 11 *in* §5*).*

v) *For the abelian cancellative monoid* $M_{\mathrm{abel},m}$ $(m = 2, 3, \dots)$, *we have* $h(M_{\mathrm{abel},m}, \deg) = \infty$ *(see Lemma* 6 *in* §5*).*

Secondly, we define a spherical growth function $P_{M,\deg}$ and a skew growth function $N_{M,\deg}$. In the previous section, we have fixed a degree map $\deg : M \to \mathbb{Z}_{\geq 0}$. Then, we define the *spherical growth function of the monoid* $(M, \deg)$ by

$$P_{M,\deg} := \sum_{u \in M} t^{\deg(u)}.$$

We define the *skew growth function of the monoid* $(M, \deg)$ by

$$N_{M,\deg}(t) := 1 + \sum_{T \in \mathrm{Tmcm}(M)} (-1)^{\#J_1+\cdots+\#J_n-n+1} \sum_{\Delta \in |T|} t^{\deg(\Delta)}. \tag{3.1}$$

REMARK 4. *In the definition* (3.1)*, we can write down directly the coefficient of the term t. Namely, we write*

$$N_{M,\deg}(t) = 1 - \#(I_0)t + \sum_{\textit{height of } T \geq 1} (-1)^{\#J_1+\cdots+\#J_n-n+1} \sum_{\Delta \in |T|} t^{\deg(\Delta)}.$$

*Therefore, if $M$ is a free monoid of rank $n$, then we have $N_{M,\deg}(t) = 1 - nt$.*

Lastly, we recall from [S1] the inversion formula for the spherical growth function of the monoid $(M, \deg)$.

THEOREM 1. *We have the inversion formula*

$$P_{M,\deg}(t) \cdot N_{M,\deg}(t) = 1.$$

## 4. Cancellativity of $G_m^+$ and $H_m^+$

In this section, for a preparation for calculations of the skew growth functions for the monoids $G_m^+$ and $H_m^+$ in §5, we prove the cancellativity of them.

**4.1. Cancellativity of $G_m^+$.** In this subsection, we show the cancellativity of the monoid $G_m^+$.

THEOREM 2. *The monoid $G_m^+$ is a cancellative monoid.*

PROOF. First, we remark the following.

PROPOSITION 1. *The left cancellativity on $G_m^+$ implies the right cancellativity.*

PROOF. Consider a map $\varphi : G_m^+ \to G_m^+$, $W \mapsto \varphi(W) := \sigma(rev(W))$, where $\sigma$ is a permutation $\binom{a,b,c}{c,b,a}$ and $rev(W)$ is the reverse of the word $W = x_1x_2\dots x_k$ ($x_i$ is a letter) given by the word $x_k \dots x_2x_1$. In view of the defining relation of $G_m^+$, $\varphi$ is well-defined and is an anti-isomorphism. If $\beta\alpha \doteqdot \gamma\alpha$, then $\varphi(\beta\alpha) \doteqdot \varphi(\gamma\alpha)$, i.e., $\varphi(\alpha)\varphi(\beta) \doteqdot \varphi(\alpha)\varphi(\gamma)$. By using the left cancellativity, we obtain $\varphi(\beta) \doteqdot \varphi(\gamma)$ and, hence, $\beta \doteqdot \gamma$.

The following is sufficient to show the left cancellativity on the monoid $G_m^+$.

Proposition 2. *Let $Y$ be a positive word in $G_m^+$ of length $r \in \mathbb{Z}_{\geq 0}$ and let $X^{(h)}$ be a positive word in $G_m^+$ of length $r-h \in \{r-m+1,\dots,r\}$.*

(i) *If $vX^{(0)} \doteq vY$ for some $v \in \{a,b,c\}$, then $X^{(0)} \doteq Y$.*

(ii) *If $aX^{(0)} \doteq bY$, then $X^{(0)} \doteq bZ$ and $Y \doteq cZ$ for some positive word $Z$.*

(iii) *If $aX^{(0)} \doteq cY$, then $X^{(0)} \doteq cZ$ and $Y \doteq aZ$ for some positive word $Z$.*

(iv-0) *If $bX^{(0)} \doteq cY$, then there exist an integer $k$ $(0 \leq k \leq r-m)$ and a positive word $Z$ such that $X^{(0)} \doteq c^k b^{m-1} a \cdot Z$ and $Y \doteq a^k b^m \cdot Z$.*

(iv-1-$a$) *There do not exist words $X^{(1)}$ and $Y$ that satisfy an equality $ba \cdot X^{(1)} \doteq cY$.*

(iv-1-$b$) *If $bb \cdot X^{(1)} \doteq cY$, then $X^{(1)} \doteq b^{m-2} a \cdot Z$ and $Y \doteq b^m \cdot Z$ for some positive word $Z$.*

(iv-1-$c$) *If $bc \cdot X^{(1)} \doteq cY$, then there exists an integer $k$ $(0 \leq k \leq r-m-1)$ and a positive word $Z$ such that $X^{(1)} \doteq c^k b^{m-1} a \cdot Z$ and $Y \doteq a^{k-1} b^m \cdot Z$.*

*If $m \geq 4$, then, for $2 \leq h \leq m-2$, we need prepare the following propositions* (iv-$h$-$a$) (iv-$h$-$b$) *and* (iv-$h$-$c$).

(iv-$h$-$a$) *There do not exist positive words $X^{(h)}$ and $Y$ that satisfy an equality $b^h a \cdot X^{(h)} \doteq cY$.*

(iv-$h$-$b$) *If $b^{h+1} \cdot X^{(h)} \doteq cY$, then $X^{(h)} \doteq b^{m-h-1} a \cdot Z$ and $Y \doteq b^m \cdot Z$ for some positive word $Z$.*

(iv-$h$-$c$) *There do not exist positive words $X^{(h)}$ and $Y$ that satisfy an equality $b^h c \cdot X^{(h)} \doteq cY$.*

(iv-$(m-1)$-$a$) *If $b^{m-1} a \cdot X^{(m-1)} \doteq cY$, then $X^{(m-1)} \doteq ba \cdot Z$ and $Y \doteq b^m c \cdot Z$ for some positive word $Z$.*

(iv-$(m-1)$-$b$) *If $b^m \cdot X^{(m-1)} \doteq cY$, then $X^{(m-1)} \doteq aZ$ and $Y \doteq b^m \cdot Z$ for some positive word $Z$.*

(iv-$(m-1)$-$c$) *There do not exist positive words $X^{(m-1)}$ and $Y$ that satisfy an equality $b^{m-1} c \cdot X^{(m-1)} \doteq cY$.*

Proof. The statement in Proposition 2 for a positive word $Y$ of word-length $r$ and $X^{(h)}$ of word-length $r-h \in \{r-m+1,\dots,r\}$ will be referred to as $\mathrm{H}_{r,h}$. We will show the general theorem by induction[1]. It is easy to show that, for $r = 0, 1$, $\mathrm{H}_{r,h}$ is true. If a positive word $U_1$ is transformed into $U_2$ by using $t$ single applications of the defining relations of $G_m^+$, then the whole transformation will be said to be of *chain-length* $t$. For the induction hypothesis, we assume

(A) $\mathrm{H}_{s,h}$ is true for $s = 0,\dots,r$ and arbitrary $h$ for transformations of all chain-lengths,

[1] For the proof, we refer to the technique of the triple induction (see proof of Proposition 4 in [I2]).

and

(B) $\mathrm{H}_{r+1,h}$ is true for $0 \le h \le m-1$ for all chain-lengths $\le t$.

We will show the claim $\mathrm{H}_{r+1,h}$ for chain-lengths $t+1$. For the sake of simplicity, we devide the proof into two steps.

**Step 1**. We shall prove the claim $\mathrm{H}_{r+1,h}$ for $h = 0$. Let $X$, $Y$ be of word-length $r+1$, and let

$$v_1 X \doteq v_2 W_2 \doteq \cdots \doteq v_{t+1} W_{t+1} \doteq v_{t+2} Y$$

be a sequence of single transformations of $t+1$ steps, where $v_1, \ldots, v_{t+2} \in \{a, b, c\}$ and $W_2, \ldots, W_{t+1}$ are positive words of length $r+1$. By the assumption $t > 1$, for any index $\tau \in \{2, \ldots, t+1\}$ we can decompose the sequence into two steps

$$v_1 X \doteq v_\tau W_\tau \doteq v_{t+2} Y,$$

in which each step satisfies the induction hypothesis (B).

If there exists $\tau_0$ such that $v_{\tau_0}$ is equal to either to $v_1$ or $v_{t+2}$, then by the induction hypothesis, $W_{\tau_0}$ is equivalent either to $X$ or to $Y$. Hence, we obtain the statement for the $v_1 X \doteq v_{t+2} Y$. Thus, we assume from now on $v_\tau \neq v_1$ and $v_\tau \neq v_{t+2}$ for $1 < \tau \le t+1$.

We suppose that $v_1 = v_{t+2}$. If there exists $\tau_0$ such that $\{v_1 = v_{t+2}, v_{\tau_0}\} \neq \{b, c\}$, then each of the equivalences says the existence of $\alpha, \beta \in \{a, b, c\}$ and positive words $Z_1$, $Z_2$ such that $X \doteq \alpha Z_1$, $W_{\tau_0} \doteq \beta Z_1 \doteq \beta Z_2$ and $Y \doteq \alpha Z_2$. Applying the induction hypothesis (A) to $\beta Z_1 \doteq \beta Z_2$, we get $Z_1 \doteq Z_2$. Hence, we obtain the statement $X \doteq \alpha Z_1 \doteq \alpha Z_2 \doteq Y$. Thus, we exclude these cases from our considerations. Next, we consider the case where $(v_1 = v_{t+2}, v_\tau) = (b, c)$ for $1 < \tau \le t+1$. Namely, we have $v_2 = \cdots = v_{t+1} = c$. Hence, we consider the following case

$$bX \doteq cW_1 \doteq \cdots \doteq cW_{t+1} \doteq bY.$$

Applying the induction hypothesis (B) to each step, we see that there exist positive words $Z_3$ and $Z_4$ such that

$$X \doteq b^{m-1} a \cdot Z_3, \qquad W_1 \doteq b^m \cdot Z_3,$$

$$W_{t+1} \doteq b^m \cdot Z_4, \qquad Y \doteq b^{m-1} a \cdot Z_4.$$

Since an equality $W_1 \doteq W_{t+1}$ holds, we see that

$$b^m \cdot Z_3 \doteq b^m \cdot Z_4.$$

By the induction hypothesis, we have $X \doteq Y$.

In the case of $(v_1 = v_{t+2}, v_\tau) = (c, b)$ for $1 < \tau \le t+1$, we can prove the statement in a similar manner.

Suppose $v_1 \neq v_{t+2}$. We consider the following three cases.

Case 1: $(v_1, v_\tau, v_{t+2}) = (a, b, c)$.

Because of the above consideration, we consider the case where $\tau = t+1$, namely

$$aX \doteq bW_{t+1} \doteq cY.$$

Applying the induction hypothesis to each step, we see that there exist positive words $Z_1$ and $Z_2$ such that

$$X \doteq bZ_1, \qquad W_{t+1} \doteq cZ_1,$$

$$W_{t+1} \doteq b^{m-1}a \cdot Z_2, \qquad Y \doteq b^m \cdot Z_2.$$

Thus, we see that $c \cdot Z_1 \doteq b^{m-1}a \cdot Z_2$. Applying the induction hypothesis (A) to this equality, we see that there exists a positive word $Z_3$ such that

$$Z_1 \doteq b^m c \cdot Z_3, \qquad Z_2 \doteq ba \cdot Z_3.$$

Hence, we have $X \doteq cb^{m+1} \cdot Z_3$ and $Y \doteq ab^{m+1} \cdot Z_3$.

Case 2: $(v_1, v_\tau, v_{t+2}) = (a, c, b)$.

We consider the case where $\tau = t+1$, namely

$$aX \doteq cW_{t+1} \doteq bY.$$

Applying the induction hypothesis to each step, we see that there exist positive words $Z_1$ and $Z_2$ such that

$$X \doteq cZ_1, \qquad W_{t+1} \doteq aZ_1,$$

$$W_{t+1} \doteq b^m \cdot Z_2, \qquad Y \doteq b^{m-1}a \cdot Z_2.$$

Thus, we see that $aZ_1 \doteq b^m \cdot Z_2$. Applying the induction hypothesis (A) to this equality, we see that there exists a positive word $Z_3$ such that

$$Z_1 \doteq b^{m+1} \cdot Z_3, \qquad Z_2 \doteq ba \cdot Z_3.$$

Hence, we have $X \doteq b \cdot b^m c \cdot Z_3$ and $Y \doteq c \cdot b^m c \cdot Z_3$.

Case 3: $(v_1, v_\tau, v_{t+2}) = (b, a, c)$.

Then, we consider the following case

$$bX \doteq aW_\tau \doteq cY.$$

Applying the induction hypothesis to each step, we see that there exist positive words $Z_1$ and $Z_2$ such that

$$X \doteq cZ_1, \qquad W_\tau \doteq bZ_1,$$

$$W_\tau \doteq cZ_2, \qquad Y \doteq aZ_2.$$

Moreover, we see that there exist a positive word $Z_3$ and an integer $k \in \mathbb{Z}_{\geq 0}$ such that

$$Z_1 \doteq c^k b^{m-1} a \cdot Z_3, \qquad Z_2 \doteq a^k b^m \cdot Z_3.$$

Thus, we have

$$X \doteq c^{k+1} b^{m-1} a \cdot Z_3, \qquad Y \doteq a^{k+1} b^m \cdot Z_3.$$

**Step 2**. We shall prove the claim $\mathrm{H}_{r+1,h}$ for $0 \leq h \leq m-1$. We will show the general claim $\mathrm{H}_{r+1,h}$ by induction on $h$. The case where $h = 0$ is proved in Step 1. First, we show the case where $h = 1$. Let $X^{(1)}$ be of word-length $r$ and $Y$ of word-length $r+1$. We consider a sequence of single transformations of $t+1$ steps

$$V \cdot X^{(1)} \doteq \cdots \doteq cY,$$

where $V$ is a positive word of length 2. We discuss the following three cases.

Case 1: $V = ba$.

We consider the following case

$$ba \cdot X^{(1)} \doteq \cdots \doteq cY. \tag{4.1}$$

By the result of Step 1, we see that there exists a positive word $Z_1$ and an integer $k \in \mathbb{Z}_{\geq 0}$ such that

$$aX^{(1)} \doteq c^k b^{m-1} a \cdot Z_1, \qquad Y \doteq a^k b^m \cdot Z_1.$$

Applying the induction hypothesis (A), we see that there exists a positive word $Z_2$ such that

$$X^{(1)} \doteq c^k \cdot Z_2, \qquad b^{m-1} a \cdot Z_1 \doteq aZ_2.$$

Moreover, we see that there exists a positive word $Z_3$ such that

$$b^{m-2} a \cdot Z_1 \doteq cZ_3, \qquad Z_2 \doteq bZ_3.$$

By the induction hypothesis, we have a contradiction. Hence, there does not exist positive words $X^{(1)}$ and $Y$ that satisfy the equality (4.1).

Case 2: $V = bb$.

We consider the following case

$$bb \cdot X^{(1)} \doteq V_2 \cdot W_2 \doteq \cdots \doteq V_{t+1} \cdot W_{t+1} \doteq cY,$$

where $V_2$ and $V_{t+1}$ are positive words. It is enough to discuss the case where $(V_2, V_{t+1}) = (bcb^m, ac)$. Applying the induction hypothesis (A) to the equality

$$bcb^m \cdot W_2 \doteq ac \cdot W_{t+1}, \tag{4.2}$$

we see that there exists a positive word $Z_1$ such that $cW_{t+1} \doteq bZ_1$. Applying the induction hypothesis, we see that there exists a positive word $Z_2$ and an integer $k \in \mathbb{Z}_{\geq 0}$ such that

$$W_{t+1} \doteq a^k b^m \cdot Z_2, \qquad Z_1 \doteq c^k b^{m-1} a \cdot Z_2. \tag{4.3}$$

Applying (4.3) to the equality (4.2), we have

$$bcb^m \cdot W_2 \doteq ac \cdot a^k b^m \cdot Z_2.$$

Moreover, we see

$$b^m \cdot W_2 \doteq c^k b^{m-1} a \cdot Z_2. \tag{4.4}$$

We consider the following two cases.

Case 2-1: $k = 0$.
There exists a positive word $Z_3$ such that

$$W_2 \doteq cZ_3, \qquad Z_2 \doteq bZ_3.$$

Thus, we have

$$X^{(1)} \doteq b^{m-1} a \cdot cZ_3 \doteq b^{m-2} a \cdot ba \cdot Z_3,$$
$$Y \doteq ab^m b \cdot Z_3 \doteq b^m \cdot baZ_3.$$

Case 2-2: $k \geq 1$.
Applying the induction hypothesis to the equality (4.4), we see that there exists a positive word $Z_3$ such that

$$W_2 \doteq a^k \cdot Z_3.$$

Thus, we consider the equality $b^m \cdot Z_3 \doteq b^{m-1} a \cdot Z_2$. We see that there exists a positive word $Z_4$ such that

$$Z_2 \doteq bZ_4, \qquad Z_3 \doteq cZ_4.$$

Thus, we have

$$X^{(1)} \doteq b^{m-1} a \cdot a^k c \cdot Z_3 \doteq b^{m-2} a \cdot ba^{k+1} Z_3,$$
$$Y \doteq aa^k b^m b \cdot Z_3 \doteq b^m \cdot ba^{k+1} Z_3.$$

Case 3: $V = bc$.
Then, we consider the following case

$$bc \cdot X^{(1)} \doteq \cdots \doteq cY.$$

By the induction hypothesis, we see that there exist a positive word $Z_1$ and an integer $k \in \mathbb{Z}_{\geq 0}$ such that

$$cX^{(1)} \doteqdot c^k b^{m-1} a \cdot Z_1, \qquad Y \doteqdot a^k b^m \cdot Z_1.$$

We consider the following two cases.

Case 3-1: $k = 0$.

By the induction hypothesis, we see that there exists a positive word $Z_2$ such that

$$X^{(1)} \doteqdot b^m c \cdot Z_2, \qquad Z_1 \doteqdot ba \cdot Z_2.$$

Thus, we have

$$X^{(1)} \doteqdot b^{m-1} a \cdot bZ_2, \qquad Y \doteqdot b^m ba \cdot Z_2 \doteqdot ab^m \cdot bZ_2.$$

Case 3-2: $k \geq 1$.

Then, we have

$$X^{(1)} \doteqdot c^{k-1} b^{m-1} a \cdot Z_1, \qquad Y \doteqdot a^k b^m \cdot Z_1.$$

Second, when $m \geq 4$, we show the claim $\mathrm{H}_{r+1,h}$ $(2 \leq h \leq m-2)$ by induction on $h$. We assume $h = 1, 2, \ldots, j$ $(j \leq m-3)$. The case where $h = 1$ has been proved. Let $X^{(j+1)}$ be of word-length $r - j$ and $Y$ of word-length $r+1$. We consider a sequence of single transformations of $t+1$ steps

$$V \cdot X^{(j+1)} \doteqdot \cdots \doteqdot cY, \tag{4.5}$$

where $V$ is a positive word of length $j+2$. We discuss the following three cases.

Case 1: $V \doteqdot bb^j a$.

Applying the induction hypothesis, we see that there exists a positive word $Z_1$ such that

$$aX^{(j+1)} \doteqdot b^{m-j-1} a \cdot Z_1, \qquad Y \doteqdot b^m \cdot Z_1.$$

By the induction hypothesis, we see that there exists a positive word $Z_2$ such that

$$X^{(j+1)} \doteqdot bZ_2, \qquad b^{m-j-2} a \cdot Z_1 \doteqdot cZ_2.$$

By the induction hypothesis, we have a contradiction. Hence, there do not exist positive words $X^{(j+1)}$ and $Y$ that satisfy the equality (4.5).

Case 2: $V \doteqdot bb^{j+1}$.

Applying the induction hypothesis, we see that there exists a positive word $Z_1$ such that

$$bX^{(j+1)} \doteqdot b^{m-j-1} a \cdot Z_1, \qquad Y \doteqdot b^m \cdot Z_1.$$

Thus, we have $X^{(j+1)} \doteqdot b^{m-j-2} a \cdot Z_1$.

Case 3: $V \doteq bb^j c$.
Applying the induction hypothesis, we see that there exists a positive word $Z_1$ such that

$$cX^{(j+1)} \doteq b^{m-j-1}a \cdot Z_1, \qquad Y \doteq b^m \cdot Z_1.$$

By the induction hypothesis, we have a contradiction. Hence, there do not exist positive words $X^{(j+1)}$ and $Y$ that satisfy the equality (4.5).

Lastly, we show the claim $\mathrm{H}_{r+1,m-1}$. Let $X^{(m-1)}$ be of word-length $r-m+2$ and $Y$ of word-length $r+1$. We consider a sequence of single transformations of $t+1$ steps

$$V \cdot X^{(m-1)} \doteq \cdots \doteq cY, \tag{4.6}$$

where $V$ is a positive word of length $m$. We discuss the following three cases.

Case 1: $V \doteq b^{m-1}a$.
By the above result, we see that there exists a positive word $Z_1$ such that

$$aX^{(m-1)} \doteq ba \cdot Z_1, \qquad Y \doteq b^m \cdot Z_1.$$

By the induction hypothesis, we see that there exists a positive word $Z_2$ such that

$$X^{(m-1)} \doteq ba \cdot Z_2, \qquad Z_1 \doteq cZ_2.$$

Thus, we have $Y \doteq b^m c \cdot Z_2$.

Case 2: $V \doteq b^{m-1}b$.
By the above result, we see that there exists a positive word $Z_1$ such that

$$bX^{(m-1)} \doteq ba \cdot Z_1, \qquad Y \doteq b^m \cdot Z_1.$$

Thus, we have $X^{(m-1)} \doteq aZ_1$.

Case 3: $V \doteq b^{m-1}c$.
By the above result, we see that there exists a positive word $Z_1$ such that

$$cX^{(m-1)} \doteq ba \cdot Z_1, \qquad Y \doteq b^m \cdot Z_1.$$

We have a contradiction. Hence, there do not exist positive words $X^{(m-1)}$ and $Y$ that satisfy the equality (4.6).

This completes the proof of Theorem 2.

**4.2. Cancellativity of $H_m^+$.** In this subsection, we show the cancellativity of the monoid $H_m^+$.

Theorem 3. *The monoid $H_m^+$ is a cancellative monoid.*

Proof. First, we remark the following.

Proposition 3. *The left cancellativity on $H_m^+$ implies the right cancellativity.*

Proof. Consider a map $\varphi : H_m^+ \to H_m^+$, $W \mapsto \varphi(W) := \sigma(rev(W))$, where $\sigma$ is a permutation $\binom{a,b,c}{c,b,a}$. By a similar arguments in the proof in Proposition 1, we can show the statement.

To prove the cancellativity of the monoid $H_m^+$, it suffices to show the following proposition.

Proposition 4. *Let $Y$ be a positive word in $H_m^+$ of length $r \in \mathbb{Z}_{\geq 0}$ and let $X^{(h)}$ be a positive word in $H_m^+$ of length $r - h \in \{2m, \dots, r\}$.*

(i) *If $vX^{(0)} \doteq vY$ for some $v \in \{a,b,c\}$, then $X^{(0)} \doteq Y$.*

(ii) *If $aX^{(0)} \doteq bY$, then $X^{(0)} \doteq bZ$ and $Y \doteq cZ$ for some positive word $Z$.*

(iii) *If $aX^{(0)} \doteq cY$, then $X^{(0)} \doteq cZ$ and $Y \doteq aZ$ for some positive word $Z$.*

(iv) *If $bX^{(0)} \doteq cY$, then there exists an integer $k$ $(0 \leq k \leq r - 2m - 2)$ and a positive word $Z$ such that $X^{(0)} \doteq c^k(ab)^m ba \cdot Z$ and $Y \doteq a^k b(ab)^m b \cdot Z$.*

(v) *If $bb \cdot X^{(1)} \doteq cY$, then $X^{(1)} \doteq c(ab)^{m-1}ba \cdot Z$ and $Y \doteq b(ab)^m b \cdot Z$ for some positive word $Z$.*

*For $2 \leq h \leq r - 2m$, we prepare the following propositions.*

(vi-$h$) *If $c^{h-1}bb \cdot X^{(h)} \doteq bY$, then $X^{(h)} \doteq c(ab)^{m-1}b \cdot Z$ and $Y \doteq (ab)^m ba^{h-1} \cdot Z$ for some positive word $Z$.*

Proof. The statement in Proposition 4 for a positive word $Y$ of word-length $r$ and $X^{(h)}$ of word-length $r - h \in \{r - 2m, \dots, r\}$ will be referred to as $\mathrm{H}_{r,h}$. We will show the general claim by induction. It is easy to show that, for $r = 0, 1$, $\mathrm{H}_{r,h}$ is true. For the induction hypothesis, we assume

(A) $\mathrm{H}_{s,h}$ is true for $s = 0, \dots, r$ and arbitrary $h$ for transformations of all chain-lengths,

and

(B) $\mathrm{H}_{r+1,h}$ is true for $0 \leq h \leq \max\{0, r + 1 - 2m\}$ for all chain-lengths $\leq t$.

We will show the claim $\mathrm{H}_{r+1,h}$ for chain-lengths $t + 1$. For the sake of simplicity, we devide the proof into two steps.

**Step 1**. We shall prove the claim $\mathrm{H}_{r+1,h}$ for $h = 0$. Let $X, Y$ be of word-length $r + 1$, and let

$$v_1 X \doteq v_2 W_2 \doteq \cdots \doteq v_{t+1} W_{t+1} \doteq v_{t+2} Y$$

be a sequence of single transformations of $t + 1$ steps, where $v_1, \dots, v_{t+2} \in \{a,b,c\}$ and $W_2, \dots, W_{t+1}$ are positive words of length $r + 1$. By the assumption $t > 1$, for any index $\tau \in \{2, \dots, t + 1\}$ we can decompose the sequence into two steps

$$v_1X \doteq v_\tau W_\tau \doteq v_{t+2}Y,$$

in which each step satisfies the induction hypothesis (B).

If there exists $\tau_0$ such that $v_{\tau_0}$ is equal to either to $v_1$ or $v_{t+2}$, then by the induction hypothesis, $W_{\tau_0}$ is equivalent either to $X$ or to $Y$. Hence, we obtain the statement for the $v_1X \doteq v_{t+2}Y$. Thus, we assume from now on $v_\tau \neq v_1$ and $v_\tau \neq v_{t+2}$ for $1 < \tau \leq t+1$.

Suppose $v_1 = v_{t+2}$. If there exists $\tau_0$ such that $\{v_1 = v_{t+2}, v_{\tau_0}\} \neq \{b, c\}$, then each of the equivalences says the existence of $\alpha, \beta \in \{a, b, c\}$ and positive words $Z_1$, $Z_2$ such that $X \doteq \alpha Z_1$, $W_{\tau_0} \doteq \beta Z_1 \doteq \beta Z_2$ and $Y \doteq \alpha Z_2$. Applying the induction hypothesis (A) to $\beta Z_1 \doteq \beta Z_2$, we get $Z_1 \doteq Z_2$. Hence, we obtain the statement $X \doteq \alpha Z_1 \doteq \alpha Z_2 \doteq Y$. Thus, we exclude these cases from our considerations. Next, we consider the case where $(v_1 = v_{t+2}, v_\tau) = (b, c)$ for $1 < \tau \leq t+1$. Namely we have $v_2 = \cdots = v_{t+1} = c$. Hence, we consider the following case

$$bX \doteq cW_1 \doteq \cdots \doteq cW_{t+1} \doteq bY.$$

Applying the induction hypothesis (B) to each step, we see that there exist positive words $Z_3$ and $Z_4$ such that

$$X \doteq (ab)^m ba \cdot Z_3, \qquad W_1 \doteq b(ab)^m b \cdot Z_3,$$

$$W_{t+1} \doteq b(ab)^m b \cdot Z_4, \qquad Y \doteq (ab)^m ba \cdot Z_4.$$

Since the equality $W_1 \doteq W_{t+1}$ holds, we see that $X \doteq Y$.

In the case of $(v_1 = v_{t+2}, v_\tau) = (c, b)$ for $1 < \tau \leq t+1$, we can prove the statement in a similar manner.

Suppose $v_1 \neq v_{t+2}$. It suffices to consider the following two cases.

Case 1: $(v_1, v_\tau, v_{t+2}) = (a, b, c)$.

Because of the above consideration, we consider the case where $\tau = t+1$, namely

$$aX \doteq bW_{t+1} \doteq cY.$$

Applying the induction hypothesis to each step, we see that there exist positive words $Z_1$ and $Z_2$ such that

$$X \doteq bZ_1, \qquad W_{t+1} \doteq cZ_1,$$

$$W_{t+1} \doteq (ab)^m ba \cdot Z_2, \qquad Y \doteq b(ab)^m b \cdot Z_2.$$

Thus, we see that $cZ_1 \doteq (ab)^m ba \cdot Z_2$. Applying the induction hypothesis (A) to this equality, we see that there exists a positive word $Z_3$ such that

$$Z_1 \doteq aZ_3, \qquad b(ab)^{m-1} ba \cdot Z_2 \doteq cZ_3.$$

Hence, we have $bbc(ab)^{m-2}ba \cdot Z_2 \doteqdot cZ_3$. Applying the induction hypothesis (A) to this equality, there exists a positive word $Z_4$ such that

$$c(ab)^{m-2}ba \cdot Z_2 \doteqdot c(ab)^{m-1}ba \cdot Z_4, \qquad Z_3 \doteqdot b(ab)^m b \cdot Z_4.$$

Hence, we have $ba \cdot Z_2 \doteqdot abba \cdot Z_4$. Moreover, we see that there exists a positive word $Z_5$ such that

$$Z_2 \doteqdot cba \cdot Z_5, \qquad Z_4 \doteqdot cZ_5.$$

Thus, we have

$$X \doteqdot bab(ab)^m bc \cdot Z_5 \doteqdot c \cdot b(ab)^m bcb \cdot Z_5,$$

$$Y \doteqdot b(ab)^m bcba \cdot Z_5 \doteqdot a \cdot b(ab)^m bcb \cdot Z_5.$$

Case 2: $(v_1, v_\tau, v_{t+2}) = (a, c, b)$.
We consider the case where $\tau = t+1$, namely

$$aX \doteqdot cW_{t+1} \doteqdot bY.$$

Applying the induction hypothesis to each step, we see that there exist positive words $Z_1$ and $Z_2$ such that

$$X \doteqdot cZ_1, \qquad W_{t+1} \doteqdot aZ_1,$$

$$W_{t+1} \doteqdot b(ab)^m b \cdot Z_2, \qquad Y \doteqdot (ab)^m ba \cdot Z_2.$$

Thus, we see that $aZ_1 \doteqdot b(ab)^m b \cdot Z_2$. Applying the induction hypothesis (A) to this equality, we see that there exists a positive word $Z_3$ such that

$$Z_1 \doteqdot bZ_3, \qquad (ab)^m b \cdot Z_2 \doteqdot cZ_3.$$

Hence, there exists a positive word $Z_4$ such that

$$b(ab)^{m-1} b \cdot Z_2 \doteqdot cZ_4, \qquad Z_3 \doteqdot aZ_4.$$

We have $bbc(ab)^{m-2}b \cdot Z_2 \doteqdot cZ_4$. Applying the induction hypothesis (A) to this equality, we see that there exists a positive word $Z_5$ such that

$$c(ab)^{m-2}b \cdot Z_2 \doteqdot c(ab)^{m-1}ba \cdot Z_5, \qquad Z_4 \doteqdot b(ab)^m b \cdot Z_5.$$

Hence, we have $Z_2 \doteqdot cba \cdot Z_5$. Thus, we obtain

$$X \doteqdot cbab(ab)^m b \cdot Z_5 \doteqdot b(ab)^m bacb \cdot Z_5,$$

$$Y \doteqdot (ab)^m bacba \cdot Z_5 \doteqdot c(ab)^m bacb \cdot Z_5.$$

**Step 2**. We shall prove the claim $\mathrm{H}_{r+1,h}$ for $1 \leq h \leq r+1-2m$. We will show the general claim $\mathrm{H}_{r+1,h}$. First, we show the case where $h = 1$. Then, we consider the following case

$$bb \cdot X^{(1)} \doteq \cdots \doteq cY.$$

By the result of Step 1, we see that there exists a positive word $Z_1$ and an integer $k \in \mathbb{Z}_{\geq 0}$ such that

$$bX^{(1)} \doteq c^k (ab)^m ba \cdot Z_1, \qquad Y \doteq a^k b(ab)^m b \cdot Z_1.$$

Thus, we have $bX^{(1)} \doteq ac^k b(ab)^{m-1} ba \cdot Z_1$. Applying the induction hypothesis (A), we see that there exists a positive word $Z_2$ such that

$$X^{(1)} \doteq cZ_2, \qquad bZ_2 \doteq c^k b(ab)^{m-1} ba \cdot Z_1 \doteq c^k bbc(ab)^{m-2} ba \cdot Z_1.$$

We consider the case where $k \geq 1$. By the induction hypothesis, we see that there exists a positive word $Z_3$ such that

$$Z_2 \doteq (ab)^m ba^k \cdot Z_3, \qquad c(ab)^{m-2} ba \cdot Z_1 \doteq c(ab)^{m-1} b \cdot Z_3.$$

Hence we have $ba \cdot Z_1 \doteq abb \cdot Z_3$ and therefore we have $aZ_1 \doteq cb \cdot Z_3$. By the induction hypothesis, there exists a positive word $Z_4$ such that

$$Z_1 \doteq cb \cdot Z_4, \qquad Z_3 \doteq cZ_4.$$

Thus, we have

$$X^{(1)} \doteq c(ab)^m ba^k c \cdot Z_4 \doteq c(ab)^{m-1} ba \cdot cba^k \cdot Z_4,$$

$$Y \doteq a^k b(ab)^m bcb \cdot Z_4 \doteq b(ab)^m b \cdot cba^k \cdot Z_4.$$

Next, we consider the case where $2 \leq k \leq r + 1 - 2m$. We consider the following case

$$c^{h-1} bb \cdot X^{(h)} \doteq \cdots \doteq bY. \tag{4.7}$$

By the result of Step 1, we see that there exists a positive word $Z_1$ and an integer $k_1 \in \mathbb{Z}_{\geq 0}$ such that

$$c^{h-2} bb \cdot X^{(h)} \doteq a^{k_1} b(ab)^m b \cdot Z_1, \qquad Y \doteq c^{k_1} (ab)^m ba \cdot Z_1.$$

By repeating the same process $h - 1$ times, there exist integers $k_2, \ldots, k_{h-1} \in \mathbb{Z}_{\geq 0}$ and a positive word $Z_{h-1}$ such that

$$bb \cdot X^{(h)} \doteq a^{k_{h-1}} \cdot b(ab)^m b \cdot Z_{h-1}.$$

Then, we have $b \cdot X^{(h)} \doteq c^{k_{h-1}} \cdot (ab)^m b \cdot Z_{h-1} \doteq ac^{k_{h-1}} \cdot b(ab)^{m-1} b \cdot Z_{h-1}$. By the induction hypothesis, there exists a positive word $Z_h$ such that

$$X^{(h)} \doteq cZ_h, \qquad c^{k_{h-1}} \cdot b(ab)^{m-1} b \cdot Z_{h-1} \doteq bZ_h.$$

Hence, we have $bZ_h \doteqdot c^{k_{h-1}} \cdot bbc(ab)^{m-2}b \cdot Z_{h-1}$. By the induction hypothesis, there exists a positive word $Z_0$ such that

$$c(ab)^{m-2}b \cdot Z_{h-1} \doteqdot c(ab)^{m-1}b \cdot Z_0, \qquad Z_h \doteqdot (ab)^m ba^{k_{h-1}} \cdot Z_0.$$

Thus, we have $bZ_{h-1} \doteqdot abb \cdot Z_0$. We obtain $Z_{h-1} \doteqdot cb \cdot Z_0$, and hence we have

$$X^{(h)} \doteqdot c(ab)^m ba^{k_{h-1}} \cdot Z_0 \doteqdot c(ab)^{m-1}b \cdot cba^{k_{h-1}} \cdot Z_0.$$

Applying this result to (4.7), we have

$$bY \doteqdot c^{h-1}bb \cdot c(ab)^{m-1}b \cdot cba^{k_{h-1}} \cdot Z_0 \doteqdot b(ab)^m ba^{h-1} \cdot cba^{k_{h-1}} \cdot Z_0.$$

Therefore we have $Y \doteqdot (ab)^m ba^{h-1} \cdot cba^{k_{h-1}} \cdot Z_0$.

This completes the proof of Theorem 3.

We have a remark on the presentation of the two monoids $G_m^+$ and $H_m^+$.

Remark 5. *Since the presentation of the monoid $G_m^+$ (resp. $H_m^+$) is not complete, the sufficient criterion for the cancellativity given in* [Deh1], [Deh2] *is not satisfied for the monoid $H_m^+$ (resp. $H_m^+$). Moreover, some procedures, called completion (*[Deh1], [Deh2]*), do not stop in finite steps in both cases. Thus, the cancellativity of them cannot be checked by the method in* [Deh1], [Deh2].

## 5. Calculations of the skew growth functions

In this section, we will calculate the skew growth functions for the monoids $G_{\mathrm{B_{ii}}}^+$, $G_m^+$, $H_m^+$ and $M_{\mathrm{abel},m}$. The datum for proving the cancellativity of the monoids are indispensable to the calculations of the skew growth functions.

**5.1. The skew growth function $N_{G_{\mathrm{B_{ii}}}^+,\deg}(t)$.** In this subsection, we present an explicit calculation of the skew growth function for the monoid $G_{\mathrm{B_{ii}}}^+$. In [I1], we have made a success in calculating the spherical growth function $P_{G_{\mathrm{B_{ii}}}^+,\deg}(t)$ by using the normal form for the monoid $G_{\mathrm{B_{ii}}}^+$. By the inversion formula, we can calculate the skew growth function $N_{G_{\mathrm{B_{ii}}}^+,\deg}(t)$. Nevertheless, we present an explicit calculation, because, in spite of the fact that the monoid is non-abelian and the height of it is infinite, we succeed in the non-trivial calculation.

First of all, we recall a fact from [I1, Section 7].

Lemma 1. *Let $X$ and $Y$ be positive words in $G_{\mathrm{B_{ii}}}^+$ of length $r \in \mathbb{Z}_{\geq 0}$.*

(i) *If $vX \doteqdot vY$ for some $v \in \{a, b, c\}$, then $X \doteqdot Y$.*

(ii) *If $aX \doteqdot bY$, then $X \doteqdot bZ$ and $Y \doteqdot cZ$ for some positive word $Z$.*

(iii) *If $aX \doteqdot cY$, then $X \doteqdot cZ$ and $Y \doteqdot aZ$ for some positive word $Z$.*

(iv) *If* $bX \doteqdot cY$, *then there exist an integer* $k \in \mathbb{Z}_{\geq 0}$ *and a positive word* $Z$ *such that* $X \doteqdot c^k ba \cdot Z$ *and* $Y \doteqdot a^k bb \cdot Z$.

Thanks to Lemma 1, we have proved the cancellativity in [S-I]. Moreover, we can prove the following Lemma.

Lemma 2. *If an equality* $bb \cdot X \doteqdot cY$ *in* $G^+_{\mathrm{B_{ii}}}$ *holds, then* $X \doteqdot aZ$ *and* $Y \doteqdot bb \cdot Z$ *for some positive word* $Z$.

Proof. Due to Lemma 1, we see that there exists an integer $k \in \mathbb{Z}_{\geq 0}$ and a positive word $Z_0$ such that

$$bX \doteqdot c^k ba \cdot Z_0, \qquad Y \doteqdot a^k bb \cdot Z_0. \tag{5.1}$$

We consider the case $k \geq 1$. Due to Lemma 1, we see that there exist an integer $i_1 \in \mathbb{Z}_{\geq 0}$ and a positive word $Z_1$ such that

$$X \doteqdot c^{i_1} ba \cdot Z_1, \qquad c^{k-1} ba \cdot Z_0 \doteqdot a^{i_1} bb \cdot Z_1.$$

Moreover, we see that there exists a positive word $Z_0^{(1)}$ such that

$$Z_0 \doteqdot c^{i_1} \cdot Z_0^{(1)}, \qquad c^{k-1} ba \cdot Z_0^{(1)} \doteqdot bb \cdot Z_1.$$

Repeating the same process $k$-times, there exist integers $i_2, \dots, i_k \in \mathbb{Z}_{\geq 0}$ and positive words $Z_0^{(k)}$ and $Z_k$ such that

$$Z_0 \doteqdot c^{i_1+i_2+\cdots+i_k} \cdot Z_0^{(k)}, \qquad ba \cdot Z_0^{(k)} \doteqdot bb \cdot Z_k.$$

Moreover, we see that there exists a positive word $Z'$ such that

$$Z_0^{(k)} \doteqdot bZ', \qquad Z_k \doteqdot cZ'.$$

Applying this result to (5.1), we have

$$bX \doteqdot c^k bac^{i_1+i_2+\cdots+i_k} b \cdot Z' \doteqdot bac^{i_1+i_2+\cdots+i_k} ba^k \cdot Z',$$

$$Y \doteqdot a^k bbc^{i_1+i_2+\cdots+i_k} b \cdot Z' \doteqdot bb \cdot c^{i_1+i_2+\cdots+i_k} ba^k \cdot Z'.$$

Thus, we have $X \doteqdot a \cdot c^{i_1+i_2+\cdots+i_k} ba^k \cdot Z'$.

As a consequence of Lemma 2, we obtain the followings.

Corollary 1. *If an equality* $bb \cdot X \doteqdot c^l \cdot Y$ *in* $G^+_{\mathrm{B_{ii}}}$ *holds for some positive integer* $l$, *then* $X \doteqdot a^l \cdot Z$ *and* $Y \doteqdot bb \cdot Z$ *for some positive word* $Z$.

Due to Corollary 1, we can solve the following equation.

Proposition 5. *If an equality* $c^i b \cdot X \doteqdot c^j b \cdot Y$ *in* $G^+_{\mathrm{B_{ii}}}$ *holds for* $0 \leq i < j$, *then there exists an integer* $k \in \mathbb{Z}_{\geq 0}$ *and a positive word* $Z$ *such that*

$$X \doteqdot c^k ba^{j-i} \cdot Z, \qquad Y \doteqdot c^k b \cdot Z.$$

Proof. Due to the cancellativity, $c^i b \cdot X \doteq c^j b \cdot Y$ if and only if $bX \doteq c^{j-i} b \cdot Y$. Thanks to Lemma 1, we see that there exist an integer $k \in \mathbb{Z}_{\geq 0}$ and a positive word $Z_1$ such that

$$X \doteq c^k ba \cdot Z_1, \qquad c^{j-i-1} b \cdot Y \doteq a^k bb \cdot Z_1.$$

Moreover, we see that there exists $Y'$

$$Y \doteq c^k \cdot Y', \qquad c^{j-i-1} b \cdot Y' \doteq bb \cdot Z_1.$$

Due to Corollary 1, there exists a positive word $Z_2$ such that

$$bY' \doteq bb \cdot Z_2, \qquad Z_1 \doteq a^{j-1-1} \cdot Z_2.$$

Thus, we have

$$X \doteq c^k ba^{j-i} \cdot Z_2, \qquad Y \doteq c^k b \cdot Z_2.$$

As a corollary of Proposition 5, we show the following lemma.

Lemma 3. *For* $0 \leq \kappa_1 < \kappa_2 < \cdots < \kappa_p$,

$$\mathrm{mcm}(\{c^{\kappa_1} b, c^{\kappa_2} b, \dots, c^{\kappa_p} b\}) = \{c^{\kappa_p} b \cdot c^k b \mid k = 0, 1, \dots\}$$

By using Lemma 3, we easily show the following.

Proposition 6. *We have* $h(G^+_{\mathrm{B_{ii}}}, \deg) = \infty$.

Proof. Due to Lemma 1, we show

$$\mathrm{mcm}(\{b, c\}) = \{cb \cdot c^k b \mid k = 0, 1, \dots\}.$$

Due to Lemma 1, for $0 \leq \kappa_1 < \kappa_2 < \cdots < \kappa_p$, we have

$$\mathrm{mcm}(\{cb \cdot c^{\kappa_1} b, cb \cdot c^{\kappa_2} b, \dots, cb \cdot c^{\kappa_p} b\}) = \{cb \cdot c^{\kappa_p} b \cdot c^k b \mid k = 0, 1, \dots\}.$$

By using Lemma 3 repeatedly, we show $h(G^+_{\mathrm{B_{ii}}}, \deg) = \infty$.

By using Lemma 3, we calculate the skew growth function. We have to consider four cases where $J_1 = \{a, b\}, \{a, c\}, \{b, c\}$ or $\{a, b, c\}$. We denote by $\mathrm{Tmcm}(G^+_{\mathrm{B_{ii}}}, J_1)$ the set of all the towers starting from a fixed $J_1$. If $J_1 = \{a, b\}$ or $\{a, c\}$, due to Lemma 1, then $\mathrm{mcm}(\{a, b\})$ and $\mathrm{mcm}(\{a, c\})$ consist of only one element, respectively. Next, we consider the case where $J_1 = \{b, c\}$. For a fixed tower $T$, if there exists an element $\Delta \in |T|$ such that $\deg(\Delta) = l + 2$, then, from Lemma 3, we see the uniqueness. For any fixed $l \in \mathbb{Z}_{>0}$, we calculate the coefficient of the term $t^{l+2}$ which is denoted by $a_l$, by counting all the signs $(-1)^{\#J_1 + \cdots + \#J_n - n + 1}$ in the definition (3.1) associated with the towers $T = (I_0, J_1, J_2, \dots, J_n)$ for which $\deg(\Delta)$ can take a value $l + 2$. To calculate the coefficient $a_l$, we consider the set

$$\mathscr{T}^l_{G^+_{\mathrm{B_{ii}}}} := \{T \in \mathrm{Tmcm}(G^+_{\mathrm{B_{ii}}}, J_1) \mid \Delta \in |T| \text{ such that } \deg(\Delta) = l+2\}.$$

By using Lemma 3 repeatedly, we show

$$\max\{\text{the height of } T \in \mathscr{T}^l_{G^+_{\mathrm{B_{ii}}}}\} = \lfloor (l+1)/2 \rfloor.$$

For $u \in \{1, \dots, \lfloor (l+1)/2 \rfloor\}$, we define the set

$$\mathscr{T}^l_{G^+_{\mathrm{B_{ii}}}, u} := \{T \in \mathscr{T}^l_{G^+_{\mathrm{B_{ii}}}} \mid \text{the height of } T = u\}.$$

Hereafter, we write simply $\mathscr{T}^l$ (resp. $\mathscr{T}_u{}^l$) for $\mathscr{T}^l_{G^+_{\mathrm{B_{ii}}}}$ (resp. $\mathscr{T}^l_{G^+_{\mathrm{B_{ii}}}, u}$). Thus, we have the decomposition:

$$\mathscr{T}^l = \bigsqcup_u \mathscr{T}_u{}^l. \tag{5.2}$$

**Claim 1.** For any $u$, we show the following equality

$$(-1)^{u-1}{}_{l-u}C_{u-1} = \sum_{T \in \mathscr{T}_u{}^l} (-1)^{\# J_1 + \cdots + \# J_u - u + 1}.$$

Proof. For the case of $u = 1$, the equality holds. For the case of $u = 2$, we calculate the sum $\sum_{T \in \mathscr{T}_2{}^l} (-1)^{\# J_2 - 1}$. By indices $0 \le \kappa_1 < \kappa_2 < \cdots < \kappa_p$, the set $J_2$ is generally written by $\{cb \cdot c^{\kappa_1} b, cb \cdot c^{\kappa_2} b, \dots, cb \cdot c^{\kappa_p} b\}$. Due to Lemma 3, we show that the maximum index $\kappa_p$ can range from 1 to $l-2$. For a fixed index $\kappa_p = \kappa \in \{1, \dots, l-2\}$, we easily show

$$\sum_{T \in \mathscr{T}_2{}^l, \kappa_p = \kappa} (-1)^{\# J_2 - 1} = -1.$$

Therefore, we show that the sum $\sum_{T \in \mathscr{T}_2{}^l} (-1)^{\# J_2 - 1} = -(l-2) = -{}_{l-2}C_{2-1}$.

We show the case for $3 \le u \le \lfloor (l+1)/2 \rfloor$ by induction on $u$. We assume the case where $u = j$. For the case of $u = j+1$, we focus our attention to the set $J_2$. Since the set $J_2$ can be written as $\{cb \cdot c^{\kappa_1} b, cb \cdot c^{\kappa_2} b, \dots, cb \cdot c^{\kappa_p} b\}$, due to Lemma 3, we show that the maximum index $\kappa_p$ can range from 1 to $l - 2j$. By the induction hypothesis, it suffices to show the following equality

$$\sum_{k=1}^{l-2j} {}_{l-j-k-1}C_{j-1} = {}_{l-j-1}C_j.$$

Therefore, we have shown the case $u = j+1$. This completes the proof.

By the decomposition (5.2), we show the following equality.

**Claim 2.** $a_l = \sum_{k=0}^{\lfloor (l-1)/2 \rfloor} (-1)^k {}_{l-k-1}C_k$.

Then, we easily show the following.

**Claim 3.** $a_{l+2} - a_{l+1} + a_l = 0$.

Proof. Since an equality ${}_{n+1}C_k - {}_nC_k = {}_nC_{k-1}$ holds, we can show our statement.

We easily show $a_1 = a_2 = 1$. Hence, the sequence $\{a_l\}_{l=1}^{\infty}$ has a period 6. Lastly, we consider the case where $J_1 = \{a, b, c\}$. For any fixed $l \in \mathbb{Z}_{>0}$, we calculate the coefficient of the term $t^{l+3}$ which is denoted by $b_l$. Since $\mathrm{mcm}(\{a, b, c\}) = \{cb \cdot c^k b \mid k = 1, 2, \ldots\}$, we can reuse Lemma 3. In a similar manner, we have the following conclusion.

**Claim 4.** $b_{l+2} - b_{l+1} + b_l = 0$.

Since $b_1 = b_2 = 1$, we also show that the sequence $\{b_l\}_{l=1}^{\infty}$ has a period 6. After all, we can calculate the skew growth function for the monoid $G_{\mathrm{B_{ii}}}^+$:

$$N_{G_{\mathrm{B_{ii}}}^+, \deg}(t) = 1 - 3t + 2t^2 + \frac{t^3}{1 - t + t^2} - \frac{t^4}{1 - t + t^2} = \frac{(1-t)^4}{1 - t + t^2}.$$

**5.2. The skew growth function $N_{G_m^+, \deg}(t)$.** In this subsection, we present an explicit calculation of the skew growth function for the monoid $G_m^+$.

First of all, we show the following proposition.

Proposition 7. *If an equation $c^i b^{m-1} \cdot X \doteq c^j b^{m-1} \cdot Y$ in $G_m^+$ holds for $0 \le i < j$, then there exists a positive word $Z$ such that*

$$X \doteq ba^{j-i} \cdot Z \qquad \text{and} \qquad Y \doteq bZ.$$

Proof. Since we have shown the cancellativity in §4, $c^i b^{m-1} \cdot X \doteq c^j b^{m-1} \cdot Y$ if and only if $b^{m-1} \cdot X \doteq c^{j-i} b^{m-1} \cdot Y$. Thanks to Proposition 2 (iv-$(m-2)$-$b$), we see that there exists a positive word $Z$ such that

$$X \doteq ba^{j-i} \cdot Z, \qquad Y \doteq bZ.$$

As a corollary of Proposition 7, we show the following lemma.

Lemma 4. *For $0 \le \kappa_1 < \kappa_2 < \cdots < \kappa_p$,*

$$\mathrm{mcm}(\{c^{\kappa_1} b^{m-1}, c^{\kappa_2} b^{m-1}, \ldots, c^{\kappa_p} b^{m-1}\}) = \{c^{\kappa_p} b^m\}$$

Thus, we obtain the following proposition.

Proposition 8. $h(G_m^+, \deg) = 2.$

By using Lemma 4, we calculate the skew growth function. We have to consider four cases where $J_1 = \{a, b\}, \{a, c\}, \{b, c\}$ or $\{a, b, c\}$. We denote by

$\mathrm{Tmcm}(G_m^+, J_1)$ the set of all the towers starting from a fixed $J_1$. If $J_1 = \{a, b\}$ or $\{a, c\}$, due to Proposition 2, then $\mathrm{mcm}(\{a, b\})$ and $\mathrm{mcm}(\{a, c\})$ consist of only one element, respectively. Next, we consider the case where $J_1 = \{b, c\}$. For any fixed $l \in \mathbf{Z}_{>0}$, we calculate the coefficient of the term $t^{m+l}$ which is denoted by $c_l$. To calculate the coefficient $c_l$, we consider the set

$$\mathscr{T}_{G_m^+}^l := \{T \in \mathrm{Tmcm}(G_m^+, J_1) \mid \varDelta \in |T| \text{ such that } \deg(\varDelta) = m + l\}.$$

For $u \in \{1, 2\}$, we define the set

$$\mathscr{T}_{G_m^+, u}^l := \{T \in \mathscr{T}_{G_m^+}^l \mid \text{the height of } T = u\}.$$

Since $\mathrm{mcm}(\{b, c\}) = \{cb \cdot c^k b^{m-1} \mid k = 0, 1, \ldots\}$, we easily show $c_1 = c_2 = 1$. Moreover, we show the following.

Proposition 9. $c_l = 0 \ (l = 3, 4, \ldots)$.

Proof. From the consideration in Claim 1 of Example 1, for $u = 2$, we also show

$$\sum_{T \in \mathscr{T}_{G_m^+, u}^l} (-1)^{\#J_1 + \cdots + \#J_u - u + 1} = -1.$$

Thus, we have $c_l = 0 \ (l = 3, 4, \ldots)$.

Lastly, we consider the case where $J_1 = \{a, b, c\}$. For any fixed $l \in \mathbf{Z}_{>0}$, we calculate the coefficient of the term $t^{m+l+1}$ which is denoted by $d_l$. In a similar way, we show $d_1 = d_2 = 1$ and $d_l = 0 \ (l = 3, 4, \ldots)$. After all, we calculate the skew growth function for the monoid $G_m^+$:

$$\begin{aligned} N_{G_m^+, \deg}(t) &= 1 - 3t + 2t^2 + (t^{m+1} + t^{m+2}) - (t^{m+2} + t^{m+3}) \\ &= (1 - t)(t^{m+2} + t^{m+1} - 2t + 1). \end{aligned}$$

Remark 6. *By the inversion formula, we are able to calculate the spherical growth function $P_{G_m^+, \deg}(t)$ through the skew growth function $N_{G_m^+, \deg}(t)$. We can not find the direct calculation of the spherical growth function $P_{G_m^+, \deg}(t)$ in the existence literatures.*

**5.3. The skew growth function $N_{H_m^+, \deg}(t)$.** In this subsection, we present an explicit calculation of the skew growth function for the monoid $H_m^+$.

First of all, we show the following proposition.

Proposition 10. *If an equality $c^i b(ab)^{m-1} ba \cdot X \doteq c^j b(ab)^{m-1} ba \cdot Y$ in $H_m^+$ holds for $0 \le i < j$, then there exists a positive word $Z$ such that*

$$X \doteq cba^{j-i} \cdot Z, \qquad Y \doteq cb \cdot Z.$$

Proof. Since we have shown the cancellativity of $H_m^+$ in §4, we show $c^i b(ab)^{m-1}ba \cdot X \doteqdot c^j b(ab)^{m-1}ba \cdot Y \Leftrightarrow b(ab)^{m-1}ba \cdot X \doteqdot c^{j-i}b(ab)^{m-1}ba \cdot Y$. Thanks to Proposition 4 (vi-$h$), we see that there exists a positive word $Z_1$ such that

$$(ab)^{m-1}ba \cdot X \doteqdot (ab)^m ba^{j-i} \cdot Z_1, \qquad c(ab)^{m-2}ba \cdot Y \doteqdot c(ab)^{m-1}b \cdot Z_1.$$

Therefore, we see that there exists a positive word $Z_2$ such that

$$X \doteqdot cba^{j-i} \cdot Z_2, \qquad Y \doteqdot cb \cdot Z_2.$$

As a corollary of Proposition 10, we show the following lemma.

Lemma 5. *For* $0 \le \kappa_1 < \kappa_2 < \cdots < \kappa_p$,

$$\begin{aligned}&\mathrm{mcm}(\{c^{\kappa_1}b(ab)^{m-1}ba, c^{\kappa_2}b(ab)^{m-1}ba, \ldots, c^{\kappa_p}b(ab)^{m-1}ba\})\\ &\quad = \{c^{\kappa_p}b(ab)^{m-1}bacb\}\end{aligned}$$

Thus, we obtain the following proposition.

Proposition 11. $h(H_m^+, \deg) = 2$.

Thanks to Lemma 5, we can calculate the skew growth function. We have to consider four cases where $J_1 = \{a,b\}, \{a,c\}, \{b,c\}$ or $\{a,b,c\}$. We denote by $\mathrm{Tmcm}(H_m^+, J_1)$ the set of all the towers starting from a fixed $J_1$. If $J_1 = \{a,b\}$ or $\{a,c\}$, due to Proposition 4, then $\mathrm{mcm}(\{a,b\})$ and $\mathrm{mcm}(\{a,c\})$ consist of only one element, respectively. Next, we consider the case where $J_1 = \{b,c\}$. For any fixed $l \in \mathbb{Z}_{>0}$, we calculate the coefficient of the term $t^{2m+3+l}$ which is denoted by $e_l$. In order to calculate the coefficient $e_l$, we consider the set

$$\mathscr{T}^l_{H_m^+} := \{T \in \mathrm{Tmcm}(H_m^+, J_1) \mid \Delta \in |T| \text{ such that } \deg(\Delta) = 2m+3+l\}.$$

For $u \in \{1,2\}$, we define the set

$$\mathscr{T}^l_{H_m^+,u} := \{T \in \mathscr{T}^l_{H_m^+} \mid \text{the height of } T = u\}.$$

Since $\mathrm{mcm}(\{b,c\}) = \{bc^k(ab)^m ba \mid k = 0,1,\ldots\}$, we easily show $e_1 = e_2 = e_3 = 1$. Moreover, we show the following.

Proposition 12. $e_l = 0$ $(l = 4,5,\ldots)$.

Proof. From the consideration in Claim 1 of Example 1, for $u = 2$, we also show

$$\sum_{T \in \mathscr{T}^l_{H_m^+,u}} (-1)^{\#J_1+\cdots+\#J_u-u+1} = -1.$$

Thus, we have $e_l = 0$ $(l = 4,5,\ldots)$.

Lastly, we consider the case where $J_1 = \{a, b, c\}$. For any fixed $l \in \mathbb{Z}_{>0}$, we calculate the coefficient of the term $t^{2m+4+l}$ which is denoted by $f_l$. In a similar way, we show $f_1 = f_2 = f_3 = 1$ and $f_l = 0$ $(l = 4, 5, \dots)$. After all, we calculate the skew growth function for the monoid $H_m^+$:

$$\begin{aligned} N_{H_m^+,\deg}(t) &= 1 - 3t + 2t^2 + (t^{2m+3} + t^{2m+4} + t^{2m+5}) - (t^{2m+4} + t^{2m+5} + t^{2m+6}) \\ &= (1-t)(t^{2m+5} + t^{2m+4} + t^{2m+3} - 2t + 1). \end{aligned}$$

Remark 7. *By the inversion formula, we are able to calculate the growth function $P_{H_m^+,\deg}(t)$ through the skew growth function $N_{H_m^+,\deg}(t)$. We can not find the direct calculation of the spherical growth function $P_{H_m^+,\deg}(t)$ in the literatures.*

### 5.4. The skew growth function $N_{M_{\mathrm{abel},m},\deg}(t)$.

In this subsection, we calculate the skew growth function for the monoid $M_{\mathrm{abel},m}$.

First of all, we easily show the following proposition.

Proposition 13. *Let $X$ and $Y$ be positive words in $M_{\mathrm{abel},m}$ of length $r \in \mathbb{Z}_{\geq 0}$.*

(i) *If $vX \doteq vY$ for some $v \in \{a, b\}$, then $X \doteq Y$.*

(ii) *If $aX \doteq bY$, then either $X \doteq a^{m-1} \cdot Z_1$ and $Y \doteq b^{m-1} \cdot Z_1$ for some positive word $Z_1$ or $X \doteq bZ_2$ and $Y \doteq aZ_2$ for some positive word $Z_2$.*

Lemma 6. *There exists a unique tower $T_n = (I_0, J_1, J_2, \dots, J_n)$ of height $n \in \mathbb{Z}_{>0}$ with the ground set $I_0 = \{a, b\}$ such that*

$$J_{2k-1} = \{a^{(k-1)m+1}, a^{(k-1)m}b\} \qquad (k = 1, \dots, \lfloor (n+1)/2 \rfloor),$$

$$J_{2k} = \{a^{km}, a^{(k-1)m+1}b\} \qquad (k = 1, \dots, \lfloor n/2 \rfloor).$$

Proof. We easily show $J_1 = \{a, b\}$ and $J_2 = \{a^m, ab\}$. Thanks to Proposition 13, we show our statement by induction on $k$.

Therefore, we immediately show $h(M_{\mathrm{abel},m}, \deg) = \infty$. Moreover, from the definition (3.1), we can calculate the skew growth function

$$N_{M_{\mathrm{abel},m},\deg}(t) = (1 - 2t + t^2)(1 + t^m + t^{2m} + \cdots) = \frac{(1-t)^2}{1-t^m}.$$

## 6. Appendix

In this section, we deal with two monoids $M_4$ and $G^+(4_1)$ whose towers do not stop on the first stage $J_1$. The skew growth functions for them can be calculated with comparative ease.

Example 1. In [Deh2], the author investigated a certain monoid that we rename to $M_4$. The presentation is the following

$$M_4 := \left\langle a, b, c, d \,\middle|\, \begin{array}{l} ab = bc = ca, \\ ba = db = ad, \\ caa = dbb \end{array} \right\rangle_{mo}.$$

By referring to Higman-Garside's method (see [G], [B-S]), we easily show the following proposition.

Proposition 14. *Let $X$ and $Y$ be positive words in $M_4$ of length $r \in \mathbb{Z}_{\geq 0}$.*

(i) *If $vX \doteq vY$ for some $v \in \{a, b, c, d\}$, then $X \doteq Y$.*

(ii) *If $aX \doteq bY$, then either $X \doteq bZ_1$ and $Y \doteq cZ_1$ for some positive word $Z_1$ or $X \doteq dZ_2$ and $Y \doteq aZ_2$ for some positive word $Z_2$.*

(iii) *If $aX \doteq cY$, then $X \doteq bZ$ and $Y \doteq aZ$ for some positive word $Z$.*

(iv) *If $aX \doteq dY$, then $X \doteq dZ$ and $Y \doteq bZ$ for some positive word $Z$.*

(v) *If $bX \doteq cY$, then $X \doteq cZ$ and $Y \doteq aZ$ for some positive word $Z$.*

(vi) *If $bX \doteq dY$, then $X \doteq aZ$ and $Y \doteq bZ$ for some positive word $Z$.*

(vii) *If $cX \doteq dY$, then $X \doteq aa \cdot Z$ and $Y \doteq bb \cdot Z$ for some positive word $Z$.*

Thanks to Proposition 14, we see that the monoid $M_4$ is a left cancellative monoid. In the monoid $M_4$, we have an anti-homomorphism $\varphi : M_4 \to M_4$, $W \mapsto \varphi(W) := \sigma(rev(W))$, where $\sigma$ is a permutation $\binom{a,b,c,d}{b,a,c,d}$ and $rev(W)$ is the reverse of the word $W = x_1 x_2 \dots x_k$ ($x_i$ is a letter) given by the word $x_k \dots x_2 x_1$. By a similar argument in §5.1, we can show that the monoid $M_4$ is a cancellative monoid. Due to Proposition 14, we can calculate the skew growth function. We have to consider the case where $J_1 = \{a, b\}$. We have $\mathrm{mcm}(\{a, b\}) = \{ab, ad\}$ and $\mathrm{mcm}(\{ab, ad\}) = \{aba\}$, and therefore $h(M_4, \deg) = 2$. From the definition (3.1), we can calculate the skew growth function for the monoid $M_4$ as follows:

$$N_{M_4, \deg}(t) = 1 - 4t + 4t^2 - t^3 = (1 - t)(1 - 3t + t^2).$$

Example 2. For the figure-eight knot, a Wirtinger presentation of the knot group $G(4_1)$ can be shown to be

$$G(4_1) \cong \left\langle a, b, c, d \,\middle|\, \begin{array}{l} ca = dc,\ bd = da, \\ ac = ba,\ db = bc \end{array} \right\rangle.$$

For this presentation, we associate the monoid defined by it, which is denoted by $G^+(4_1)$. By referring to Higman-Garside's method (see [G], [B-S]), we easily show the following proposition.

Proposition 15. *Let $X$ and $Y$ be positive words in $G^+(4_1)$ of length $r \in \mathbb{Z}_{\geq 0}$.*

(i) *If $vX \doteqdot vY$ for some $v \in \{a,b,c,d\}$, then $X \doteqdot Y$.*

(ii) *If $aX \doteqdot bY$, then $X \doteqdot cZ$ and $Y \doteqdot aZ$ for some positive word $Z$.*

(iii) *There do not exist positive words $X$ and $Y$ that satisfy an equation $aX \doteqdot cY$.*

(iv) *There do not exist positive words $X$ and $Y$ that satisfy an equation $aX \doteqdot dY$.*

(v) *There do not exist positive words $X$ and $Y$ that satisfy an equation $bX \doteqdot cY$.*

(vi) *If $bX \doteqdot dY$, then either $X \doteqdot dZ_1$ and $Y \doteqdot aZ_1$ for some positive word $Z_1$ or $X \doteqdot cZ_2$ and $Y \doteqdot bZ_2$ for some positive word $Z_2$.*

(vii) *If $cX \doteqdot dY$, then $X \doteqdot aZ$ and $Y \doteqdot cZ$ for some positive word $Z$.*

Thanks to Proposition 15, we see that the monoid $G^+(4_1)$ is a left cancellative monoid. In the monoid $G^+(4_1)$, we have an anti-homomorphism $\varphi : G^+(4_1) \to G^+(4_1)$, $W \mapsto \varphi(W) := \sigma(rev(W))$, where $\sigma$ is a permutation $\binom{a,b,c,d}{b,c,d,a}$ and $rev(W)$ is the reverse of the word $W = x_1x_2\dots x_k$ ($x_i$ is a letter) given by the word $x_k\dots x_2x_1$. By a similar argument in §5.1, we can show that the monoid $G^+(4_1)$ is a cancellative monoid. Due to Proposition 15, we easily have $h(G^+(4_1), \deg) = 2$. From the definition (3.1), we can calculate the skew growth function for the monoid $G^+(4_1)$ as follows:

$$N_{G^+(4_1),\deg}(t) = 1 - 4t + 4t^2 - t^3 = (1-t)(1-3t+t^2).$$


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