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Extremality of quaternionic Jørgensen inequality

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Abstract. Let $SL(2, \mathbb{H})$ be the group of 2×2 quaternionic matrices with Dieudonné determinant one. The group $SL(2, \mathbb{H})$ acts on the five dimensional hyperbolic space by isometries. We investigate extremality of Jørgensen type inequalities in $SL(2, \mathbb{H})$. Along the way, we derive Jørgensen type inequalities for quaternionic Möbius transformations which extend earlier inequalities obtained by Waterman and Kellerhals.

1. Introduction

In the theory of Fuchsian groups, one of the important old problems is the "discreteness problem": given two elements in $PSL(2, \mathbb{R})$, to decide whether the group generated by them is discrete. For an elaborate account on this problem, see Gilman [11]. Algorithmic solutions to this problem were given by Rosenberger [20], Gilman and Maskit [12], Gilman [11]. The Jørgensen inequality [7] is a major result related to this problem. Jørgensen [7] obtained an inequality that the generators of a discrete, non-elementary, two-generator subgroup of $SL(2, \mathbb{C})$ necessarily satisfy. Wada [30] used this inequality to provide an effective algorithm that helps the software OPTi to test discreteness of subgroups, as well as to draw deformation spaces of discrete groups.

A two-generator discrete subgroup of isometries of the hyperbolic space is called *extreme group* if it satisfies equality in the Jørgensen inequality. Investigation of extreme groups in $SL(2, \mathbb{C})$ was initiated by Jørgensen and Kikka [8]. This work was followed by attempts to classify the two-generator extreme groups in $SL(2, \mathbb{C})$, for eg. see [12, 14]. In a series of papers, Sato et al. [21]–[26] have investigated this problem in great detail and provided a conjectural list of the parabolic-type extreme groups. Callahan [4] has provided a counter example to that conjecture. Callahan has also classified all non-compact arithmetic extreme groups which were not in the list of Sato et al. The

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problem of classifying parabolic-type Jørgensen groups in $SL(2, \mathbb{C})$ is still open. Recently, Vesnin and Masley [29] have investigated extremality of other Jørgensen type inequalities in $SL(2, \mathbb{C})$. Vesnin [2] has raised the problem of classifying all hyperbolic 3-orbifold groups that satisfy extremality in Jørgensen type inequalities obtained by Gehring-Martin [10] and Tan [27].

The problem of classifying extreme Jørgensen groups in higher dimension has not seen much attempt till date. The aim of this paper is to address this problem for Jørgensen type inequalities in $SL(2, \mathbb{H})$. Here $\mathbb{H}$ is the division ring of the real quaternions and $SL(2, \mathbb{H})$ is the group of 2×2 quaternionic matrices with Dieudonné determinant 1. It is well-known that $SL(2, \mathbb{H})$ acts on the five dimensional real hyperbolic space $\mathbf{H}^5$ by the Möbius transformations (or linear fractional transformations), for a proof see [13]. The isometries of $\mathbf{H}^5$ are classified by their fixed points, as elliptic, parabolic and hyperbolic (or loxodromic). This classification can be characterized algebraically by conjugacy invariants of the isometries, see [18, 19, 13, 3] for more details.

The Jørgensen inequality has been generalized in higher dimensions by Martin [17] who formulated it using the upper half space or the unit ball model of the hyperbolic n-space in $\mathbb{R}^{n+1}$. Hence, in Martin's generalization, the isometries are real matrices of rank $n + 1$. Ahlfors [1] used Clifford algebras to investigate higher dimensional Möbius groups. In this approach, the isometry group of the hyperbolic n-space can be identified with a group of 2×2 matrices over the Clifford numbers, see Ahlfors [1], Waterman [31]. Using the Clifford algebraic formalism, a generalization of Jørgensen inequality was obtained by Waterman [31]. However, it may be difficult to deal with the Clifford matrices due to the non-commutative and non-associative structure of the Clifford numbers.

Using the real quaternions there is an intermediate approach between the complex numbers and the Clifford numbers. This approach should provide the closest generalization of the low dimensional results for four and five dimensional Möbius groups. The Clifford group that acts by isometries on the hyperbolic 4-space, is a proper subgroup of $SL(2, \mathbb{H})$. So, Waterman's result restricts to this case. Kellerhals [15] has used this quaternionic Clifford group to investigate collars in $\mathbf{H}^4$. Recently, Tan et al. [28] have obtained a generalization of the classical Delambre-Gauss formula for right-angles hexagons in hyperbolic 4-space using the quaternionic Clifford group of Ahlfors and Waterman.

The Clifford group that acts on $\mathbf{H}^5$, however, is not a subgroup of $SL(2, \mathbb{H})$. In fact, the group $SL(2, \mathbb{H})$ is not in the list of the Clifford groups of Ahlfors and Waterman. However, following the approach of Waterman, it is not hard to formulate Jørgensen type inequalities for pairs of isometries in $SL(2, \mathbb{H})$. Kellerhals [16] derived Jørgensen inequality for two-generator discrete subgroups in $SL(2, \mathbb{H})$, where one the of the generators is either unipotent parabolic or hyperbolic.

Using similar methods as that of Waterman, we give here slightly generalized versions of the Jørgensen inequalities in $SL(2, \mathbb{H})$ where, one of the generators is either semisimple or, fixes a point on the boundary, see Theorem 2 and Theorem 3 in Section 3. The quaternionic formulations of the inequalities of Kellerhals and Waterman are derived as corollaries, see Corollary 1 and Corollary 7 respectively. We formulate a Jørgensen type inequality for strictly hyperbolic elements that is very close to the original formulation of Jørgensen, see Corollary 2. We recall here that a strictly hyperbolic element or a stretch is conjugate to a diagonal matrix that has real diagonal entries different from 0, 1 or -1. As corollaries we obtain two weaker versions of the inequality for subgroups having one generator semisimple.

We investigate the extremality of these Jørgensen inequalities in Section 4. We extend the results of Jørgensen and Kikka in the quaternionic set up, see Theorem 5, Corollary 8 and Theorem 6. We also obtain necessary conditions for a two-generator subgroup of $SL(2, \mathbb{H})$ to be extremal, see Corollaries 11 and 12.

2. Preliminaries

2.1. The quaternions. Let $\mathbb{H}$ denote the division ring of quaternions. Recall that every element of $\mathbb{H}$ is of the form $a_0 + a_1 i + a_2 j + a_3 k$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$, and i, j, k satisfy relations: $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$ and $ijk = -1$. Any $a \in \mathbb{H}$ can be written as $a = a_0 + a_1 i + a_2 j + a_3 k = (a_0 + a_1 i) + (a_2 + a_3 i) j = z + wj$, where $z = a_0 + a_1 i$, $w = a_2 + a_3 i \in \mathbb{C}$. For $a \in \mathbb{H}$, with $a = a_0 + a_1 i + a_2 j + a_3 k$, we define $\Re(a) = a_0$ the real part of a and $\Im(a) = a_1 i + a_2 j + a_3 k =$ the imaginary part of a. Also define the conjugate of a as $\bar{a} = \Re(a) - \Im(a)$. If $\Re(a) = 0$, then we call a as a vector in $\mathbb{H}$ which we can identify with $\mathbb{R}^3$. The norm of a is $|a| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$.

2.1.1. *Useful properties.* We note the following properties of the quaternions that will help us further:

(1) For $x \in \mathbb{R}$, $a \in \mathbb{H}$, we have $ax = xa$.

(2) For $a \in \mathbb{C}$, $aj = j\bar{a}$.

(3) For $a, b \in \mathbb{H}$, $|ab| = |a|\,|b| = |ba|$ and if $a \neq 0$, then $a^{-1} = \frac{\bar{a}}{|a|^2}$.

Two quaternions a, b are said to be *similar* if there exists a non-zero quaternion c such that $b = c^{-1}ac$ and we write it as $a \backsim b$. Obviously, '$\backsim$' is an equivalence relation on $\mathbb{H}$ and denote $[a]$ as the class of a. It is easy to verify that $a \backsim b$ if and only if $\Re(a) = \Re(b)$ and $|a| = |b|$. Equivalently, $a \backsim b$

if and only if $\Re(a) = \Re(b)$ and $|\Im(a)| = |\Im(b)|$. Thus the similarity class of every quaternion a contains a pair of complex conjugates with absolute-value $|a|$ and real part equal to $\Re(a)$. Let a is similar to $re^{i\theta}$, $\theta \in [-\pi, \pi]$. In most cases, we will adopt the convention of calling $|\theta|$ as the *argument* of a and will denote it by $\arg(a)$. According to this convention, $\arg(a) \in [0, \pi]$, unless specified otherwise.

Suppose a quaternion q is conjugate to a complex number $z = re^{i\alpha}$. Since $\Re(q) = \Re(z)$ and $|q| = |z|$, it follows that $|\Im q| = |\Im z| = |r \sin \alpha|$, i.e. $|\sin \alpha| = \frac{|\Im q|}{|q|}$.

2.2. Matrices over the quaternions. Let $\mathrm{M}(2, \mathbb{H})$ denotes the set of all 2×2 matrices over the quaternions. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we can associate the 'quaternionic determinant' $\det(A) = |ad - aca^{-1}b|$. A matrix $A \in \mathrm{M}(2, \mathbb{H})$ is invertible if and only if $\det(A) \neq 0$. Also, note that for $A, B \in \mathrm{M}(2, \mathbb{H})$, $\det(AB) = \det(A)\det(B)$. Now set

$$\mathrm{SL}(2, \mathbb{H}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}(2, \mathbb{H}) : \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = |ad - aca^{-1}b| = 1 \right\}.$$

The group $\mathrm{SL}(2, \mathbb{H})$ acts as the orientation-preserving isometry group of the hyperbolic 5-space $\mathbf{H}^5$. We identify the extended quaternionic plane $\hat{\mathbb{H}} = \mathbb{H} \cup \{\infty\}$ with the conformal boundary $\mathbb{S}^4$ of the hyperbolic 5-space. The group $\mathrm{SL}(2, \mathbb{H})$ acts on $\hat{\mathbb{H}}$ by Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : Z \mapsto (aZ + b)(cZ + d)^{-1}.$$

The action is extended over $\mathbf{H}^5$ by Poincaré extensions.

2.3. Classification of elements of $\mathrm{SL}(2, \mathbb{H})$**.** Every element A of $\mathrm{SL}(2, \mathbb{H})$ has a fixed point on the closure of the hyperbolic space $\overline{\mathbf{H}}^5$. This gives us the usual trichotomy of elliptic, parabolic and hyperbolic (or loxodromic) elements in $\mathrm{SL}(2, \mathbb{H})$. Further, it follows from the Lefschetz fixed point theorem that every element of $\mathrm{SL}(2, \mathbb{H})$ has a fixed point on the conformal boundary. Up to conjugacy, we can take that fixed point to be ∞, and hence, every element in $\mathrm{SL}(2, \mathbb{H})$ is conjugate to an upper-triangular matrix.

We would like to note here that an elliptic or hyperbolic element A is conjugate to a matrix of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix},$$

where $\lambda, \mu \in \mathbb{C}$. If $|\lambda| = |\mu| (= 1)$ then A is elliptic. Otherwise it is hyperbolic. In the hyperbolic case, $|\lambda| \neq 1 \neq |\mu|$ and $|\lambda|\,|\mu| = 1$. A hyperbolic or loxodromic element will be called *strictly hyperbolic* if it is conjugate to a real diagonal (non-identity) matrix. A parabolic isometry is conjugate to an element of the form

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \qquad |\lambda| = 1.$$

For more details of the classification and algebraic criteria to detect them, see [3, 13, 18, 19].

2.4. Conjugacy invariants. According to Foreman [6], the following three functions are conjugacy invariants of $\mathrm{SL}(2, \mathbb{H})$: for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{H})$,

$$\begin{aligned} \beta = \beta_A &= |d|^2 \Re(a) + |a|^2 \Re(d) - \Re(\bar{a}bc) - \Re(bc\bar{d}) \\ &= \Re[(ad - bc)\bar{a} + (da - cb)\bar{d}], \\ \gamma = \gamma_A &= |a|^2 + |d|^2 + 4\Re(a)\Re(d) - 2\Re(bc) \\ &= |a|^2 + |d|^2 + 2[\Re(a\bar{d}) + \Re(ad)] - 2\Re(bc) \\ &= |a + d|^2 + 2\Re(ad - bc), \\ \delta = \delta_A &= \Re(a + d). \end{aligned}$$

Parker and Short [19] defined another two quantities for each $A \in \mathrm{SL}(2, \mathbb{H})$ as follows:

$$\begin{aligned} \sigma = \sigma_A &= cac^{-1}d - cb, \qquad \text{when } c \neq 0, \\ &= bdb^{-1}a, \qquad \text{when } c = 0,\, b \neq 0, \\ &= (d - a)a(d - a)^{-1}d, \qquad \text{when } b = c = 0,\, a \neq d, \\ &= a\bar{a}, \qquad \text{when } b = c = 0,\, a = d \\ \tau = \tau_A &= cac^{-1} + d, \qquad \text{when } c \neq 0 \\ &= bdb^{-1} + a, \qquad \text{when } c = 0,\, b \neq 0 \\ &= (d - a)a(d - a)^{-1} + d, \qquad \text{when } b = c = 0,\, a \neq d \\ &= a + \bar{a}, \qquad \text{when } b = c = 0,\, a = d. \end{aligned}$$

It can be proved that in each case $|\sigma|^2 = \alpha = 1$, where

$$\alpha = \alpha_A = |a|^2|d|^2 + |b|^2|c|^2 - 2\Re(a\bar{c}d\bar{b}).$$

We are going to show that $\sqrt{\alpha} = \det(A) = |ad - aca^{-1}b| = |\sigma|$.

LEMMA 1. *If* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}(2, \mathbb{H})$, *then* $\sqrt{\alpha} = det(A) = |ad - aca^{-1}b| = |\sigma|$.

PROOF. We observe that

$$\begin{aligned} (\det(A))^2 &= |ad - aca^{-1}b|^2 = (ad - aca^{-1}b)\overline{(ad - aca^{-1}b)} \\ &= (ad - aca^{-1}b)(\bar{d}\bar{a} - \bar{b}\bar{a}^{-1}\bar{c}\bar{a}) \\ &= |a|^2|d|^2 + |b|^2|c|^2 - ad\bar{b}\bar{a}^{-1}\bar{c}\bar{a} - aca^{-1}b\bar{d}\bar{a} \\ &= |a|^2|d|^2 + |b|^2|c|^2 - 2\Re(aca^{-1}b\bar{d}\bar{a}) \\ &= |a|^2|d|^2 + |b|^2|c|^2 - 2\Re(c\bar{a}b\bar{d}) = |a|^2|d|^2 + |b|^2|c|^2 - 2\Re(a\bar{c}d\bar{b}) = \alpha. \end{aligned}$$

This completes the proof.

2.5. Some observations. It can be checked that $\alpha = \alpha_A = |l_{ij}|^2 = |r_{ij}|^2$, $1 \le i, j \le 2$, where l_{ij}, r_{ij} are defined as follows:

$$\begin{aligned} l_{11} &= da - dbd^{-1}c & l_{12} &= bdb^{-1}a - bc \\ l_{21} &= cac^{-1}d - cb & l_{22} &= ad - aca^{-1}b \\ r_{11} &= ad - bd^{-1}cd & r_{12} &= db^{-1}ab - cb \\ r_{21} &= ac^{-1}dc - bc & r_{22} &= da - ca^{-1}ba \end{aligned}$$

THEOREM 1 ([16]). *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}(2, \mathbb{H})$ *be such that* $\det(M) \neq 0$. *Then* M *is invertible*

$$M^{-1} = \begin{pmatrix} l_{11}^{-1}d & -l_{12}^{-1}b \\ -l_{21}^{-1}c & l_{22}^{-1}a \end{pmatrix} = \begin{pmatrix} dr_{11}^{-1} & -br_{12}^{-1} \\ -cr_{21}^{-1} & ar_{22}^{-1} \end{pmatrix}.$$

2.6. Notations. For our convenience we use the following notations:

$$\begin{aligned} d^{\sim} &= l_{11}^{-1}d, & c^{\sim} &= l_{21}^{-1}c, & b^{\sim} &= l_{12}^{-1}b, & a^{\sim} &= l_{22}^{-1}a, \\ d_{\sim} &= dr_{11}^{-1}, & c_{\sim} &= cr_{21}^{-1}, & b_{\sim} &= br_{12}^{-1}, & a_{\sim} &= ar_{22}^{-1}. \end{aligned}$$

Kellerhals has proved some interesting properties of these numbers given by following lemma:

Lemma 2 ([16]). *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}(2, \mathbb{H})$ *be invertible. Then we have the following properties:*

(1) $ad_{\sim} - bc_{\sim} = 1 = da_{\sim} - cb_{\sim}$, $d^{\sim}a - b^{\sim}c = 1 = a^{\sim}d - c^{\sim}b$.

(2) $ad^{\sim} - bc^{\sim} = 1 = da^{\sim} - cb^{\sim}$, $d_{\sim}a - b_{\sim}c = 1 = a_{\sim}d - c_{\sim}b$.

(3) $ab^{\sim} = ba^{\sim}$, $cd^{\sim} = dc^{\sim}$, $a^{\sim}c = c^{\sim}a$, $b^{\sim}d = d^{\sim}b$.

(4) $ab_{\sim} = ba_{\sim}$, $cd_{\sim} = dc_{\sim}$, $a_{\sim}c = c_{\sim}a$, $b_{\sim}d = d_{\sim}b$.

3. Jørgensen inequality for $\mathrm{SL}(2, \mathbb{H})$

The following proposition gives a Jørgensen inequality for a two-generator subgroup of $\mathrm{SL}(2, \mathbb{H})$ that has a semisimple generator.

Theorem 2. *Let* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $T = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, λ *is not similar to* μ, *generate a discrete non-elementary subgroup of* $\mathrm{SL}(2, \mathbb{H})$. *Then*

$$\{(\Re\lambda - \Re\mu)^2 + (|\Im\lambda| + |\Im\mu|)^2\}(1 + |bc|) \geq 1.$$

Proof. Let us suppose that

$$K_0 = \{(\Re\lambda - \Re\mu)^2 + (|\Im\lambda| + |\Im\mu|)^2\}(1 + |bc|) < 1.$$

Consider the Shimizu-Leutbecher sequence defined inductively by

$$S_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad S_{n+1} = \begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix} = S_n T S_n^{-1}.$$

Now,

$$S_{n+1} = S_n T S_n^{-1} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} d_n^{\sim} & -b_n^{\sim} \\ -c_n^{\sim} & a_n^{\sim} \end{pmatrix} \tag{3.1}$$

$$= \begin{pmatrix} a_n\lambda & b_n\lambda \\ c_n\lambda & d_n\lambda \end{pmatrix} \begin{pmatrix} d_n^{\sim} & -b_n^{\sim} \\ -c_n^{\sim} & a_n^{\sim} \end{pmatrix} \tag{3.2}$$

$$= \begin{pmatrix} a_n\lambda d_n^{\sim} - b_n\mu c_n^{\sim} & -a_n\lambda b_n^{\sim} + b_n\mu a_n^{\sim} \\ c_n\lambda d_n^{\sim} - d_n\mu c_n^{\sim} & -c_n\lambda b_n^{\sim} + d_n\mu a_n^{\sim} \end{pmatrix} \tag{3.3}$$

$$= \begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix}. \tag{3.4}$$

Thus,

$$a_{n+1} = a_n\lambda d_n^{\sim} - b_n\mu c_n^{\sim}, \qquad b_{n+1} = -a_n\lambda b_n^{\sim} + b_n\mu a_n^{\sim},$$

$$c_{n+1} = c_n\lambda d_n^{\sim} - d_n\mu c_n^{\sim}, \qquad d_{n+1} = -c_n\lambda b_n^{\sim} + d_n\mu a_n^{\sim}.$$

Now, we have

$$|b_{n+1}|\,|c_{n+1}| = |(-a_n\lambda b_n^{\sim} + b_n\mu a_n^{\sim})(c_n\lambda d_n^{\sim} - d_n\mu c_n^{\sim})|$$
$$= |a_n b_n c_n d_n|\,|\lambda - a_n^{-1} b_n \mu a_n^{\sim} b_n^{\sim -1}|\,|\lambda - c_n^{-1} d_n \mu c_n^{\sim} d_n^{\sim -1}|.$$

By an easy computation, we see that

$$\begin{aligned}|\lambda - a_n^{-1} b_n \mu a_n^{\sim} b_n^{\sim -1}| &= |\Re\lambda + \Im\lambda - \Re\mu - a_n^{-1} b_n(\Im\mu) a_n^{\sim} b_n^{\sim -1}|, \quad \text{since } a_n b_n^{\sim} = b_n a_n^{\sim}\\ &= |(\Re\lambda - \Re\mu) + \Im\lambda - a_n^{-1} b_n (\Im\mu) a_n^{\sim} b_n^{\sim -1}|\\ &= \sqrt{(\Re\lambda - \Re\mu)^2 + |\Im\lambda - a_n^{-1} b_n(\Im\mu) a_n^{\sim} b_n^{\sim -1}|^2}\\ &\le \sqrt{(\Re\lambda - \Re\mu)^2 + (|\Im\lambda| + |\Im\mu|)^2}.\end{aligned}$$

Similarly, we may deduce that

$$|\lambda - c_n^{-1} d_n \mu c_n^{\sim} d_n^{\sim -1}| \le \sqrt{(\Re\lambda - \Re\mu)^2 + (|\Im\lambda| + |\Im\mu|)^2}.$$

Therefore,

$$|b_{n+1}|\,|c_{n+1}| \le |a_n b_n c_n d_n|\{(\Re\lambda - \Re\mu)^2 + (|\Im\lambda| + |\Im\mu|)^2\}. \tag{3.5}$$

Since, $|a_n d_n| \le 1 + |b_n c_n|$, this implies

$$|b_{n+1}|\,|c_{n+1}| \le \{(\Re\lambda - \Re\mu)^2 + (|\Im\lambda| + |\Im\mu|)^2\}(1 + |b_n c_n|)|b_n c_n|. \tag{3.6}$$

Since, $K_0 = \{(\Re\lambda - \Re\mu)^2 + (|\Im\lambda| + |\Im\mu|)^2\}(1 + |bc|) < 1$, by using induction process we have the relation, $|b_{n+1} c_{n+1}| \le K_0^n |bc| \Rightarrow b_n c_n^{\sim} \to 0$, *as* $n \to \infty$, and so, $a_n d_n^{\sim} = 1 + b_n c_n^{\sim} \to 1$, *as* $n \to \infty$. Since, $|a_{n+1}| = |a_n \lambda d_n^{\sim} - b_n \mu c_n^{\sim}|$, $|d_{n+1}| = |-c_n\lambda b_n^{\sim} + d_n \mu a_n^{\sim}|$, we have

$$\begin{aligned}|\lambda|\,|a_n d_n^{\sim}| - |\mu|\,|b_n c_n^{\sim}| \le |a_{n+1}| &\le |\lambda|\,|a_n d_n^{\sim}| + |\mu|\,|b_n c_n^{\sim}|\\ &\Rightarrow |a_{n+1}| \to |\lambda| \qquad \text{as } n \to \infty.\end{aligned}$$

Similarly, we have $|d_{n+1}| \to |\mu|$, as $n \to \infty$. Again we have

$$\begin{aligned}|b_{n+1}| &= |-a_n\lambda b_n^{\sim} + b_n \mu a_n^{\sim}| = |a_n b_n^{\sim}|\,|\lambda - a_n^{-1} b_n \mu a_n^{\sim} b_n^{\sim -1}|\\ &\le |a_n b_n| \sqrt{(\Re\lambda - \Re\mu)^2 + (|\Im\lambda| + |\Im\mu|)^2}\\ &\le K_0 |a_n|\,|b_n| \to K_0|b_n|, \qquad \text{since } |a_n| \to 1\\ &\le K_0^n |b| \to 0, \qquad \text{since } K_0 < 1.\end{aligned}$$

Thus, for all positive integers, $|b_n| \to 0$ as $n \to \infty$, i.e. $b_n \to 0$ as $n \to \infty$. Similarly, we may show that $c_n \to 0$ as $n \to \infty$. Thus, the sequence S_n has a convergent subsequence and since the subgroup $\langle A, B\rangle$ is discrete, so we arrive at a contradiction. This proves the theorem.

Corollary 1. *Let* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $T = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, λ *is not similar to* μ, *generate a discrete non-elementary subgroup* $\langle S, T\rangle$ *of* $\mathrm{SL}(2, \mathbb{H})$. *Then*

$$2(\cosh \tau - \cos(\alpha + \beta))(1 + |bc|) \geq 1,$$

where $\alpha = arg(\lambda)$, $\beta = arg(\mu)$, $\tau = 2 \log|\lambda|$.

Proof. Without loss of generality, assume $|\lambda| = r \geq 1$. Observe that,

$$\begin{aligned}
&(\Re\lambda - \Re\mu)^2 + (|\Im\lambda| + |\Im\mu|)^2 \\
&\quad = \left(r \cos \alpha - \frac{1}{r} \cos \beta\right)^2 + \left(r|\sin \alpha| + \frac{1}{r}|\sin \beta|\right)^2 \\
&\quad = r^2 + \frac{1}{r^2} - 2(\cos \alpha \cos \beta - |\sin \alpha|\,|\sin \beta|) \\
&\quad = 2(\cosh \tau - \cos(\alpha + \beta)), \qquad \text{where } r = e^{\tau/2},\ \tau \geq 0.
\end{aligned}$$

This completes the proof.

Remark 1. *Kellerhals* [15, Proposition 3] *proved the above result assuming* T *hyperbolic, i.e.* $\tau \neq 0$. *However, it follows from above that Kellerhals's result carry over to the elliptic case as well, i.e. when* $\tau = 0$.

The Theorem 2 *also extends Waterman's Theorem* 9 *in* [31] *when restricted to the quaternionic set up. Note that* $\mathrm{SL}(2, \mathbb{H})$ *is not a Clifford group and hence, Theorem* 9 *of Waterman does not restrict to* $\mathrm{SL}(2, \mathbb{H})$. *For example, the element*

$$T = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\phi} \end{pmatrix},$$

does not belong to the Clifford group $\mathrm{SL}(2, C_2)$, *see* [31, p. 95], *but it belongs to the group* $\mathrm{SL}(2, \mathbb{H})$. *This class of elements are also covered by Theorem* 2.

The next theorem generalizes the Jørgensen's inequality in $\mathrm{SL}(2, \mathbb{H})$ for strictly hyperbolic elements with some given conditions. The formulation resembles the original inequality by Jørgensen.

Corollary 2. *Let* $A, B \in \mathrm{SL}(2, \mathbb{H})$ *be such that both* A *and the commutator* $[A, B]$ *are strictly hyperbolic. If* $\langle A, B\rangle$ *is a non-elementary discrete subgroup of* $\mathrm{SL}(2, \mathbb{H})$*, then*

$$|\delta_A^2 - 4| + |\delta_{ABA^{-1}B^{-1}} - 2| \geq 1.$$

Proof. Let $A = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}$, where $k > 1$ and $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$. Then $\delta_A = k + k^{-1}$ implies $|\delta_A^2 - 4| = |(k + k^{-1})^2 - 4| = |k - k^{-1}|^2$.

Now, we have

$$AB = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ k^{-1}c & k^{-1}d \end{pmatrix}$$

and

$$\begin{aligned} ABA^{-1}B^{-1} &= \begin{pmatrix} ka & kb \\ k^{-1}c & k^{-1}d \end{pmatrix}\begin{pmatrix} k^{-1} & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} c^{-1}d\sigma^{-1}c & -a^{-1}b\sigma^{-1}cac^{-1} \\ -\sigma^{-1}c & \sigma^{-1}cac^{-1} \end{pmatrix} \\ &= \begin{pmatrix} a & k^2b \\ k^{-2}c & d \end{pmatrix}\begin{pmatrix} c^{-1}d\sigma^{-1}c & -a^{-1}b\sigma^{-1}cac^{-1} \\ -\sigma^{-1}c & \sigma^{-1}cac^{-1} \end{pmatrix} \\ &= \begin{pmatrix} ac^{-1}d\sigma^{-1}c - k^2b\sigma^{-1}c & (k^2-1)b\sigma^{-1}cac^{-1} \\ (k^{-2}-1)d\sigma^{-1}c & d\sigma^{-1}cac^{-1} - k^{-2}ca^{-1}b\sigma^{-1}cac^{-1} \end{pmatrix}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \delta_{ABA^{-1}B^{-1}} &= \Re(ac^{-1}d\sigma^{-1}c - k^2b\sigma^{-1}c) + \Re(d\sigma^{-1}cac^{-1} - k^{-2}ca^{-1}b\sigma^{-1}cac^{-1}) \\ &= \Re(ac^{-1}d\sigma^{-1}c) - k^2\Re(b\sigma^{-1}c) + \Re(d\sigma^{-1}cac^{-1}) \\ &\quad - k^{-2}\Re(ca^{-1}b\sigma^{-1}cac^{-1}) \\ &= 2\Re(cac^{-1}d\bar{\sigma}) - (k^2 + k^{-2})\Re(b\bar{\sigma}c) \\ &= 2(1 + \Re(cb\bar{\sigma})) - (k^2 + k^{-2})\Re(b\bar{\sigma}c), \qquad \text{since } \sigma = cac^{-1}d - cb. \\ &= 2 - (k^2 + k^{-2} - 2)\Re(b\bar{\sigma}c) \\ &= 2 - (k - k^{-1})^2\Re(b\bar{\sigma}c). \end{aligned}$$

This implies that $|\delta_{ABA^{-1}B^{-1}} - 2| = |k - k^{-1}|^2|\Re(b\bar{\sigma}c)|$. Since $ABA^{-1}B^{-1}$ is strictly hyperbolic, we have

$$b\bar{\sigma}c = b\bar{d}c\bar{a} - |bc|^2 \Rightarrow \Re(b\bar{\sigma}c) = \Re(b\bar{d}c\bar{a}) - |bc|^2 = \Re(a\bar{c}d\bar{b}) - |bc|^2.$$

Also, we have $b\bar{\sigma}cac^{-1} = 0 \Rightarrow b\overline{(cac^{-1}d - cb)}cac^{-1} = 0 \Rightarrow |b|^2(|ad|^2 - \bar{b}a\bar{c}d) = 0 \Rightarrow |ad|^2 = \bar{b}a\bar{c}d$, since $bc \neq 0$, for otherwise $\langle A, B\rangle$ becomes elementary.

This shows that $b\bar{\sigma}c = |ad|^2 - |bc|^2 = \Re(b\bar{\sigma}c)$. Thus,

$$|\delta_A^2 - 4| + |\delta_{ABA^{-1}B^{-1}} - 2| = |k - k^{-1}|^2(1 + |bc|).$$

Now the theorem follows from Theorem 5.

The following two corollaries give weaker versions of Theorem 2.

Corollary 3. *Let* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $T = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ *generate a non-elementary discrete subgroup of* $\mathrm{SL}(2, \mathbb{H})$.

Then we have

$$\beta(T)L^k \geq 1,$$

where

$$\beta(T) = \sup_{e, f \neq 0, \infty} |(\lambda - e\mu e^{-1})(\lambda - f\mu f^{-1})|,$$

$$L = 1 + |\mu| \qquad \textit{and} \qquad k = [1 + |bc|] + 1,$$

[.] *denotes the greatest integer function.*

Proof. Since $L > 1$, $k > 2$, note that $1 + |bc| \leq k \leq L^k$. Let

$$K = (\Re\lambda - \Re\mu)^2 + (|\Im\lambda| + |\Im\mu|)^2.$$

Using conjugation if necessary, suppose without loss of generality that λ, μ are complex numbers. Note that, both $\beta(T)$ and K are invariant if we conjugate the matrix T in the above theorems to a diagonal matrix in $\mathrm{SL}(2, \mathbb{H})$ over the complex numbers. Note that

$$|\lambda - j\mu j^{-1}| = \sqrt{(\Re\lambda - \Re\mu)^2 + (|\Im\lambda|^2 + |\Im\mu|^2)},$$

hence, $K \leq \beta(T)$.

Further note that a diagonal element $T \in \mathrm{SL}(2, \mathbb{H})$ can be conjugated to a diagonal matrix $T' \in \mathrm{SL}(2, \mathbb{C})$ and, the conjugation can be done using a diagonal element in $\mathrm{SL}(2, \mathbb{H})$. So, given $\langle S, T \rangle$ as in the above results, if we conjugate it to $\langle DSD^{-1}, DTD^{-1} \rangle$, where D a diagonal matrix in $\mathrm{SL}(2, \mathbb{H})$, then the conjugation makes DTD^{-1} a diagonal matrix over $\mathbb{C}$. And further, it is easy to check that if $DTD^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, then $|b'c'| = |bc|$. Thus conjugation of $\langle S, T \rangle$ by a diagonal matrix in $\mathrm{SL}(2, \mathbb{H})$ does not change the left hand sides of the above inequalities.

Since $\langle S, T \rangle$ is discrete, non-elementary, $K(1 + |bc|) > 1$. Hence

$$\beta(T)L^k \geq K(1 + |bc|) \geq 1.$$

This completes the proof.

COROLLARY 4. *Suppose that* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $T = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ *generate a non-elementary discrete subgroup of* $\mathrm{SL}(2, \mathbb{H})$. *Then we have*

$$\beta(T)(1 + |bc|) \geq 1,$$

where $\beta(T) = \sup_{e, f \neq 0, \infty} |(\lambda - e\mu e^{-1})(\lambda - f\mu f^{-1})|$.

The next theorem gives Jørgensen inequality for a two-generator subgroup where one of the generators fixes ∞.

THEOREM 3. *Let* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $T = \begin{pmatrix} \lambda & \eta \\ 0 & \mu \end{pmatrix}$, $\Re\lambda = \Re\mu \neq 0$, $|\lambda| \leq 1 \leq |\mu|$, *generate a non-elementary discrete subgroup of* $\mathrm{SL}(2, \mathbb{H})$. *Suppose* $S(\lambda, \mu) = |\mu|(|\Im\lambda| + |\Im\mu|) \leq \frac{1}{4\sqrt{2}}$. *Then we have*

$$|c|\sqrt{|\tau_0|\,|t_0|} \geq \frac{1 + \sqrt{1 - 4\sqrt{2}S(\lambda, \mu)}}{2},$$

where $\tau_0 = \lambda(-c^{-1}d) + \eta + (c^{-1}d)\mu$ *and,* $t_0 = \lambda(ac^{-1}) + \eta - (ac^{-1})\mu$.

PROOF. Let $\alpha = \arg\lambda$, $\beta = \arg\mu$. Denote $r = |\lambda|$, where we see that $r^2 \cos\alpha = \cos\beta$. Consider the Shimizu-Leutbecher sequence

$$S_0 = S, \qquad S_{n+1} = S_n T S_n^{-1}, \qquad \text{where } S_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}.$$

Now, we have

$$\begin{aligned} S_{n+1} &= S_n T S_n^{-1} \\ &= \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} \lambda & \eta \\ 0 & \mu \end{pmatrix} \begin{pmatrix} d_n^{\sim} & -b_n^{\sim} \\ -c_n^{\sim} & a_n^{\sim} \end{pmatrix} \\ &= \begin{pmatrix} a_n\lambda d_n^{\sim} - a_n\eta c_n^{\sim} - b_n\mu c_n^{\sim} & -a_n\lambda b_n^{\sim} + a_n\eta a_n^{\sim} + b_n\mu a_n^{\sim} \\ c_n\lambda d_n^{\sim} - c_n\eta c_n^{\sim} - d_n\mu c_n^{\sim} & -c_n\lambda b_n^{\sim} + c_n\eta a_n^{\sim} + d_n\mu a_n^{\sim} \end{pmatrix}. \end{aligned}$$

Define τ_n, t_n by

$$\tau_n = \lambda(-c_n^{-1}d_n) + \eta + (c_n^{-1}d_n)\mu, \tag{3.7}$$

$$t_n = \lambda(a_n c_n^{-1}) + \eta - (a_n c_n^{-1})\mu. \tag{3.8}$$

Since $\Re\lambda = \Re\mu$ by assumption, using this we obtain

$$\tau_n = \Im\lambda(-c_n^{-1}d_n) + \eta + (c_n^{-1}d_n)\Im\mu, \tag{3.9}$$

$$t_n = \Im\lambda(a_n c_n^{-1}) + \eta - (a_n c_n^{-1})\Im\mu. \tag{3.10}$$

We see that

$$\begin{aligned} c_{n+1} &= c_n\lambda d_n^{\sim} - c_n c_n^{\sim} - d_n\lambda c_n^{\sim} \\ &= c_n(\Im\lambda(d_n^{\sim} c_n^{\sim -1}) - \eta - (c_n^{-1} d_n)\Im\mu)c_n^{\sim} \\ &= -c_n\{\Im\lambda(-c_n^{-1} d_n) + \eta + (c_n^{-1} d_n)\Im\mu\}c_n^{\sim} \\ &= -c_n\tau_n c_n^{\sim} \\ &\Rightarrow |c_{n+1}| = |\tau_n c_n|\,|c_n|. \end{aligned}$$

$$\begin{aligned} d_{n+1} &= -c_n\lambda b_n^{\sim} + c_n\eta a_n^{\sim} + d_n\mu a_n^{\sim} \\ &= \Re\lambda(d_n a_n^{\sim} - c_n b_n^{\sim}) + c_n\{\Im\lambda(-b_n^{\sim} a_n^{\sim -1}) + \eta + (c_n^{-1} d_n)\Im\mu\}a_n^{\sim} \\ &= \Re\lambda + c_n\{\Im\lambda(a_n^{-1} c_n^{\sim -1} - c_n^{-1} d_n) + \eta + (c_n^{-1} d_n)\Im\mu\}a_n^{\sim} \\ &= r\cos\alpha + c_n\tau_n a_n^{\sim} + c_n\Im\lambda a_n^{-1} c_n^{\sim -1} a_n^{\sim}. \end{aligned}$$

By similar computations, we have

$$a_{n+1} = r\cos\alpha - a_n\tau_n c_n^{\sim} + c_n^{\sim -1}\Im\mu c_n^{\sim}.$$

Using above equalities, we see that

$$\begin{aligned} \tau_{n+1} &= \Im\lambda(-c_{n+1}^{-1} d_{n+1}) + \eta + (c_{n+1}^{-1} d_{n+1})\Im\mu \\ &= \Im\lambda\{c_n^{\sim -1}\tau_n^{-1} c_n^{-1}(r\cos\alpha + c_n\tau_n a_n^{\sim} + c_n\Im\lambda a_n^{-1} c_n^{\sim -1} a_n^{\sim})\} \\ &\quad + \eta - \{c_n^{\sim -1}\tau_n^{-1} c_n^{-1}(r\cos\alpha + c_n\tau_n a_n^{\sim} + c_n\Im\lambda a_n^{-1} c_n^{\sim -1} a_n^{\sim})\}\Im\mu \\ &= t_n + r\cos\alpha\,\Im\lambda c_n^{\sim -1}\tau_n^{-1} c_n^{-1} + \Im\lambda c_n^{\sim -1}\tau_n^{-1}\Im\lambda a_n^{-1} c_n^{\sim -1} a_n^{\sim} \\ &\quad - r\cos\alpha c_n^{\sim -1}\tau_n^{-1} c_n^{-1}\Im\mu - c_n^{\sim -1}\tau_n^{-1}\Im\lambda a_n^{-1} c_n^{\sim -1} a_n^{\sim}\Im\mu \end{aligned}$$

$$\Rightarrow |\tau_{n+1}| \le |t_n| + \frac{(r^2|\sin\alpha| + |\sin\beta|)(|\cos\alpha| + |\sin\alpha|)}{|\tau_n c_n^2|},\ \text{using},\ |a_n^{\sim}| = |a_n|\,|l_{22}^{-1}|,$$

$$|l_{22}^{-1}| = \det S_n = 1,\ \text{and},\ |\Im\lambda| = |\lambda|\,|\sin\alpha|,\ \ |\Im\mu| = |\mu|\,|\sin\beta|$$

$$\Rightarrow |\tau_{n+1} c_{n+1}| \le |\tau_n c_n|\,|t_n c_n| + \sqrt{2}S(\lambda,\mu),\ \text{since},\ |\cos\alpha| + |\sin\alpha| \le \sqrt{2},\ \text{where},$$

$$\begin{aligned} S(\lambda,\mu) &= (|\sin\alpha| + |\mu|^2|\sin\beta|) \\ &= \left(\frac{|\Im\lambda|}{|\lambda|} + |\mu|\,|\Im\mu|\right) \\ &= |\mu|(|\Im\lambda| + |\Im\mu|). \end{aligned}$$

Similarly, we have $|t_{n+1}c_{n+1}| \leq |\tau_n c_n|\,|t_n c_n| + \sqrt{2}S(\lambda,\mu)$, and

$$|d_{n+1}| \leq |\tau_n c_n|\,|a_n| + 2r,$$

$$|a_{n+1}| \leq |\tau_n c_n|\,|a_n| + \frac{2}{r},$$

$$|b_{n+1}| \leq |a_n|^2 + rS(\lambda,\mu)|a_n|\,|b_n|.$$

Consider the sequence

$$x_0 = |c|\sqrt{|\tau_0|\,|t_0|}, \qquad x_{n+1} = x_n^2 + \sqrt{2}S(\lambda,\mu).$$

If $0 \leq x_0 < \frac{1+\sqrt{1-4\sqrt{2}S(\lambda,\mu)}}{2} \leq 1$, then $\{x_n\}$ is a monotonically decreasing sequence of real numbers and is bounded above by $\frac{1+\sqrt{1-4\sqrt{2}S(\lambda,\mu)}}{2}$ and converges to $\frac{1-\sqrt{1-4\sqrt{2}S(\lambda,\mu)}}{2}$. Hence

$$|t_n c_n| < \frac{1+\sqrt{1-4\sqrt{2}S(\lambda,\mu)}}{2} \leq 1, \qquad \text{and}$$

$$|\tau_n c_n| < \frac{1+\sqrt{1-4\sqrt{2}S(\lambda,\mu)}}{2} \leq 1.$$

One a subsequence $|t_n c_n|$ and $|\tau_n c_n|$ converges to values at most $\frac{1-\sqrt{1-4\sqrt{2}S(\lambda,\mu)}}{2}$. Hence on a subsequence $|a_n|$, $|b_n|$, $|c_n|$, $|d_n|$ converge. In particular, $|c_n| \to 0$. Note that $c_n \neq 0$ unless $c = 0$, but c can not be zero as the group $\langle S, T\rangle$ is non-elementary by assumption. Thus the theorem follows.

COROLLARY 5. *Suppose* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $T = \begin{pmatrix} \lambda & \eta \\ 0 & \mu \end{pmatrix}$, *where* $\Re\lambda = \Re\mu \neq 0$, $\eta \neq 0$, $|\lambda| \leq 1 \leq |\mu|$, *generate a non-elementary discrete subgroup of* $\mathrm{SL}(2,\mathbb{H})$. *Suppose*

$$S'(\lambda,\mu) = \frac{|\mu|}{|\eta|^2}(|\Im\lambda| + |\Im\mu|).$$

Then we have

$$|c|\sqrt{|\tau_0'|\,|t_0'|} \geq \frac{1+\sqrt{1-4\sqrt{2}|\eta|^2 S'(\lambda,\mu)}}{2|\eta|},$$

where

$$\tau_0' = \lambda(-c^{-1}d)\eta^{-1} + 1 + (c^{-1}d)\mu\eta^{-1} \qquad \textit{and}, \qquad t_0' = \lambda(ac^{-1})\eta^{-1} + 1 - (ac^{-1})\mu\eta^{-1}.$$

PROOF. If $\eta \neq 0$, we write $\tau_0 = \tau_0'\eta$ and $t_0 = t_0'\eta$. Then the result follows from the inequality in Theorem 3.

Corollary 6. *Suppose* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $T = \begin{pmatrix} \lambda & \eta \\ 0 & \mu \end{pmatrix}$, $\Re\lambda = \Re\mu = 0$, $|\lambda| \le 1 \le |\mu|$, *generate a non-elementary discrete subgroup of* $\mathrm{SL}(2, \mathbb{H})$. *Suppose* $S(\lambda, \mu) = |\mu|(|\Im\lambda| + |\Im\mu|) \le \frac{1}{4}$. *Then we have*

$$|c|\sqrt{|\tau_0|\,|t_0|} \ge \frac{1 + \sqrt{1 - 4S(\lambda, \mu)}}{2},$$

where $\tau_0 = \lambda(-c^{-1}d) + \eta + (c^{-1}d)\mu$ *and,* $t_0 = \lambda(ac^{-1}) + \eta - (ac^{-1})\mu$.

Proof. In this case, we proceed as in the proof of the previous theorem. The only difference from the previous proof is essentially the following bound:

$$|\tau_{n+1}| \le |t_n| + \frac{(r^2|\sin\alpha| + |\sin\beta|)|\sin\alpha|}{|\tau_n c_n^2|}, \text{ using, } |\tilde{a_n}| = |a_n|\,|l_{22}^{-1}|,$$

$$|l_{22}^{-1}| = \det S_n = 1, \text{ and, } |\Im\lambda| = |\lambda|\,|\sin\alpha|, \ |\Im\mu| = |\mu|\,|\sin\beta|$$

$$\Rightarrow |\tau_{n+1}c_{n+1}| \le |\tau_n c_n|\,|t_n c_n| + S(\lambda, \mu), \text{ where}$$

$$\begin{aligned} S(\lambda, \mu) &= (|\sin\alpha| + |\mu|^2|\sin\beta|) \\ &= \left(\frac{|\Im\lambda|}{|\lambda|} + |\mu|\,|\Im\mu|\right) \\ &= |\mu|(|\Im\lambda| + |\Im\mu|). \end{aligned}$$

Noting this bound, the rest is similar.

Given any parabolic transformation in $\mathrm{SL}(2, \mathbb{H})$, it is conjugate to a transformation of the form

$$T = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \qquad |\lambda| = 1,$$

and moreover, one can choose $\Re(\lambda) = 0$ up to conjugacy. Using Corollary 6, this recovers Waterman's result [31, Theorem 8] in $\mathrm{SL}(2, \mathbb{H})$.

Corollary 7. *If* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $T = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $|\lambda| = 1$ *generate a non-elementary discrete subgroup of* $\mathrm{SL}(2, \mathbb{H})$ *with* T *parabolic fixing* ∞, *then*

$$|c|\sqrt{|T(ac^{-1}) - ac^{-1}|}\sqrt{|T(-c^{-1}d) - (-c^{-1}d)|} \ge \frac{1 + \sqrt{1 - 8|\Im\lambda|}}{2}.$$

Proof. Note that $T(ac^{-1}) = (\lambda(ac^{-1}) + 1)\lambda^{-1}$ and $T(-c^{-1}d) = (\lambda(-c^{-1}d) + 1)\lambda^{-1}$. Now, $T(ac^{-1}) - (ac^{-1}) = (\lambda(ac^{-1}) + 1 - (ac^{-1})\lambda)\lambda^{-1} =$

$t_0\lambda^{-1}$. Similarly, $T(-c^{-1}d) - (-c^{-1}d) = \tau_o\lambda^{-1}$. Since $|\lambda| = 1$, the result follows.

Recently, Erlandsson and Zakeri [5] have proved a geometric version of Theorem 3. Their geometric inequality does not depend on any quantity like $S(\lambda, \mu)$. Also, in the asymptotic case, it covers some of the two-generators groups whose discreteness remain inconclusive by Corollary 7. However, the inequality of Erlandsson and Zakeri does not involve the algebraic coefficients of the matrices. The above results give a more explicit algorithm involving the matrix coefficients to test discreteness.

Using similar argument as in the proof of Theorem 3, we can also prove the following theorem that gives Jørgensen inequality for a two-generator subgroup where one of the generators has a fixed point 0.

THEOREM 4. *Suppose* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $T = \begin{pmatrix} \lambda & 0 \\ \eta & \mu \end{pmatrix}$, *where* $\Re\lambda = \Re\mu = \kappa$, $|\lambda| \le 1 \le |\mu|$, *generate a non-elementary discrete subgroup of* $\mathrm{SL}(2, \mathbb{H})$. *Suppose,* $S(\lambda, \mu) = |\mu|(|\Im\lambda| + |\Im\mu|) \le \varepsilon$. *Then*

$$|c|\sqrt{|\tau_0|\,|t_0|} \ge \frac{1 + \sqrt{1 - \varepsilon^{-1}S(\lambda, \mu)}}{2},$$

where $\tau_0 = \mu(-b^{-1}a) + \eta + (b^{-1}a)\lambda$, $t_0 = \mu(db^{-1}) + \eta - (db^{-1})\lambda$ *and* $\varepsilon = \frac{1}{4\sqrt{2}}$ *or* $\frac{1}{4}$ *depending upon* $\kappa \ne 0$ *or* $\kappa = 0$.

4. Extremality of Jørgensen inequalities

The following theorem generalizes Theorem-1 of Jørgensen-Kikka [8].

THEOREM 5. *Let* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $T = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \in \mathrm{SL}(2, \mathbb{H})$. *Suppose,* $\langle S, T \rangle$ *is discrete, non-elementary and for* $\alpha = arg(\lambda)$, $\beta = arg(\mu)$, $\tau = 2\log|\lambda|$,

$$\{(\Re\lambda - \Re\mu)^2 + (|\Im\lambda| + |\Im\mu|)^2\}(1 + |bc|) = 1.$$

We consider the Shimizu-Leutbechar sequence

$$S_0 = S, \qquad S_{n+1} = S_n T S_n^{-1}.$$

Then T *and* $S_{n+1} = S_n T S_n^{-1}$ *generate a non-elementary discrete group and*

$$\{(\Re\lambda - \Re\mu)^2 + (|\Im\lambda| + |\Im\mu|)^2\}(1 + |b_n c_n|) = 1.$$

PROOF. We consider the Shimizu-Leutbechar sequence

$$S_0 = S, \qquad S_{n+1} = S_n T S_n^{-1}.$$

From relation (3.1) we get

$$a_{n+1} = a_n \lambda \tilde{d}_n - b_n \mu \tilde{c}_n, \qquad b_{n+1} = -a_n \lambda \tilde{b}_n + b_n \mu \tilde{a}_n$$

$$c_{n+1} = c_n \lambda \tilde{d}_n - d_n \mu \tilde{c}_n, \qquad d_{n+1} = -c_n \lambda \tilde{b}_n + d_n \mu \tilde{a}_n$$

and we also have

$$\begin{aligned} |b_{n+1}|\,|c_{n+1}| &= |(-a_n \lambda \tilde{b}_n + b_n \mu \tilde{a}_n)(c_n \lambda \tilde{d}_n - d_n \mu \tilde{c}_n)| \\ &= |a_n b_n c_n d_n|\,|\lambda - a_n^{-1} b_n \mu \tilde{a}_n \tilde{b}_n^{-1}|\,|\lambda - c_n^{-1} d_n \mu \tilde{c}_n \tilde{d}_n^{-1}|. \end{aligned}$$

This implies (see (3.6) in the proof of Theorem 2)

$$|b_{n+1} c_{n+1}| \le \{(\Re\lambda - \Re\mu)^2 + (|\Im\lambda| + |\Im\mu|)^2\}(1 + |b_n c_n|)|b_n c_n|. \tag{4.1}$$

Let

$$K = (\Re\lambda - \Re\mu)^2 + (|\Im\lambda| + |\Im\mu|)^2. \tag{4.2}$$

Construct the sequence w_n where

$$w_0 = |bc|, \qquad w_n = |b_n c_n|.$$

It follows from (4.1) that $w_{n+1} \le K w_n (1 + w_n)$. Now note that $K(1 + w_0) = 1$. Now $w_0 \neq 0$, for otherwise, S and T will have a common fixed point. Hence $K < 1$.

Observe that

$$1 \le K(1 + w_1) \le K(1 + K w_0 (1 + w_0)) \le K(1 + w_0) = 1,$$

and hence $K(1 + w_1) = 1$. By induction it follows that $K(1 + w_n) = 1$ for all $n \ge 0$. Since $K < 1$, it follow that $w_n \neq 0$ for all n and hence the result follows.

The following corollary generalizes Theorem-2 of Jørgensen and Kikka [8].

Corollary 8. *Let* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $T = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \in \mathrm{SL}(2, \mathbb{H})$. *If* $\langle S, T \rangle$ *is discrete, non-elementary and*

$$\{(\Re\lambda - \Re\mu)^2 + (|\Im\lambda| + |\Im\mu|)^2\}(1 + |bc|) = 1,$$

then T *is elliptic of order at least seven.*

Proof. If possible suppose T is hyperbolic. As in the above proof, it follows from the extremal relation that $K < 1$. Now, let $\arg\lambda = \alpha$ and $\arg\mu = \beta$. Then we have

$$
\begin{aligned}
K &= (\Re\lambda - \Re\mu)^2 + (|\Im\lambda| + |\Im\mu|)^2 \\
&= |\lambda|^2 + |\mu|^2 + 2(|\Im\lambda|\,|\Im\mu| - \Re\lambda\Re\mu) \\
&= |\lambda|^2 + |\mu|^2 + 2|\sin\alpha|\,|\sin\beta| - \cos\alpha\cos\beta \\
&= |\lambda|^2 + |\mu|^2 - 2\cos(\alpha+\beta).
\end{aligned}
$$

Let $|\lambda| = e^{\tau/2}$. Then using $\cosh(\tau) = \frac{e^{\tau}+e^{-\tau}}{2}$, observe that

$$
\begin{aligned}
K &= e^{\tau} + e^{-\tau} - 2\cos(\alpha+\beta) \\
&\geq e^{\tau} + e^{-\tau} + 2 = (e^{\tau/2} + e^{-\tau/2})^2.
\end{aligned}
$$

Since $e^{\tau/2} + e^{-\tau/2} > 1$, this implies $K > 1$. This is a contradiction. Hence T must be elliptic.

Since T is elliptic, $\tau = 0$. Now, $K = 1$ implies, $\cos(\alpha+\beta) > \frac{1}{2}$. Thus $0 < \alpha + \beta < \frac{\pi}{3}$. This implies that the order of T must be at least seven. This completes the proof.

COROLLARY 9. *Let* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $T = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \in \mathrm{SL}(2, \mathbb{H})$. *Suppose* $\langle S, T\rangle$ *is discrete, non-elementary and*

$$\beta(T)(1 + |bc|) = 1,$$

then T *is elliptic of order at least seven.*

PROOF. Suppose, up to conjugacy, λ, μ are complex numbers. Then $K \leq \beta(T)$. Since $\langle S, T\rangle$ is discrete, we must have $K(1 + |bc|) \geq 1$. Hence the equality in the hypothesis implies $K(1 + |bc|) = 1$. The result now follows from the above corollary.

COROLLARY 10. *Let* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $T = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \in \mathrm{SL}(2, \mathbb{H})$. *If* $\langle S, T\rangle$ *is discrete, non-elementary and*

$$\beta(T)L^k = 1,$$

where $k = [1 + |bc|] + 1 > 2$ *and* $L = 1 + |\mu| > 1$, *then* T *is elliptic of order at least seven.*

PROOF. Up to conjugacy, we assume λ, μ are complex numbers. It is enough to show that $K(1 + |bc|) = 1$, where $K = (\Re\lambda - \Re\mu)^2 + (|\Im\lambda| + |\Im\mu|)^2$. Since the subgroup $\langle S, T\rangle$ generates a discrete non-elementary subgroup of $\mathrm{SL}(2, \mathbb{H})$, then we have $K(1 + |bc|) \geq 1$. Now note that

$$K \leq \beta(T) = \frac{1}{L^k} \leq \frac{1}{1 + |bc|},$$

This imples $K(1+|bc|) \le 1$. Hence, $K(1+|bc|) = 1$. The result now follows from Corollary 8.

The following characterizes non-extreme groups.

Corollary 11. *Let* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $T = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ *generate a discrete non-elementary subgroup of* $\mathrm{SL}(2, \mathbb{H})$. *Suppose*

$$||ad| - 1| > \left(\cot^2\left(\frac{\alpha+\beta}{2}\right) - 3\right).$$

Then $\langle S, T \rangle$ *is not an extreme group.*

Proof. If possible suppose $\langle S, T \rangle$ satisfy equality in Jørgensen inequality. Note that, it follows from the equality in Jørgensen inequality that

$$|bc| = \frac{1-K}{K}, \tag{4.3}$$

where $K = (\Re\lambda - \Re\mu)^2 + (|\Im\lambda| + |\Im\mu|)^2$. The condition $|\sigma| = |ad - aca^{-1}b| = 1$ implies

$$\begin{aligned} 1 &\le |ad| + |bc| \\ \Rightarrow |ad| &\ge 1 - |bc| \\ &= K(1+|bc|) - |bc| \\ &= K + (K-1)\frac{(1-K)}{K} \\ &= 2 - \frac{1}{K}. \end{aligned}$$

This implies

$$|ad| \ge 1 - |bc|. \tag{4.4}$$

Also we have from $|\sigma| = 1$ that $|ad| - |bc| \le 1$. This implies $|ad| \le 1 + |bc|$. Combining this with (4.4) we get

$$||ad| - 1| \le |bc|.$$

Now we see that $K = 2(1 - \cos(\alpha+\beta))$ and

$$\begin{aligned} |bc| = \frac{1-K}{K} &= \frac{2\cos(\alpha+\beta) - 1}{2 - 2\cos(\alpha+\beta)} \\ &= \frac{\cos^2\left(\frac{\alpha+\beta}{2}\right) - 3\sin^2\left(\frac{\alpha+\beta}{2}\right)}{4\sin^2\left(\frac{\alpha+\beta}{2}\right)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\cot^2\left(\frac{\alpha+\beta}{2}\right) - 3}{4} \\
&\le \left(\cot^2\left(\frac{\alpha+\beta}{2}\right) - 3\right).
\end{aligned}
$$

Hence, we have

$$
||ad| - 1| \le \left(\cot^2\left(\frac{\alpha+\beta}{2}\right) - 3\right),
$$

which is a contradiction. This proves the result.

THEOREM 6. *Suppose* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $T = \begin{pmatrix} \lambda & \eta \\ 0 & \mu \end{pmatrix}$, $\Re\lambda = \Re\mu = \kappa$, $|\lambda| \le 1 \le |\mu|$, *generate a non-elementary discrete subgroup of* $\mathrm{SL}(2, \mathbb{H})$. *Suppose*

$$
|c|\sqrt{|\tau_0|\,|t_0|} = \frac{1 + \sqrt{1 - \varepsilon^{-1} S(\lambda, \mu)}}{2},
$$

where $S(\lambda, \mu) = |\mu|(|\Im\lambda| + |\Im\mu|) \le \varepsilon$ *and,* $\varepsilon = \frac{1}{4\sqrt{2}}$ *or* $\frac{1}{4}$ *depending on* $\kappa \neq 0$ *or* $\kappa = 0$. *We consider the Shimizu-Leutbechar sequence*

$$
S_0 = S, \qquad S_{n+1} = S_n T S_n^{-1}.
$$

Then, for each n, $\langle S_n, T\rangle$ *is a non-elementary discrete subgroup of* $\mathrm{SL}(2, \mathbb{H})$ *and*

$$
|c_n|\sqrt{|\tau_n|\,|t_n|} = \frac{1 + \sqrt{1 - \varepsilon^{-1} S(\lambda, \mu)}}{2}.
$$

where $\tau_n = \lambda(-c_n^{-1} d_n) + \eta + (c_n^{-1} d_n)\mu$, $t_n = \lambda(a_n c_n^{-1}) + \eta - (a_n c_n^{-1})\mu$.

PROOF. We prove the result assuming $\kappa \neq 0$. The case $\kappa = 0$ is similar.

Consider the Shimizu-Leutbecher sequence $S_0 = S$, $S_{n+1} = S_n T S_n^{-1}$, where $S_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. Now, we have

$$
\begin{aligned}
S_{n+1} &= S_n T S_n^{-1} \\
&= \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} \lambda & \eta \\ 0 & \mu \end{pmatrix} \begin{pmatrix} d_n^{\sim} & -b_n^{\sim} \\ -c_n^{\sim} & a_n^{\sim} \end{pmatrix} \\
&= \begin{pmatrix} a_n\lambda d_n^{\sim} - a_n\eta c_n^{\sim} - b_n\mu c_n^{\sim} & -a_n\lambda b_n^{\sim} + a_n\eta a_n^{\sim} + b_n\mu a_n^{\sim} \\ c_n\lambda d_n^{\sim} - c_n\eta c_n^{\sim} - d_n\mu c_n^{\sim} & -c_n\lambda b_n^{\sim} + c_n\eta a_n^{\sim} + d_n\mu a_n^{\sim} \end{pmatrix}
\end{aligned}
$$

Define τ_n, t_n by

$$\tau_n = \lambda(-c_n^{-1}d_n) + \eta + (c_n^{-1}d_n)\mu \tag{4.5}$$

$$t_n = \lambda(a_n c_n^{-1}) + \eta - (a_n c_n^{-1})\mu \tag{4.6}$$

We see that

$$\begin{aligned} c_{n+1} &= c_n\lambda d_n^{\sim} - c_n\eta c_n^{\sim} - d_n\mu c_n^{\sim} \\ &= -c_n(\lambda(-d_n^{\sim}c_n^{\sim -1}) + \eta + (c_n^{-1}d_n)\mu)c_n^{\sim} \\ &= -c_n\tau_n c_n^{\sim}. \end{aligned}$$

Thus $|c_{n+1}| = |\tau_n c_n|\,|c_n|$. Similarly, we have

$$\begin{aligned} |d_{n+1}| &\le |\tau_n c_n|\,|a_n| + 2r, \\ |a_{n+1}| &\le |\tau_n c_n|\,|a_n| + \frac{2}{r}. \end{aligned}$$

Also, $|b_{n+1}| \le |a_n|^2 + rS(\lambda,\mu)|a_n|\,|b_n|$, as in the proof of Theorem 3. We have

$$\begin{aligned} |\tau_{n+1}c_{n+1}| &\le |\tau_n c_n|\,|t_n c_n| + \sqrt{2}S(\lambda,\mu), \\ |t_{n+1}c_{n+1}| &\le |\tau_n c_n|\,|t_n c_n| + \sqrt{2}S(\lambda,\mu). \end{aligned}$$

Consider the sequence

$$x_0 = |c|\sqrt{|\tau_0|\,|t_0|}, \qquad x_{n+1} = x_n^2 + \sqrt{2}S(\lambda,\mu), \qquad \text{where } S(\lambda,\mu) \le \frac{1}{4\sqrt{2}}.$$

Note that $\{x_n\}$ is a monotonically decreasing sequence of real numbers and is bounded above by $\frac{1+\sqrt{1-4\sqrt{2}S(\lambda,\mu)}}{2}$. By the hypothesis $x_0 = \frac{1+\sqrt{1-4\sqrt{2}S(\lambda,\mu)|}}{2}$. Hence $\{x_n\}$ must be a constant sequence. In particular, $c_n \ne 0$ for all n and hence, S_n and T can not have a common fixed point. Thus $\langle S_n, T\rangle$ is non-elementary.

Corollary 12. *Let* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $T = \begin{pmatrix} \lambda & \eta \\ 0 & \mu \end{pmatrix}$, *where* $\Re\lambda = \Re\mu$, $|\lambda| \le 1 \le |\mu|$, *generate a discrete, non-elementary subgroup of* $\mathrm{SL}(2,\mathbb{H})$. *Suppose*

$$\begin{aligned} \tau_0 &= \lambda(-c^{-1}d) + \eta + (c^{-1}d)\mu, \\ t_0 &= \lambda(ac^{-1}) + \eta - (ac^{-1})\mu. \end{aligned}$$

If

$$\frac{|\tau_0 - t_0|}{|\tau_0 t_0|} > |\bar{c}d + a\bar{c}|,$$

then $\langle S, T \rangle$ *is not extreme.*

PROOF. Let $\sigma_0 = \frac{1+\sqrt{1-\varepsilon^{-1}S(\lambda,\mu)|}}{2}$. Suppose $\langle S, T \rangle$ is extreme. Then $|c|^2|\tau_0 t_0| = \sigma_0^2$. Note that

$$\begin{aligned}\tau_0 - t_0 &= -\lambda(c^{-1}d + ac^{-1}) + (c^{-1}d + ac^{-1})\mu \\ &= \Im\lambda(c^{-1}d + ac^{-1}) + (c^{-1}d + ac^{-1})\Im\mu.\end{aligned}$$

Thus

$$\begin{aligned}|\tau_0 - t_0| &\le (|\Im\lambda| + |\Im\mu|)(|c^{-1}d + ac^{-1}|) \\ &\le (|\Im\lambda| + |\Im\mu|)\frac{1}{|c|^2}|\bar{c}d + a\bar{c}|.\frac{|c|^2|\tau_0 t_0|}{\sigma_0^2}.\end{aligned}$$

This implies

$$\frac{|\tau_0 - t_0|}{|\tau_0 t_0|} \le S(\lambda, \mu)\frac{|\bar{c}d + a\bar{c}|}{\sigma_0^2}.$$

Now note that $S(\lambda, \mu) \le \frac{1}{4}$ and $\sigma_0 \ge \frac{1}{2}$, hence $\frac{S(\lambda,\mu)}{\sigma_0^2} \le 1$. Thus, we have

$$\frac{|\tau_0 - t_0|}{|\tau_0 t_0|} \le |\bar{c}d + a\bar{c}|.$$

This proves the result.

If we choose $T = \begin{pmatrix} \lambda & 0 \\ \eta & \mu \end{pmatrix}$, then analogous result to Theorem 6 follows using similar arguments as above. In particular, we have the following.

COROLLARY 13. *Let* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $T = \begin{pmatrix} \lambda & 0 \\ \eta & \mu \end{pmatrix}$, $\Re\lambda = \Re\mu$, *generate a discrete, non-elementary subgroup of* $\mathrm{SL}(2, \mathbb{H})$. *Suppose*

$$\begin{aligned}\tau_0 &= \mu(-b^{-1}a) + \eta + (b^{-1}a)\lambda, \\ t_0 &= \mu(db^{-1}) + \eta - (db^{-1})\lambda.\end{aligned}$$

If

$$\frac{|\tau_0 - t_0|}{|\tau_0 t_0|} > |\bar{b}d + a\bar{b}|,$$

then $\langle S, T \rangle$ *is not extreme.*

4.1. Examples of extreme groups. Let us consider the following elements in $\mathrm{SL}(2,\mathbb{H})$: $S=\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$, $T=\begin{pmatrix} \lambda & c^{-1}j \\ 0 & \mu \end{pmatrix}$, $|c|\geq 1$ and $\Im\lambda=\Im\mu=0$. Suppose that the subgroup $\langle S,T\rangle$ in $\mathrm{SL}(2,\mathbb{H})$ is non-elementary and discrete. For instance, if $a=d=c=1$ and $\lambda=\mu=1$ then this is the case. We see that $\tau_0=c^{-1}j$, $t_0=c^{-1}j$ and so we have $|c|\sqrt{|\tau_0|\,|t_0|}=1$. Also observe that $S(\lambda,\mu)=0$ and $\frac{1+\sqrt{1-4\sqrt{2}S(\lambda,\mu)}}{2}=\frac{2}{2}=1$. So, we have $|c|\sqrt{|\tau_0|\,|t_0|}=1=\frac{1+\sqrt{1-4\sqrt{2}S(\lambda,\mu)}}{2}$.

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References

[1] L. V. Ahlfors, Möbius transformations and Clifford numbers, Differential Geometry and Complex Analysis (Springer, 1985), 65–73.

[2] V. G. Bardakov, K. Gongopadhyay, M. Singh, A. Vesnin and J. Wu, Some problems on knots, braids, and automorphism groups, Sib. Elektron. Mat. Izv. 12 (2015), 394–405, electronic only.

[3] W. Cao, On the classification of four-dimensional Móbius transformations, Proc. Edinb. Math. Soc. (2) 50 (2007), no. 1, 49–62.

[4] J. Callahan, Jørgensen number and arithmeticity, Conform. Geom. Dyn. 13 (2009), 160–186.

[5] V. Erlandsson and S. Zakeri, A discreteness criterion for groups containing parabolic isometries, Geometry, groups and dynamics, 235–242, Contemp. Math., 639, Amer. Math. Soc., 2015.

[6] B. Foreman, Conjugacy invariants of $\mathrm{SL}(2,\mathbb{H})$, Linear Algebra Appl. 381 (2004), 25–35.

[7] T. Jørgensen, On discrete groups of Möbius transformations, Amer. J. Math. 98, no. 3 (1976), 739–749.

[8] T. Jørgensen and M. Kikka, Some extreme discrete groups, Ann. Acad. Scient. Fenn. Volume 1 (1975), 245–248.

[9] F. W. Gehring and G. J. Martin, Stability and extremality in Jørgensen's inequality, Complex Variables Theory Appl. 12: 1–4, (1989), 277–282.

[10] F. W. Gehring and G. J. Martin, Iteration theory and inequalities for Kleinian groups, Bull. Amer. Math. Soc. (N.S.), 21:1 (1989), 57–63.

[11] J. Gilman, Two-generator discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$, Mem. Amer. Math. Soc. 561, 204 p. (1995).

[12] J. Gilman and B. Maskit, An algorithm for 2-generator Fuchsian groups, Mich. Math. J. 38, No. 1 (1991), 13–32.
[13] K. Gongopadhyay, Algebraic characterization of the isometries of the hyperbolic 5-space, Geom. Dedicata, Volume 144, Issue 1 (2010), 157–170.
[14] F. González-Acuña and A. Ramírez, Jørgensen subgroups of the Picard group, Osaka J. Math. 44 (2), (2007), 471–482.
[15] R. Kellerhals, Collars in $\mathrm{PSL}(2, \mathbb{H})$, Ann. Acad. Sci. Fenn. Math. 26 (2001), 51–72.
[16] R. Kellerhals, Quaternions and some global properties of hyperbolic 5-manifolds, Canad. J. Math. 55 (2003), 1080–1099.
[17] G. J. Martin, On discrete Möbius groups in all dimensions: A generalization of Jørgensen's inequality, Acta Math, Volume 163, Issue 1 (1989), 253–289.
[18] J. R. Parker, Hyperbolic Spaces, Jyäskylä Lectures in Mathematics 2, 2008.
[19] J. R. Parker and I. Short, Conjugacy classification of quaternionic Möbius transformations, Comput. Methods Funct. Theory 9 (2009), 13–25.
[20] G. Rosenberger, All generating pairs of all two-generator Fuchsian groups, Arch. Math. 46 (1986), 198–204.
[21] H. Sato, One-parameter families of extreme discrete groups for Jørgensen's inequality, Kra, Irwin (ed.) et al., In the tradition of Ahlfors and Bers. Proceedings of the first Ahlfors-Bers colloquium, Contemp. Math. 256 (2000), 271–287, American Mathematical Society (AMS).
[22] H. Sato, The Picard group, the figure-eight knot group and Jørgensen groups, RIMS Kokyuroku 1223 (2001), 37–42.
[23] H. Sato, Jørgensen's inequality for classical Schottky groups of real type. II, J. Math. Soc. Japan 53, No. 4 (2001), 791–811.
[24] H. Sato, M. Oichi and C. Li, Jørgensen groups of parabolic type I (finite case), Comput. Methods Funct. Theory 5, No. 2 (2005), 409–430.
[25] H. Sato, M. Oichi and C. Li, Jørgensen groups of parabolic type II (Countably infinite case), Osaka J. Math. 41 (2004), 491–506.
[26] H. Sato, M. Oichi and C. Li, Jørgensen groups of parabolic type III (Uncountably infinite case), Kodai Math. J. 28, No. 2 (2005), 248–264.
[27] D. Tan, On two-generator discrete groups of Möbius transformations, Proc. Amer. Math. Soc., 106:3 (1989), 763–770.
[28] S. P. Tan, Y. L. Wong and Y. Zhang, Delambre-Gauss formulas for augmented, right-angled hexagons in hyperbolic 4-space, Adv. Math. 230 (2012), no. 3, 927–956.
[29] A. Yu. Vesnin and A. V. Masley, On Jørgensen numbers and their analogs for groups of figure-eight orbifolds, Siberian Mathematical Journal, Vol. 55, no. 5 (2014), 807–816.
[30] M. Wada, OPTi's algorithm for discreteness determination, Experimental Mathematics, Volume 15, Issue 1 (2006), 61–66.
[31] P. L. Waterman, Möbius groups in several dimensions, Adv. Math. 101 (1993), 87–113.

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