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On prolongations of second-order regular overdetermined systems with two independent and one dependent variables

Dedicated to Professor Keizo Yamaguchi on his sixtieth birthday

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Abstract. The purpose of this paper is to investigate the geometric structure of regular overdetermined systems of second order with two independent and one dependent variables from the point of view of the rank two prolongation. Utilizing this prolongation, we characterize the type of overdetermined systems and clarify the specificity for each type. We also give systematic methods for constructing the geometric singular solutions by analyzing a decomposition of this prolongation. As an application, we determine the geometric singular solutions of Cartan's overdetermined system.

1. Introduction

The subject of this present paper is geometric study of partial differential equations which are called second-order regular overdetermined systems with two independent and one dependent variables. For these overdetermined systems, various pioneering works have been given by many researchers (cf. [4], [11], [8], [24], [29]). In particular, the study of overdetermined involutive systems has significant results. E. Cartan [4] characterized overdetermined involutive systems by the condition that these admit a one dimensional Cauchy characteristic system. He also found out a systematic method for constructing regular solutions (see Definition 5) of involutive systems. Recently, these considerations have been reformulated as the theory of PD-manifolds by Yamaguchi (cf. [24], [29]). In addition, Kakie (cf. [6], [7]) studied the existence of regular solutions in C^{∞}-category and Cauchy problem for involutive systems by using the theory of characteristics.

In this paper, we investigate the geometric theory of regular overdetermined systems by analyzing the rank two prolongation. The aim is to provide

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the following two results. The first result is to clarify the difference between the prolongations with the transversality condition and the rank two prolongations for our overdetermined systems. The second result is to give two systematic methods for constructing geometric singular solutions (see Definition 5) utilizing the obtained difference. In the field of geometry of differential systems, regular (i.e. nonsingular) solutions are examined usually, for example Cartan-Kähler theory, Cauchy problem, etc. In contrast, we study singular solutions mainly. The significance of this research is that the notion of our singular solutions corresponds to wave front propagation appearing in mathematical physics, hence various applications are expected. In this context, we give two systematic methods for constructing singular solutions by using the characterization of the rank two prolongation. Here, we explain this rank two prolongation. Roughly speaking, this concept expresses a certain fibration obtained by collecting integral elements which are the candidates for the tangent spaces of the graphs of solutions. For the general theory of the rank two prolongation, see [3], [8], [13]. This rank two prolongation can be regraded as a generalization of the prolongation with the transversality condition. The prolongation with the transversality condition corresponds to exterior derivation of given differential equations for the independent variables, and it has been used to construct regular solutions (cf. [2], [4], [9], [10], [24]). However, in the present paper, we treat the rank two prolongation, because geometric singular solutions cannot be constructed utilizing the ordinary prolongation with the transversality condition. Hence, we must use the rank two prolongation for the discussion of singular solutions. In this situation, through a precise analysis of the rank two prolongation in our geometric setting, we clarify the mechanism that makes singularity appear.

Let us now proceed to the description of the various sections and explain the main results in this paper. In section 2, we prepare some terminology and notation for the study of differential systems. In section 3, we introduce our setting and define the rank two prolongation. For regular overdetermined systems, we can use the classification into contact invariant four types (subcategories) consisting of involutive type, two finite types, and torsion type under the symbol algebra (see section 3). According to this classification, we determine the topology of each fiber of the rank two prolongation for regular overdetermined systems (**Theorem 1**). As a direct consequence of this characterization, we obtain the specific difference between the prolongation with the transversality condition and the rank two prolongation (**Corollary 1**). By using this characterization, we can obtain a deep understanding for regular overdetermined systems including the singularity. Actually, we can show that there do not exist singular solutions for subcategories consisting of two finite types. Moreover, in the involutive case, we note that a systematic method of the explicit construction of singular solutions can be given by analyzing this characterization more carefully. In section 4, we study the algebraic structures associated with canonical systems $\hat{D}$ on the rank two prolongations $\Sigma(R)$ of (locally) involutive systems. More precisely, we clarify the bracket structure of nilpotent graded Lie algebras (symbol algebras) defined for the rank two prolongations by using a decomposition (**Proposition 1**). Here, it is known that the symbol algebras are fundamental invariants of (weakly-regular) differential systems or filtered manifolds (cf. [22], [24], [14]). Proposition 1 means that the difference of the brackets of these graded Lie algebras corresponds to the singularity of the solutions from the algebraic viewpoint. We also have the tower structure of these involutive systems by successive prolongations (**Theorem 2**). In section 5, we provide two systematic methods to construct the geometric singular solutions of involutive systems. Moreover, we apply these methods to Cartan's overdetermined system in order to demonstrate the usefulness of our methods. Consequently, we can give an explicit integral representation of geometric singular solutions of this system by using our methods.

2. Differential systems and symbol algebras

In this section, we prepare some terminology and notation for the study of differential systems. For more details, refer to [22] and [25].

2.1. Derived systems, weak derived systems and Cauchy characteristic systems. Let D be a differential system on a manifold R. In general, by a differential system (R, D), we mean a distribution D on R, that is, D is a subbundle of the tangent bundle TR of R. The sheaf of sections to D is denoted by $\mathscr{D} = \Gamma(D)$. The derived system ∂D of a differential system D is defined, in terms of sections, by $\partial\mathscr{D} := \mathscr{D} + [\mathscr{D}, \mathscr{D}]$. In general, ∂D is obtained as a subsheaf of the tangent sheaf of R. Moreover, higher derived systems $\partial^k D$ are defined successively by $\partial^k \mathscr{D} := \partial(\partial^{k-1}\mathscr{D})$, where we set $\partial^0 D = D$ by convention. On the other hand, the k-th weak derived systems $\partial^{(k)} D$ of D are defined inductively by $\partial^{(k)}\mathscr{D} := \partial^{(k-1)}\mathscr{D} + [\mathscr{D}, \partial^{(k-1)}\mathscr{D}]$.

Definition 1. A differential system D is called regular (resp. weakly regular), if $\partial^k D$ (resp. $\partial^{(k)} D$) is a subbundle for each k.

These derived systems are also interpreted by using annihilators as follows: Let $D = \{\varpi_1 = \cdots = \varpi_s = 0\}$ be a differential system on R. We denote by $D^{\perp}$ the annihilator subbundle of D in T^*R, that is,

$$D^{\perp}(x) := \{\omega \in T_x^*R \mid \omega(X) = 0 \text{ for any } X \in D(x)\} = \operatorname{span}\{(\varpi_1)_x, \ldots, (\varpi_s)_x\}.$$

Then the annihilator $(\partial D)^{\perp}$ of the first derived system of D is given by $(\partial D)^{\perp} = \{\varpi \in D^{\perp} \mid d\varpi \equiv 0 \pmod{D^{\perp}}\}$. Moreover the annihilator $(\partial^{(k+1)}D)^{\perp}$ of the $(k+1)$-st weak derived system of D is given by

$$(\partial^{(k+1)}D)^{\perp} = \{\varpi \in (\partial^{(k)}D)^{\perp} \mid d\varpi \equiv 0 \ (\mathrm{mod}(\partial^{(k)}D)^{\perp}, \\ (\partial^{(p)}D)^{\perp} \wedge (\partial^{(q)}D)^{\perp}, 2 \leq p, q \leq k-1)\}.$$

We set $D^{-1} := D$, $D^{-k} := \partial^{(k-1)}D$ $(k \geq 2)$, for a weakly regular differential system D. Then we have ([22, Proposition 1.1]):

(T1) There exists a unique positive integer μ such that

$$D^{-1} \subset D^{-2} \subset \cdots \subset D^{-k} \subset \cdots \subset D^{-(\mu-1)} \subset D^{-\mu} = D^{-(\mu+1)} = \cdots;$$

(T2) $[\mathscr{D}^p, \mathscr{D}^q] \subset \mathscr{D}^{p+q}$ for all $p, q < 0$.

Let D be a differential system on R defined by local 1-forms $\varpi_1, \dots, \varpi_s$ such that $\varpi_1 \wedge \cdots \wedge \varpi_s \neq 0$ at each point, where s is the corank of D: $D = \{\varpi_1 = \cdots = \varpi_s = 0\}$. Then the Cauchy characteristic system $Ch(D)$ is defined at each point $x \in R$ by

$$Ch(D)(x) := \{X \in D(x) \mid X \rfloor d\varpi_i \equiv 0 \pmod{\varpi_1, \dots, \varpi_s} \text{ for } i = 1, \dots, s\},$$

where $\rfloor$ denotes the interior product (i.e., $X \rfloor d\varpi(Y) = d\varpi(X, Y)$).

2.2. Symbol algebra of regular differential system. Let (R, D) be a weakly regular differential system such that $TR = D^{-\mu} \supset D^{-(\mu-1)} \supset \cdots \supset D^{-1} =: D$. For all $x \in R$, we set $\mathfrak{g}_{-1}(x) := D^{-1}(x) = D(x)$, $\mathfrak{g}_p(x) := D^p(x)/D^{p+1}(x)$ $(p = -2, -3, \dots, -\mu)$, and $\mathfrak{m}(x) := \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x)$. Then, $\dim \mathfrak{m}(x) = \dim R$ holds. We set $\mathfrak{g}_p(x) = \{0\}$ when $p \leq -\mu - 1$. For $X \in \mathfrak{g}_p(x)$, $Y \in \mathfrak{g}_q(x)$, the Lie bracket $[X, Y] \in \mathfrak{g}_{p+q}(x)$ is defined as follows: Let ϖ_p be the projection of $D^p(x)$ onto $\mathfrak{g}_p(x)$ and $\tilde{X} \in \mathscr{D}^p$, $\tilde{Y} \in \mathscr{D}^q$ be any extensions such that $\varpi_p(\tilde{X}_x) = X$ and $\varpi_q(\tilde{Y}_x) = Y$. Then $[\tilde{X}, \tilde{Y}] \in \mathscr{D}^{p+q}$, and we define $[X, Y] := \varpi_{p+q}([\tilde{X}, \tilde{Y}]_x) \in \mathfrak{g}_{p+q}(x)$. It does not depend on the choice of the extensions because of the equation

$$[f\tilde{X}, g\tilde{Y}] = fg[\tilde{X}, \tilde{Y}] + f(\tilde{X}g)\tilde{Y} - g(\tilde{Y}f)\tilde{X} \qquad (f, g \in C^{\infty}(R)).$$

The Lie algebra $\mathfrak{m}(x)$ is a nilpotent graded Lie algebra. We call $(\mathfrak{m}(x), [\,,\,])$ the *symbol algebra* of (R, D) at x. Note that the symbol algebra $(\mathfrak{m}(x), [\,,\,])$ satisfies the generating conditions $[\mathfrak{g}_p, \mathfrak{g}_{-1}] = \mathfrak{g}_{p-1}$ $(p < 0)$. For two differential systems (R, D) and (R', D'), we define (local) isomorphisms ϕ (or (local) contact transformations) from R to R' by (local) diffeomorphisms $\phi : R \to R'$ satisfying $\phi_* D = D'$. It is well-known that the symbol algebra is a fundamental invariant of differential systems under contact transformations. Namely, if there exists a (local) contact transformation $\phi : R \to R'$, then we obtain a graded Lie algebra isomorphism $\mathfrak{m}(x) \cong \mathfrak{m}(\phi(x))$ at each point x (cf. [22], [24]).

2.3. Filtered manifolds and symbol algebras. Morimoto introduced the notion of a filtered manifold as a generalization of weakly regular differential systems ([14]). We define a filtered manifold (R, F) by a pair of a manifold R and a tangential filtration F. Here, a tangential filtration F on R is a sequence $\{F^p\}_{p<0}$ of subbundles of the tangent bundle TR and the following conditions are satisfied:

(M1) $TR = F^k = \cdots = F^{-\mu} \supset \cdots \supset F^p \supset F^{p+1} \supset \cdots \supset F^0 = \{0\}$,

(M2) $[\mathscr{F}^p, \mathscr{F}^q] \subset \mathscr{F}^{p+q}$ for all $p, q < 0$,

where $\mathscr{F}^p = \Gamma(F^p)$ is the space of sections of F^p.

Let (R, F) be a filtered manifold. For $x \in R$, we set $\mathfrak{f}_p(x) := F^p(x)/F^{p+1}(x)$ and $\mathfrak{f}(x) := \bigoplus_{p<0} \mathfrak{f}_p(x)$. For $X \in \mathfrak{f}_p(x)$, $Y \in \mathfrak{f}_q(x)$, the Lie bracket $[X, Y] \in \mathfrak{f}_{p+q}(x)$ is defined as follows: Let ϖ_p be the projection of $F^p(x)$ onto $\mathfrak{f}_p(x)$ and $\tilde{X} \in \mathscr{F}^p$, $\tilde{Y} \in \mathscr{F}^q$ be any extensions such that $\varpi_p(\tilde{X}_x) = X$ and $\varpi_q(\tilde{Y}_x) = Y$. Then $[\tilde{X}, \tilde{Y}] \in \mathscr{F}^{p+q}$, and we define $[X, Y] := \varpi_{p+q}([\tilde{X}, \tilde{Y}]_x) \in \mathfrak{f}_{p+q}(x)$. It does not depend on the choice of the extensions. The Lie algebra $\mathfrak{f}(x)$ is also a nilpotent graded Lie algebra. We call $(\mathfrak{f}(x), [\,,])$ the *symbol algebra* of (R, F) at x. In general it does not satisfy the generating conditions. Suppose (R, F) and (R', F') are filtered manifolds. Then, (local) isomorphisms (or (local) contact transformations) between (R, F) and (R', F') are defined by (local) diffeomorphisms $\phi : R \to R'$ such that $\phi_* F^p = F^{p\prime}$. It is known that this symbol algebra is also a fundamental invariant of filtered manifolds under contact transformations. Namely, if there exists a (local) contact transformation $\phi : R \to R'$, then we obtain a graded Lie algebra isomorphism $\mathfrak{f}(x) \cong \mathfrak{f}(\phi(x))$ at each point x.

3. Rank two prolongations of regular overdetermined systems

In this section, we provide a fundamental characterization of the rank two prolongations for regular overdetermined systems of second order of codimension two with two independent and one dependent variables. First, we introduce our setting to discuss contact geometry of regular overdetermined systems of second order. Let $J^2(\mathbb{R}^2, \mathbb{R})$ be the 2-jet space:

$$J^2(\mathbb{R}^2, \mathbb{R}) := \{(x, y, z, p, q, r, s, t)\}. \tag{1}$$

This space has the canonical system $C^2 = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ given by the annihilators:

$$\varpi_0 := dz - p\,dx - q\,dy, \qquad \varpi_1 := dp - r\,dx - s\,dy, \qquad \varpi_2 := dq - s\,dx - t\,dy.$$

This jet space is also constructed geometrically as the Lagrange-Grassmann bundle over the standard contact five dimensional manifold. For more details, see [29]. On the 2-jet space, we consider the following partial differential equations which are called overdetermined systems:

$$F(x, y, z, p, q, r, s, t) = G(x, y, z, p, q, r, s, t) = 0, \tag{2}$$

where F and G are smooth functions on $J^2(\mathbb{R}^2, \mathbb{R})$. We set $R = \{F = G = 0\} \subset J^2(\mathbb{R}^2, \mathbb{R})$ and restrict the canonical differential system C^2 to R. We denote by D this restricted system $C^2|_R$. In general, D is not constant rank. Therefore, we assume the following condition which is called the regularity condition for overdetermined systems. Two vectors (F_r, F_s, F_t) and (G_r, G_s, G_t) are linearly independent on R. Then, R is a submanifold of codimension two, and the restriction $\pi_1^2|_R : R \to J^1(\mathbb{R}^2, \mathbb{R})$ of the natural projection $\pi_1^2 : J^2(\mathbb{R}^2, \mathbb{R}) \to J^1(\mathbb{R}^2, \mathbb{R})$ is a submersion. Due to the property, restricted 1-forms $\varpi_i|_R$ on R are linearly independent. Hence $D = \{\varpi_0|_R = \varpi_1|_R = \varpi_2|_R = 0\}$ is a rank three system on R. For brevity, we denote by ϖ_i each restricted generator 1-form $\varpi_i|_R$ of D in the following. These differential systems (R, D) are not weakly regular, in general. Hence, we need to take an appropriate filtration on R to discuss contact geometry of overdetermined systems in terms of symbol algebras. So, we take the filtration $F = \{F^p\}_{p<0}$ given by $F^{-3} = TR$, $F^{-2} = (\pi_1^2|_R)_*^{-1} C^1$, $F^{-1} = D$, where C^1 is the canonical contact system on 1-jet space $J^1(\mathbb{R}^2, \mathbb{R})$. According to the discussion of the previous section, we define the contact transformations ϕ between two overdetermined systems R and R' by local diffeomorphisms $\phi : R \to R'$ satisfying $\phi_* F^p = F^{p\prime}$.

Definition 2. We call R or (R, F) (geometric) regular overdetermined systems of second order.

In this situation, we investigate contact geometry of regular overdetermined systems (R, F) of second order.

Next, we define the rank two prolongations of differential systems. This notion is necessary to research singular solutions.

Definition 3. Let (R, D) be a differential system given by $D = \{\varpi_1 = \cdots = \varpi_s = 0\}$. An n-dimensional *integral element* of D at $x \in R$ is an n-dimensional subspace v of $T_x R$ such that $\varpi_i|_v = d\varpi_i|_v = 0$ $(i = 1, \ldots, s)$. Namely, n-dimensional integral elements are the candidates for the tangent spaces at x to n-dimensional integral manifolds of D. Let (R, F) be a regular overdetermined system of second order. Then we define the *rank two pro-*

longation $\Sigma(R)$ of (R, F) by

$$\Sigma(R) := \bigcup_{x \in R} \Sigma_x, \tag{3}$$

where $\Sigma_x = \{v \subset T_x R \,|\, v$ is a two dimensional integral element of $D(x)\}$.

Let $p : \Sigma(R) \to R$ be the projection. We define the canonical system $\hat{D}$ on $\Sigma(R)$ by

$$\hat{D}(u) := p_*^{-1}(u) = \{v \in T_u(\Sigma(R)) \,|\, p_*(v) \in u\},$$

where $u \in \Sigma(R)$.

This space $\Sigma(R)$ is a subset of the following Grassmann bundle over R

$$J(D, 2) := \bigcup_{x \in R} J_x, \tag{4}$$

where $J_x := \{v \subset T_x R \,|\, v$ is a two dimensional subspace of $D(x)\}$. In general, the rank two prolongations $\Sigma(R)$ have singular points, that is, $\Sigma(R)$ is not smooth. For a differential system (R, D), we define another prolongation which is called the *prolongation with the transversality condition*:

$$R^{(1)} = \bigcup_{x \in R} R_x^{(1)}, \tag{5}$$

where

$$R_x^{(1)} = \{v \subset T_x R \,|\, v \text{ is a two dimensional integral element of } D(x) \text{ transversal to } \mathrm{Ker}(\pi_1^2|_R)_*\}.$$

Let $p^{(1)} : R^{(1)} \to R$ be the projection. Then we also define the canonical system $D^{(1)}$ on $R^{(1)}$ by

$$D^{(1)}(u) := p_*^{(1)^{-1}}(u) = \{v \in T_u R^{(1)} \,|\, p_*^{(1)}(v) \in u\},$$

where $u \in R^{(1)}$. For this notion, there exist many results related to the characterization of higher-order jet spaces (see [2], [4], [24]). In this section, we clarify the difference between those two prolongations for our regular overdetermined systems.

Next, we explain a classification of the type of overdetermined systems in terms of the structure equation or the corresponding symbol algebra. Let (R, F) be a regular overdetermined system. If R does not have torsion, that is, the fibration $p^{(1)} : R^{(1)} \to R$ is onto, then the structure equation of this system is one of the following three cases ([29, the case of codim $\mathfrak{f} = 2$ of Case $n = 2$ in pages 346–347]):

(I) There exists a coframe $\{\varpi_0, \varpi_1, \varpi_2, \omega_1, \omega_2, \pi\}$ around $w \in R$ such that $D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ and the following structure equation holds at w:

$$\begin{aligned} d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 && \mod \varpi_0, \\ d\varpi_1 &\equiv 0 && \mod \varpi_0, \varpi_1, \varpi_2, \\ d\varpi_2 &\equiv \omega_2 \wedge \pi && \mod \varpi_0, \varpi_1, \varpi_2. \end{aligned} \tag{6}$$

(II) There exists a coframe $\{\varpi_0, \varpi_1, \varpi_2, \omega_1, \omega_2, \pi\}$ around $w \in R$ such that $D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ and the following structure equation holds at w:

$$\begin{aligned} d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 && \mod \varpi_0, \\ d\varpi_1 &\equiv \omega_2 \wedge \pi && \mod \varpi_0, \varpi_1, \varpi_2, \\ d\varpi_2 &\equiv \omega_1 \wedge \pi && \mod \varpi_0, \varpi_1, \varpi_2. \end{aligned} \tag{7}$$

(III) There exists a coframe $\{\varpi_0, \varpi_1, \varpi_2, \omega_1, \omega_2, \pi\}$ around $w \in R$ such that $D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ and the following structure equation holds at w:

$$\begin{aligned} d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 && \mod \varpi_0, \\ d\varpi_1 &\equiv \omega_1 \wedge \pi && \mod \varpi_0, \varpi_1, \varpi_2, \\ d\varpi_2 &\equiv \omega_2 \wedge \pi && \mod \varpi_0, \varpi_1, \varpi_2. \end{aligned} \tag{8}$$

Now we consider the case where torsion exists, that is, $p^{(1)} : R^{(1)} \to R$ is not onto. In fact, then the structure equation (or the symbol algebra) of torsion type has the unique normal form by the obtained result in [18] (this fact follows from the technique of the proof of [16, Theorem 3.3]). Namely, if R has torsion at $w \in R$, we have the following structure equation at w.

(IV) There exists a coframe $\{\varpi_0, \varpi_1, \varpi_2, \omega_1, \omega_2, \pi\}$ around $w \in R$ such that $D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ and the following structure equation holds at w:

$$\begin{aligned} d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 && \mod \varpi_0, \\ d\varpi_1 &\equiv \omega_1 \wedge \omega_2 && \mod \varpi_0, \varpi_1, \varpi_2, \\ d\varpi_2 &\equiv \omega_2 \wedge \pi && \mod \varpi_0, \varpi_1, \varpi_2. \end{aligned} \tag{9}$$

Here, the type (I), (II), (III) and (IV) correspond to second-order overdetermined systems of involutive type, finite type, finite type and torsion type, respectively (cf. [28], [29]).

Definition 4. We call overdetermined systems of the above four types, overdetermined systems of type (k), where $k = \mathrm{I}$, II, III, IV.

Remark 1. *The structures of prolongations* $R^{(1)}$ *with the transversality condition for* R *of type* (I), (II), (III) *and* (IV) *are well-known* (cf. [24], [29]). *Indeed,* $R^{(1)} \to R$ *is an* $\mathbb{R}$*-bundle for the type* (I). *Moreover* $R^{(1)}$ *is diffeomorphic to* R *for the type* (II) *or* (III), *and* $R^{(1)}$ *is empty for the type* (IV).

As stated above, the main purpose of this section is to clarify the difference between $R^{(1)}$ and $\Sigma(R)$. For this purpose, we first consider the case of type (I).

Lemma 1. *Let* R *be an overdetermined system of type* (I). *Then the rank two prolongation* $\Sigma(R)$ *is a smooth submanifold of* $J(D,2)$. *Moreover, it is an* S^1*-bundle over* R.

Proof. Let $\Pi : J(D,2) \to R$ be the projection and U an open set in R. Then $\Pi^{-1}(U)$ is covered by three open sets in $J(D,2)$, that is,

$$\Pi^{-1}(U) = U_{\omega_1\omega_2} \cup U_{\omega_1\pi} \cup U_{\omega_2\pi}, \tag{10}$$

where

$$U_{\omega_1\omega_2} := \{v \in \Pi^{-1}(U) \mid \omega_1|_v \wedge \omega_2|_v \neq 0\},$$

$$U_{\omega_1\pi} := \{v \in \Pi^{-1}(U) \mid \omega_1|_v \wedge \pi|_v \neq 0\},$$

$$U_{\omega_2\pi} := \{v \in \Pi^{-1}(U) \mid \omega_2|_v \wedge \pi|_v \neq 0\}.$$

We explicitly describe the defining equation of $\Sigma(R)$ in terms of the inhomogeneous Grassmann coordinate of fibers in $U_{\omega_1\omega_2}$, $U_{\omega_1\pi}$, $U_{\omega_2\pi}$. First we consider it on $U_{\omega_1\omega_2}$. For $w \in U_{\omega_1\omega_2}$, w is a two dimensional subspace of $D(v)$, where $p(w) = v$. Hence, by restricting π to w, we can introduce the inhomogeneous coordinate p_i^1 $(i = 1,2)$ of fibers of $J(D,2)$ around w with $\pi|_w = p_1^1(w)\omega_1|_w + p_2^1(w)\omega_2|_w$. Moreover, w satisfies $d\varpi_2|_w \equiv 0$ in (6). Hence, we show that

$$d\varpi_2|_w \equiv \omega_2|_w \wedge \pi|_w \equiv p_1^1(w)\omega_2|_w \wedge \omega_1|_w.$$

Thus we obtain the defining equations $p_1^1 = 0$ of $\Sigma(R)$ in $U_{\omega_1\omega_2}$ of $J(D,2)$. Then dp_1^1 does not vanish on $\{p_1^1 = 0\}$. We next consider on $U_{\omega_1\pi}$. By restricting ω_2 to $w \in U_{\omega_1\pi}$, we can introduce the inhomogeneous coordinate p_i^2 $(i = 1,2)$ of fibers of $J(D,2)$ around w with $\omega_2|_w = p_1^2(w)\omega_1|_w + p_2^2(w)\pi|_w$. Then we show that

$$d\varpi_2|_w \equiv \omega_2|_w \wedge \pi|_w \equiv p_1^2(w)\omega_1|_w \wedge \pi|_w.$$

We have the defining equation $p_1^2 = 0$ such that dp_1^2 does not vanish on $\{p_1^2 = 0\}$. Finally we consider on $U_{\omega_2\pi}$. By restricting ω_1 to $w \in U_{\omega_2\pi}$, we can introduce the inhomogeneous coordinate p_i^3 $(i = 1, 2)$ of fibers of $J(D, 2)$ around w with $\omega_1|_w = p_1^3(w)\omega_2|_w + p_2^3(w)\pi|_w$. Moreover, w satisfies $d\varpi_2|_w \equiv 0$. However we have

$$d\varpi_2|_w \equiv \omega_2|_w \wedge \pi|_w \neq 0.$$

Thus, there does not exist an integral element, that is, $U_{\omega_2\pi} \cap p^{-1}(U) = \emptyset$.

Summarizing these discussions, we conclude that $\Sigma(R)$ is a submanifold in $J(D, 2)$, and it has the covering

$$p^{-1}(U) = P_{\omega_1\omega_2} \cup P_{\omega_1\pi} = \{p_1^1 = 0\} \cup \{p_1^2 = 0\}, \tag{11}$$

where $P_{\omega_1\omega_2} := p^{-1}(U) \cap U_{\omega_1\omega_2}$ and $P_{\omega_1\pi} := p^{-1}(U) \cap U_{\omega_1\pi}$.

Next, we show that the topology of each fiber of $\Sigma(R)$ is S^1.

Let w be a point in $P_{\omega_1\pi} \subset p^{-1}(U)$. Here, if $w \notin P_{\omega_1\omega_2}$, then we have $p_2^2 = 0$ because of the condition $\omega_1 \wedge \omega_2 = 0$. Thus we show that $p_2^2 \neq 0$ on $P_{\omega_1\omega_2} \cap P_{\omega_1\pi}$ and $p_2^2 = 0$ on $P_{\omega_1\pi} \backslash P_{\omega_1\omega_2}$. Thus the topology of each fiber of $\Sigma(R)$ is S^1.

Next, we consider the case of type (II).

LEMMA 2. *Let R be an overdetermined system of type* (II). *Then the rank two prolongation $\Sigma(R)$ is diffeomorphic to R.*

REMARK 2. *We emphasize that (R, D) and $(\Sigma(R), \hat{D})$ are different as differential systems. Indeed D is a rank three differential system on R, but $\hat{D}$ is a rank two differential system on $\Sigma(R)$.*

PROOF. In this situation we also use the covering (10) of $\Pi^{-1}(U)$ for the Grassmann bundle $J(D, 2)$ and explicitly describe the defining equation of $\Sigma(R)$ in terms of the inhomogeneous Grassmann coordinate of fibers in $U_{\omega_1\omega_2}$, $U_{\omega_1\pi}$ and $U_{\omega_2\pi}$. First we consider it on $U_{\omega_1\omega_2}$. For $w \in U_{\omega_1\omega_2}$, w is a two dimensional subspace of $D(v)$, where $p(w) = v$. Hence, by restricting π to w, we can introduce the inhomogeneous coordinate p_i^1 of fibers of $J(D, 2)$ around w with $\pi|_w = p_1^1(w)\omega_1|_w + p_2^1(w)\omega_2|_w$. Moreover w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$ in (7). Thus we get

$$d\varpi_1|_w \equiv \omega_2|_w \wedge \pi|_w \equiv p_1^1(w)\omega_2|_w \wedge \omega_1|_w,$$

$$d\varpi_2|_w \equiv \omega_1|_w \wedge \pi|_w \equiv p_2^1(w)\omega_1|_w \wedge \omega_2|_w.$$

In this way, we obtain the defining equations $p_1^1 = p_2^1 = 0$ of $\Sigma(R)$ in $U_{\omega_1\omega_2}$ of $J(D, 2)$. Hence we have the only trivial integral element. Next we consider on $U_{\omega_1\pi}$. In the same way, by restricting ω_2 to w, we can introduce the inhomogeneous coordinate p_i^2 of fibers of $J(D,2)$ around w with $\omega_2|_w = p_1^2(w)\omega_1|_w + p_2^2(w)\pi|_w$. Moreover w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$. However we have $d\varpi_2|_w \equiv \omega_1|_w \wedge \pi|_w \neq 0$. Hence there does not exist any integral element. Finally we consider on $U_{\omega_2\pi}$. In this situation, by restricting ω_1 to w, we can also introduce the inhomogeneous coordinate p_i^3 of fibers of $J(D,2)$ around w with $\omega_1|_w = p_1^3(w)\omega_2|_w + p_2^3(w)\pi|_w$. Moreover w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$. However, we have $d\varpi_1|_w \equiv \omega_2|_w \wedge \pi|_w \neq 0$. Hence there does not exist any integral element. Therefore, $\Sigma(R)$ is a section of the Grassmann bundle $J(D,2)$ over R.

Next we consider the case of type (III). We have the following assertion by the same argument as in the proof of Lemma 2.

Lemma 3. *Let R be an overdetermined system of type* (III). *Then the rank two prolongation $\Sigma(R)$ is diffeomorphic to R.*

Remark 3. *Two differential systems (R,D) and $(\Sigma(R),\hat{D})$ are different in the sense explained in Remark* 2.

Finally we consider the case of type (IV). We also have the following claim by the same argument as in the proof of Lemma 2.

Lemma 4. *Let R be an overdetermined system of type* (IV). *Then, the rank two prolongation $\Sigma(R)$ is diffeomorphic to R.*

Summarizing these lemmas in this section, we obtain the following theorem.

Theorem 1. *Let R be a second-order regular overdetermined system of codimension two for two independent and one dependent variables. Then we obtain the non-trivial rank two prolongation $\Sigma(R)$ only when R is involutive* (i.e. *type* (I)). *Moreover, in this case, the rank two prolongation $\Sigma(R)$ is an S^1-bundle over R.*

We obtain the following statement by combining Theorem 1 and Remark 1.

Corollary 1. *Let R be a second-order regular overdetermined system of codimension two for two independent and one dependent variables. Then we have*

$$R^{(1)} = \Sigma(R) \Leftrightarrow \text{(II)},\ \text{(III)},$$

$$R^{(1)} \neq \Sigma(R) \Leftrightarrow \text{(I)},\ \text{(IV)}.$$

From now on, by analyzing this specified differences of both prolongations more deeply, we obtain an understanding for overdetermined systems including the construction of singular solutions.

4. Rank two prolongations for overdetermined systems of type (I)

In this section, because of further detailed analysis of the rank two prolongations $\Sigma(R)$ for overdetermined systems R of type (I) (i.e. involutive systems), we investigate an algebraic structure of these rank two prolongations. More precisely, we define a certain filtration structure $\{F^p\}_{p=-1}^{-4}$ on $\Sigma(R)$ and calculate explicitly the bracket structures of the symbol algebras. The motivation of this research is to clarify the difference between the rank two prolongation and the prolongation with the transversality condition from the algebraic viewpoint by using nilpotent graded Lie algebras. Hence, from the obtained results in the previous section, we treat only overdetermined systems of type (I). Namely, we do not treat equations of two finite types and torsion type as research objects for the above motivation. For this purpose, we consider the decomposition

$$\Sigma(R) = \Sigma_0 \cup \Sigma_1, \tag{12}$$

where $\Sigma_i = \{w \in \Sigma(R) \mid \dim(w \cap \text{fiber}) = i\}$ $(i = 0, 1)$. Here "fiber" means that the fiber of $TR \supset D \to TJ^1$. Namely, it is the one dimensional fiber of $TR \to TJ^1$ which is included in D. For the covering of the fibration $p : \Sigma(R) \to R$, we have

$$\Sigma_0|_{p^{-1}(U)} = P_{\omega_1\omega_2}, \qquad \Sigma_1|_{p^{-1}(U)} = P_{\omega_1\pi} \backslash P_{\omega_1\omega_2}.$$

We emphasize that the set Σ_0 is an open subset in $\Sigma(R)$ which is isomorphic to $R^{(1)}$ and Σ_1 is a codimension one submanifold in $\Sigma(R)$. Now we describe the canonical system $\hat{D}$ of rank three on each open set in the following. On $P_{\omega_1\omega_2}$, we have the expression $\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_\pi = 0\}$, where $\varpi_\pi = \pi - p_2^1\omega_2$ and p_2^1 is a fiber coordinate. On $P_{\omega_1\pi}$, we have the expression $\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_{\omega_2} = 0\}$, where $\varpi_{\omega_2} = \omega_2 - p_2^2\pi$ and p_2^2 is a fiber coordinate. Utilizing this decomposition, we investigate the symbol algebra of $\Sigma(R)$. We introduce the sequence $T\Sigma(R) \supset ((\pi_1^2)|_R \circ p)_*^{-1}(C^1) \supset p_*^{-1}(D) \supset \hat{D}$, where $\pi_1^2 : J^2 \to J^1$ and C^1 is the canonical system on J^1. In the discussion of the proof of Proposition 1, we will show that this sequence becomes a filtration on $T\Sigma(R)$. Hence we can define the symbol algebra of $\Sigma(R)$. We obtain the following statement for this symbol algebra.

Proposition 1. *For any point $w \in \Sigma_0$, the symbol algebra $\mathfrak{f}^0(w)$ is isomorphic to*

$$\mathfrak{f}^0 := \mathfrak{f}_{-4} \oplus \mathfrak{f}_{-3} \oplus \mathfrak{f}_{-2} \oplus \mathfrak{f}_{-1},$$

whose bracket relations are given by

$$[X_{p_2^1}, X_{\omega_2}] = X_\pi, \qquad [X_\pi, X_{\omega_2}] = X_2, \qquad [X_1, X_{\omega_1}] = [X_2, X_{\omega_2}] = X_0,$$

and the other brackets are trivial. Here $\{X_0, X_1, X_2, X_{\omega_1}, X_{\omega_2}, X_\pi, X_{p_2^1}\}$ *is a basis of* $\mathfrak{k}^0$ *and*

$$\mathfrak{k}_{-1} = \{X_{\omega_1}, X_{\omega_2}, X_{p_2^1}\}, \qquad \mathfrak{k}_{-2} = \{X_\pi\}, \qquad \mathfrak{k}_{-3} = \{X_1, X_2\}, \qquad \mathfrak{k}_{-4} = \{X_0\}.$$

For any point $w \in \Sigma_1$*, the symbol algebra* $\mathfrak{k}^1(w)$ *is isomorphic to*

$$\mathfrak{k}^1 := \mathfrak{k}_{-4} \oplus \mathfrak{k}_{-3} \oplus \mathfrak{k}_{-2} \oplus \mathfrak{k}_{-1},$$

whose bracket relations are given by

$$[X_{p_2^2}, X_\pi] = X_{\omega_2}, \qquad [X_\pi, X_{\omega_2}] = X_2, \qquad [X_1, X_{\omega_1}] = X_0,$$

and the other brackets are trivial. Here $\{X_0, X_1, X_2, X_{\omega_1}, X_{\omega_2}, X_\pi, X_{p_2^2}\}$ *is a basis of* $\mathfrak{k}^1$ *and*

$$\mathfrak{k}_{-1} = \{X_{\omega_1}, X_\pi, X_{p_2^2}\}, \qquad \mathfrak{k}_{-2} = \{X_{\omega_2}\}, \qquad \mathfrak{k}_{-3} = \{X_1, X_2\}, \qquad \mathfrak{k}_{-4} = \{X_0\}.$$

Remark 4. *Thanks to the property* $\Sigma_0 \cong R^{(1)}$*, the symbol algebra* $\mathfrak{k}^0$ *is isomorphic to the symbol algebra of* $R^{(1)}$ *associated with the similar filtration on* $TR^{(1)}$ *based on the canonical system* $D^{(1)}$.

Proof. We first prove the assertion for the symbol algebra on Σ_0. We recall that the canonical system $\hat{D}$ on $P_{\omega_1\omega_2}$ is given by the expression $\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_\pi = 0\}$, where $\varpi_\pi = \pi - p_2^1\omega_2$. Then the structure equation of $\hat{D}$ on $P_{\omega_1\omega_2}$ can be written as

$$\begin{aligned} d\varpi_i &\equiv 0 && \operatorname{mod} \varpi_0, \varpi_1, \varpi_2, \varpi_\pi, \\ d\varpi_\pi &\equiv \omega_2 \wedge (dp_2^1 + f\omega_1) && \operatorname{mod} \varpi_0, \varpi_1, \varpi_2, \varpi_\pi, \end{aligned} \tag{13}$$

where f is an appropriate function. Hence we have $\partial\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\} = p_*^{-1}(D)$. The structure equation of $\partial\hat{D}$ is equal to the structure equation (6) of (R, D). Here we set $F^{-4} := T\Sigma(R)$, $F^{-3} := ((\pi_1^2)|_R \circ p)_*^{-1}(C^1) = \{\varpi_0 = 0\}$, $F^{-2} := \partial\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$, $F^{-1} := \hat{D}$. Moreover, for $w \in \Sigma_0$, we set $\mathfrak{k}_{-1}(w) := F^{-1}(w) = \hat{D}(w)$, $\mathfrak{k}_{-2}(w) := F^{-2}(w)/F^{-1}(w)$, $\mathfrak{k}_{-3}(w) := F^{-3}(w)/F^{-2}(w)$, $\mathfrak{k}_{-4}(w) := F^{-4}(w)/F^{-3}(w)$, and

$$\mathfrak{k}^0(w) = \mathfrak{k}_{-4}(w) \oplus \mathfrak{k}_{-3}(w) \oplus \mathfrak{k}_{-2}(w) \oplus \mathfrak{k}_{-1}(w).$$

Then we have a filtration structure $\{F^p\}_{p=-1}^{-4}$ on $\Sigma(R)$ around $w \in \Sigma_0$. By the definition of symbol algebras associated with filtration structures in Section 2,

$\mathfrak{f}^0(w)$ has the structure of a nilpotent graded Lie algebra. We consider the bracket relation of $\mathfrak{f}^0(w)$. We take a coframe around $w \in \Sigma_0$ given by

$$\{\varpi_0, \varpi_1, \varpi_2, \varpi_\pi, \omega_1, \omega_2, \varpi_{p_2^1} := dp_2^1 + f\omega_1\}, \tag{14}$$

and the dual frame

$$\{X_0, X_1, X_2, X_\pi, X_{\omega_1}, X_{\omega_2}, X_{p_2^1}\}. \tag{15}$$

Then, the structure equations of each subbundle in the filtration $\{F^p\}_{p=-1}^{-4}$ can be written by the coframe (14).

$$\begin{aligned} d\varpi_i &\equiv \quad 0 && \bmod \varpi_0, \varpi_1, \varpi_2, \varpi_\pi, \\ d\varpi_\pi &\equiv \omega_2 \wedge \varpi_{p_2^1} && \bmod \varpi_0, \varpi_1, \varpi_2, \varpi_\pi, \end{aligned}$$

$$\begin{aligned} d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 && \bmod \varpi_0, \\ d\varpi_1 &\equiv \quad 0 && \bmod \varpi_0, \varpi_1, \varpi_2, \\ d\varpi_2 &\equiv \qquad\quad \omega_2 \wedge \varpi_\pi && \bmod \varpi_0, \varpi_1, \varpi_2. \end{aligned}$$

$$d\varpi_0 \equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 \qquad \bmod \varpi_0, \varpi_1 \wedge \varpi_2, \varpi_1 \wedge \varpi_\pi, \varpi_2 \wedge \varpi_\pi.$$

We set

$$[X_{\omega_2}, X_{p_2^1}] = AX_\pi, \qquad (A \in \mathbb{R}).$$

Then

$$\begin{aligned} d\varpi_\pi(X_{\omega_2}, X_{p_2^1}) &= X_{\omega_2}\varpi_\pi(X_{p_2^1}) - X_{p_2^1}\varpi_\pi(X_{\omega_2}) - \varpi_\pi([X_{\omega_2}, X_{p_2^1}]), \\ &= -\varpi_\pi([X_{\omega_2}, X_{p_2^1}]) = -A. \end{aligned}$$

On the other hand

$$\begin{aligned} d\varpi_\pi(X_{\omega_2}, X_{p_2^1}) &= \omega_2(X_{\omega_2})\varpi_{p_2^1}(X_{p_2^1}) - \varpi_{p_2^1}(X_{\omega_2})\omega_2(X_{p_2^1}), \\ &= 1. \end{aligned}$$

Therefore, $A = -1$. The other brackets are also obtained by the same argument and the definition of the symbol algebra associated with the filtration structure. Thus we have the bracket relation of $\mathfrak{f}^0$.

We next prove the assertion for the symbol algebra on Σ_1. We recall that Σ_1 is locally given by $P_{\omega_1\pi} \backslash P_{\omega_1\omega_2}$. Thus we calculate on $P_{\omega_1\pi} \backslash P_{\omega_1\omega_2} = \{p_2^2 = 0\} \subset P_{\omega_1\pi}$. Then the canonical system $\hat{D}$ is given by $\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_{\omega_2} = 0\}$, where $\varpi_{\omega_2} = \omega_2 - p_2^2\pi$. Note that $\varpi_{\omega_2} = \omega_2$ on Σ_1. The structure equation of $\hat{D}$ at a point $w \in \Sigma_1 = \{p_2^2 = 0\}$ is

$$
\begin{aligned}
d\varpi_i &\equiv 0 && \mod \varpi_0, \varpi_1, \varpi_2, \varpi_{\omega_2}, \\
d\varpi_{\omega_2} &\equiv \pi \wedge (dp_2^2 + f\omega_1) && \mod \varpi_0, \varpi_1, \varpi_2, \varpi_{\omega_2},
\end{aligned} \tag{16}
$$

where f is an appropriate function. Hence we have $\partial\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\} = p_*^{-1}(D)$. The structure equation of $\partial\hat{D}$ is equal to the structure equation (6) of (R, D). Here, we take the filtration which is the same as in the case of $\mathfrak{f}^0$. Then we have the symbol algebra $\mathfrak{f}^1(w)$ at a point $w \in \Sigma_1$ given by

$$\mathfrak{f}^1(w) = \mathfrak{f}_{-4}(w) \oplus \mathfrak{f}_{-3}(w) \oplus \mathfrak{f}_{-2}(w) \oplus \mathfrak{f}_{-1}(w).$$

We consider the bracket relation of $\mathfrak{f}^1(w)$. We take a coframe around $w \in \Sigma_1$ given by

$$\{\varpi_0, \varpi_1, \varpi_2, \varpi_{\omega_2}, \omega_1, \pi, \varpi_{p_2^2} := dp_2^2 + f\omega_1\}, \tag{17}$$

and the dual frame

$$\{X_0, X_1, X_2, X_{\omega_2}, X_{\omega_1}, X_\pi, X_{p_2^2}\}. \tag{18}$$

Then the structure equations of each subbundle in the filtration $\{F^p\}_{p=-1}^{-4}$ can be written by the coframe (17).

$$
\begin{aligned}
d\varpi_i &\equiv 0 && \mod \varpi_0, \varpi_1, \varpi_2, \varpi_{\omega_2}, \\
d\varpi_{\omega_2} &\equiv \pi \wedge \varpi_{p_2^2} && \mod \varpi_0, \varpi_1, \varpi_2, \varpi_{\omega_2},
\end{aligned}
$$

$$
\begin{aligned}
d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 && \mod \varpi_0, \\
d\varpi_1 &\equiv 0 && \mod \varpi_0, \varpi_1, \varpi_2, \\
d\varpi_2 &\equiv \omega_2 \wedge \pi && \mod \varpi_0, \varpi_1, \varpi_2.
\end{aligned}
$$

$$d\varpi_0 \equiv \omega_1 \wedge \varpi_1 \qquad \mod \varpi_0, \varpi_1 \wedge \varpi_2, \varpi_1 \wedge \varpi_{\omega_2}, \varpi_2 \wedge \varpi_{\omega_2}.$$

By the definition of the symbol algebra associated with the filtration structure and the same argument as in the case of $\mathfrak{f}^0$, we obtain the bracket relation of $\mathfrak{f}^1$.

Here, we mention the bracket relations of the two symbol algebras $\mathfrak{f}^0$ and $\mathfrak{f}^1$. First we first show that these two symbol algebras do not satisfy the generating condition. We next emphasize a difference between $\mathfrak{f}^0$ and $\mathfrak{f}^1$ in the following. On one hand, the symbol algebra $\mathfrak{f}^0$ has a one dimensional direction spanned by X_{ω_2} which generates the highest degree component $\mathfrak{f}_{-4}$. Precisely speaking, the direction X_π generates one dimensional subspaces of $\mathfrak{f}_{-2}$, $\mathfrak{f}_{-3}$ and $\mathfrak{f}_{-4}$. On the other hand, the symbol algebra $\mathfrak{f}^1$ does not have a direction which generates the highest degree component.

Now, we mention a tower structure constructed by successive rank two prolongations of overdetermined systems of type (I). We define the k-th rank two prolongation $(\Sigma^k(R), \hat{D}^k)$ of (R, D) as follows.

$$(\Sigma^k(R), \hat{D}^k) := (\Sigma(\Sigma^{k-1}(R)), \hat{\hat{D}}^{k-1}) \qquad (k = 1, 2, \ldots),$$

where $(\Sigma^0(R), \hat{D}^0) := (R, D)$.

Theorem 2. *Let R be a regular overdetermined system of type* (I). *Then the k-th rank two prolongation $\Sigma^k(R)$ of R is also an S^1-bundle over $\Sigma^{k-1}(R)$.*

Proof. From the expressions (13) or (16) of the structure equations of $\Sigma(R)$, we easily show that the k-th rank two prolongation $\Sigma^k(R)$ can be defined successively. Then we have the assertion by using the same argument as in the proof of Lemma 1 successively.

5. Geometric singular solutions of overdetermined systems of type (I)

In this section, we provide the methods for constructing geometric singular solutions of overdetermined systems of type (I). We first define the notion of geometric singular solutions for regular PDEs ([16], [17]).

Definition 5. Let R be a second-order regular PDE in $J^2(\mathbb{R}^2, \mathbb{R})$. For a two dimensional integral manifold S of the canonical system $D := C^2|_R$ on R, if the restriction $\pi_1^2|_R : R \to J^1(\mathbb{R}^2, \mathbb{R})$ of the natural projection $\pi_1^2 : J^2 \to J^1$ is an immersion on an open dense subset in S, then we call S a *geometric solution* of R. If all points of a geometric solution S are immersion points, then we call S a *regular solution*. On the other hand, if a geometric solution S has a nonimmersion point, then we call S a *singular solution*.

From the definition, the image $\pi_1^2(S)$ of a geometric solution S by the projection π_1^2 is a Legendrian surface in $J^1(\mathbb{R}^2, \mathbb{R})$ (i.e. $\varpi_0|_{\pi_1^2(S)} = d\varpi_0|_{\pi_1^2(S)} = 0$). From the proof of Lemmas 2 and 3, we can show that there does not exist any singular solution for equations of type (II) nor of type (III). On the other hand, Lemma 4 says the possibility of the existence of a singular solution for equations of torsion type (IV). Of course, there does not exist any regular solution for equations of torsion type. In this context, from now on, we discuss the method of the construction of singular solutions only when equations are of type (I) (i.e. involutive).

Let R be an overdetermined system of type (I) and $p : \Sigma(R) \to R$ be the rank two prolongation. For this system R, we can give two methods of the construction of geometric singular solutions which are given by the following.

(i) We construct a singular solution of R by the projection of a special solution of $(\Sigma(R), \hat{D})$.

(ii) We construct a singular solution of R as a lifted solution by using the reduced fibration $\pi_B^R : R \to B$. Here, the reduced space B is a quotient space (leaf space) by the Cauchy characteristic systems $Ch(D)$ of D.

In the approach (i), a special solution which gives a singular solution of R is constructed in the higher dimensional space $\Sigma(R)$. On the other hand, in the approach (ii), a special solution which gives a singular solution of R is constructed in the lower dimensional space B. More precisely, we illustrate the character of these two methods and those differences in the following.

We first explain the principle of the approach (i). We recall the decomposition (12) of the rank two prolongation $\Sigma(R)$ of R. Here $(P_{\omega_1\omega_2}, \hat{D})$ is the rank two prolongation with the independence condition $\omega_1 \wedge \omega_2 \neq 0$. In general, for a given second-order regular overdetermined system $R = \{F = G = 0\}$ with two independent variables x, y, this prolongation corresponds to a third-order PDE system which is obtained by partial differentiations of $F = G = 0$ with respect to x and y. If we construct a solution S of the system $(P_{\omega_1\omega_2}, \hat{D})$, S is a regular solution by the definition of Σ_0. On the other hand, if we construct a solution S of $\Sigma(R)$ passing through $\Sigma_1 = \{p_2^2 = 0\} \subset P_{\omega_1\pi}$, S is a singular solution of R from the decomposition (12). The approach (i) is a method to construct singular solutions utilizing this principle.

We next explain the principle of the approach (ii). In fact, E. Cartan [4] characterized the overdetermined system R of type (I) by the condition that R admits a one dimensional Cauchy characteristic system. From the description (6) of the structure equation of R, we obtain the expression $Ch(D) = \{\varpi_0 = \varpi_1 = \varpi_2 = \omega_2 = \pi = 0\}$. This system $Ch(D)$ gives a one dimensional foliation. Hence, a leaf space $B := R/Ch(D)$ is locally a five dimensional manifold. For this fibration $\pi_B^R : R \to B$, it is well-known that there exists a rank two differential system D_B on the quotient space B (cf. [4], [19], [29]). Hence, if we construct an integral curve of the rank two differential system (B, D_B), we obtain a lifted integral surface S of R by using the fibration $\pi_B^R : R \to B$. In this situation, our strategy is to find a special integral curve of (B, D_B) which gives a lifted singular solution. This principle corresponds to the method of characteristics. Namely, the approach (ii) is a theory of reduction into ordinary differential equations. As a matter of course, the approach (ii) can be applicable to an equation only when it has a nontrivial Cauchy characteristic system. On the other hand, the approach (i) can be applicable to a wider class of equations. For example, second-order single equations with two independent and one dependent variables have trivial Cauchy characteristic systems. In particular, for the elliptic case, there does not exist even a Monge characteristic system. Hence, the approach (ii) cannot be applicable to such a class of single equations. However, the approach (i) can be applicable to the class of single equations. Indeed, in [17], integral representations of singular solutions of typical equations have been constructed. In this sense, the approach (ii) can be regarded as a method to treat a special case.

To demonstrate the usefulness of these methods, we construct singular solutions of an important equation in the following subsection.

5.1. Singular solutions of Cartan's overdetermined system. We consider Cartan's overdetermined system

$$R = \left\{ r = \frac{t^3}{3}, s = \frac{t^2}{2} \right\}.$$

The canonical system D on R is given by

$$\varpi_0 = dz - p\,dx - q\,dy, \qquad \varpi_1 = dp - \frac{t^3}{3}\,dx - \frac{t^2}{2}\,dy, \qquad \varpi_2 = dq - \frac{t^2}{2}\,dx - t\,dy,$$

and the structure equation of D is given by

$$\begin{aligned} d\varpi_0 &\equiv dx \wedge dp + dy \wedge dq, && \operatorname{mod} \varpi_0, \\ d\varpi_1 &\equiv -t^2\,dt \wedge dx - t\,dt \wedge dy, && \operatorname{mod} \varpi_0, \varpi_1, \varpi_2, \\ d\varpi_2 &\equiv -t\,dt \wedge dx - dt \wedge dy, && \operatorname{mod} \varpi_0, \varpi_1, \varpi_2. \end{aligned} \tag{19}$$

We take a new coframe

$$\{\varpi_0, \hat{\varpi}_1 := \varpi_1 - t\varpi_2,\ \varpi_2, \pi := dt,\ \omega_1 := dx,\ \omega_2 := t\,dx + dy\}.$$

For this coframe, the above structure equation is rewritten as

$$\begin{aligned} d\varpi_0 &\equiv \omega_1 \wedge \hat{\varpi}_1 + \omega_2 \wedge \varpi_2, && \operatorname{mod} \varpi_0, \\ d\hat{\varpi}_1 &\equiv 0 && \operatorname{mod} \varpi_0, \hat{\varpi}_1, \varpi_2, \\ d\varpi_2 &\equiv \omega_2 \wedge \pi, && \operatorname{mod} \varpi_0, \hat{\varpi}_1, \varpi_2. \end{aligned} \tag{20}$$

Hence this equation R is an overdetermined system of type (I).

We first construct singular solutions of R by using the approach (i). For this purpose, we need to prepare the rank two prolongation $\Sigma(R)$ of R in terms of the Grassmann bundle $\Pi : J(D,2) \to R$. For any open set $U \subset R$, $\Pi^{-1}(U)$ is covered by three open sets in $J(D,2)$ such that $\Pi^{-1}(U) = U_{xy} \cup U_{xt} \cup U_{yt}$, where

$$\begin{aligned} U_{xy} &:= \{w \in \Pi^{-1}(U) \mid dx|_w \wedge dy|_w \neq 0\}, \\ U_{xt} &:= \{w \in \Pi^{-1}(U) \mid dx|_w \wedge dt|_w \neq 0\}, \\ U_{yt} &:= \{w \in \Pi^{-1}(U) \mid dy|_w \wedge dt|_w \neq 0\}. \end{aligned}$$

We next explicitly describe the defining equation of $\Sigma(R)$ in terms of the inhomogeneous Grassmann coordinate of fibers in U_{xy}, U_{xt}, U_{yt}. First, we consider it in U_{xy}. For $w \in U_{xy}$, w is a two dimensional subspace of $D(v)$, where $p(w) = v$. Hence, by restricting dt to w, we can introduce the inhomogeneous coordinate p_i^1 of fibers of $J(D,2)$ around w with $dt|_w = p_1^1(w)dx|_w + p_2^1(w)dy|_w$. Moreover w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$. Hence we have

$$d\varpi_1|_w \equiv (t^2 p_2^1(w) - tp_1^1(w))dx|_w \wedge dy|_w,$$

$$d\varpi_2|_w \equiv (tp_2^1(w) - p_1^1(w))dx|_w \wedge dy|_w.$$

In this way, we obtain the defining equations $f = 0$ of $\Sigma(R)$ in U_{xy} of $J(D,2)$, where $f = p_1^1 - tp_2^1$. Then df does not vanish on $\{f = 0\}$. Next we consider in U_{xt}. For $w \in U_{xt}$, w is a two dimensional subspace of $D(v)$, where $p(w) = v$. Hence, by restricting dy to w, we can introduce the inhomogeneous coordinate p_i^2 of fibers of $J(D,2)$ around w with $dy|_w = p_1^2(w)dx|_w + p_2^2(w)dt|_w$. Moreover w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$. In this situation, it is sufficient to consider the condition $d\varpi_2|_w \equiv (t + p_1^2(w))dx|_w \wedge dt|_w \equiv 0$. Then, for the defining function $g = t + p_1^2$ of $\Sigma(R)$ in U_{xt}, dg does not vanish on $\{g = 0\}$. Finally, we consider in U_{yt}. For $w \in U_{yt}$, w is a two dimensional subspace of $D(v)$, where $p(w) = v$. Hence, by restricting dx to w, we can introduce the inhomogeneous coordinate p_i^3 of fibers of $J(D,2)$ around w with $dx|_w = p_1^3(w)dy|_w + p_2^3(w)dt|_w$. Moreover w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$. Here, $d\varpi_2|_w \equiv (1 + tp_1^3(w))dy|_w \wedge dt|_w$. Then, for the defining function $h = 1 + tp_1^3$ of $\Sigma(R)$ in U_{yt}, dh does not vanish on $\{h = 0\}$. Therefore, we have the covering for the fibration $p : \Sigma(R) \to R$ such that $p^{-1}(U) = P_{xy} \cup P_{xt} \cup P_{yt}$, where $P_{xy} := p^{-1}(U) \cap U_{xy}$, $P_{xt} := p^{-1}(U) \cap U_{xt}$ and $P_{yt} := p^{-1}(U) \cap U_{yt}$. However this covering is not essential.

Proposition 2. *Let R be Cartan's overdetermined system and U be an open set on R. Then we have*

$$p^{-1}(U) = P_{xy} \cup P_{xt}. \tag{21}$$

Proof. We show that $P_{yt} \subset P_{xt}$. Let w be any point in P_{yt}. Then we have

$$dx|_w \wedge dt|_w = -\frac{1}{t}(w)dy|_w \wedge dt|_w.$$

Here, if $w \notin P_{xt}$, then we have the condition $1/t(w) = 0$. However there does not exist such a point w.

We have the following description of the canonical system $\hat{D}$ of rank three: For P_{xy}, $\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_t = 0\}$, where $\varpi_t = dt - ta\, dx - a\, dy$

and a is a fiber coordinate. For P_{xt}, $\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_y = 0\}$, where $\varpi_y = dy + t\,dx - b\,dt$ and b is a fiber coordinate. The decomposition $\Sigma(R) = \Sigma_0 \cup \Sigma_1$ is given by $\Sigma_0|_{p^{-1}(U)} = P_{xy}$, $\Sigma_1|_{p^{-1}(U)} = P_{xt} \backslash P_{xy}$, respectively.

By using the approach (i), we construct geometric singular solutions of $(\Sigma(R), \hat{D})$ passing through Σ_1. Let $\iota : S \hookrightarrow P_{xt}$ be a graph defined by

$$(x, y(x,t), z(x,t), p(x,t), q(x,t), t, b(x,t)).$$

If S is an integral submanifold of $(P_{xt}, \hat{D})$, then the following conditions are satisfied:

$$\iota^*\varpi_0 = (z_x - p - qy_x)dx + (z_t - qy_t)dt = 0, \tag{22}$$

$$\iota^*\varpi_1 = \left(p_x - \frac{t^3}{3} - \frac{t^2}{2}y_x\right)dx + \left(p_t - \frac{t^2}{2}y_t\right)dt = 0, \tag{23}$$

$$\iota^*\varpi_2 = \left(q_x - \frac{t^2}{2} - ty_x\right)dx + (q_t - ty_t)dt = 0, \tag{24}$$

$$\iota^*\varpi_y = (y_x + t)dx + (y_t - b)dt = 0. \tag{25}$$

From these conditions, we have

$$z_x - p + qt = 0, \qquad z_t - bq = 0, \tag{26}$$

$$p_x + \frac{t^3}{6} = 0, \qquad p_t - \frac{bt^2}{2} = 0, \tag{27}$$

$$q_x + \frac{t^2}{2} = 0, \qquad q_t - bt = 0, \tag{28}$$

$$y_x + t = 0, \qquad y_t - b = 0. \tag{29}$$

We have $y = -tx + y_0(t)$ from (29). Note that the condition passing through Σ_1 is $y_t(0,0) = y_0'(0) = 0$. From (28), we have $q = -t^2x/2 + ty_0(t) - \int y_0(t)dt$. From (27), we have $p = -t^3x/6 + t^2y_0(t)/2 - \int ty_0(t)dt$. From (26), we have

$$\begin{aligned} z = \frac{x^2t^3}{6} - x\left\{\frac{t^2y_0(t)}{2} + \int ty_0(t)dt - t\int y_0(t)dt\right\} + \frac{ty_0^2(t)}{2} \\ + \frac{1}{2}\int y_0^2(t)dt - y_0(t)\int y_0(t)dt. \end{aligned}$$

Consequently, we obtain the explicit integral representation of a singular solution in the coordinate space $\mathbb{R}^3 = \{(x, y, z)\}$.

$$\left(x, -xt + y_0(t), \frac{x^2t^3}{6} - x\left\{\frac{t^2y_0(t)}{2} + \int ty_0(t)dt - t\int y_0(t)dt\right\}\right.$$
$$\left. + \frac{ty_0^2(t)}{2} + \frac{1}{2}\int y_0^2(t)dt - y_0(t)\int y_0(t)dt\right), \tag{30}$$

where $y_0(t)$ is a function on S depending only t and which satisfies $y_0'(0) = 0$. From this condition $y_0'(0) = 0$, this solution has a singularity at the origin in $J^1(\mathbb{R}^2, \mathbb{R})$.

We next construct geometric singular solutions by using the approach (ii). For Cartan's overdetermined system, there exists a famous reduction to a rank two distribution D_B on a five dimensional manifold B. Here, the reduced space (B, D_B) is the flat G_2-model space of $(2,3,5)$-distributions ([4], [27], [28]). Cartan obtained an explicit integral representation of a general solution of Cartan's overdetermined system based on this reduction $R \to B$. For modern interpretations of this reduction and related topics, (e.g. symmetry, quadrature, duality, etc.), various references exist (e.g. [1], [4], [19], [26]). In this paper, we refer to [19] which is an exposition about the theory of characteristics based on this reduction. From the structure equation (20), we have

$$\begin{aligned} Ch(D) &= \{\varpi_0 = \hat{\varpi}_1 = \varpi_2 = \omega_2 = \pi = 0\} \\ &= \mathrm{span}\left\{\frac{\partial}{\partial x} - t\frac{\partial}{\partial y} + (p - tq)\frac{\partial}{\partial z} - \frac{t^3}{6}\frac{\partial}{\partial p} - \frac{t^2}{2}\frac{\partial}{\partial q}\right\}. \end{aligned}$$

Hence we have a local coordinate $(x_1, x_2, x_3, x_4, x_5)$ on the leaf space $B := R/Ch(D)$ given by

$$x_1 := z - xp + xqt + \frac{1}{6}x^2t^3, \qquad x_2 := p - qt + \frac{1}{2}yt^2 + \frac{1}{6}t^3x,$$
$$x_3 := -q + \frac{1}{2}yt, \qquad x_4 := y + xt, \qquad x_5 := -t.$$

Conversely, R is locally an $\mathbb{R}$-bundle on B. If we take a coordinate function λ of the fiber $\mathbb{R}$, then the coordinate (x, y, z, p, q, t) is expressed in terms of the coordinate $(x_1, x_2, x_3, x_4, x_5, \lambda)$ defined by

$$\begin{aligned} &x = \lambda, \qquad y = x_4 + \lambda x_5, \qquad z = x_1 + \lambda x_2 - \frac{1}{2}\lambda x_4(x_5)^2 - \frac{1}{6}\lambda^2(x_5)^3, \\ &p = x_2 + x_3x_5 + \frac{1}{6}\lambda(x_5)^3, \qquad q = -x_3 - \frac{1}{2}x_4x_5 - \frac{1}{2}\lambda(x_5)^2, \qquad t = -x_5. \end{aligned} \tag{31}$$

On the base space B, we consider a differential system $D_B = \{\alpha_1 = \alpha_2 = \alpha_3 = 0\}$ of rank two given by

$$\alpha_1 = dx_1 + \left(x_3 + \frac{1}{2}x_4x_5\right)dx_4, \qquad \alpha_2 = dx_2 + \left(x_3 - \frac{1}{2}x_4x_5\right)dx_5,$$

$$\alpha_3 = dx_3 + \frac{1}{2}(x_4\, dx_5 - x_5\, dx_4).$$

For the projection $p : R \to B$, generator 1-forms of D and D_B are related as follows:

$$\varpi_0 := p^*\alpha_1 + xp^*\alpha_2, \qquad \varpi_1 := p^*\alpha_2 - xp^*\alpha_3, \qquad \varpi_2 := -p^*\alpha_3.$$

Thus, (B, D_B) is the reduced space of (R, D), that is, $(B, D_B) = (p(R), p_*D)$. Utilizing this fibration, we will give the explicit representation of singular solutions in the following. From the description $t = -x_5$ in (31), we take a parameter τ as $x_5 = \tau$. By solving ordinary differential equations given by $\alpha_1 = \alpha_2 = \alpha_3 = 0$, we obtain the following integral curve $c(\tau)$ of D_B:

$$\begin{aligned} x_1 &= \int\left\{\varphi' \int \varphi\, d\tau - \varphi\varphi'\tau\right\}d\tau, \qquad x_2 = \int\left\{\int \varphi\, d\tau\right\}d\tau, \\ x_3 &= -\frac{1}{2}\int(\varphi - \tau\varphi')d\tau, \qquad x_4 = \varphi(\tau), \qquad x_5 = \tau. \end{aligned} \tag{32}$$

where $\varphi(\tau)$ is an arbitrary smooth function of τ. Here, we assume the condition $\varphi'(0) = 0$ to construct a singular solution which has a singularity at the origin in $J^1(\mathbb{R}^2, \mathbb{R})$. Then, from the relations (31) and (32), we obtain the following integral representation of a singular solution in the coordinate space $\mathbb{R}^3 = \{(x, y, z)\}$.

$$\begin{aligned} &\left(x, -xt + \varphi(-t), \frac{x^2t^3}{6} - x\left\{\frac{t^2\varphi(-t)}{2} + \int t\varphi(-t)dt - t\int \varphi(-t)dt\right\}\right. \\ &\qquad \left. + \frac{t\varphi^2(-t)}{2} + \frac{1}{2}\int \varphi^2(-t)dt - \varphi(-t)\int \varphi(-t)dt\right). \end{aligned} \tag{33}$$

This integral representation is equal to (30) obtained by the approach (i).

REMARK 5. *In the integral* (32), *if we take a parameter* τ *as* $x_4 = \tau$, *then we have the regular solution in* (33) (cf. [4], [19]).

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References

[1] I. Anderson, B. Kruglikov, Rank 2 distributions of Monge equations: Symmetries, equivalences, extensions, Advances in Math. vol. 228, Issue 3 (2011), pp. 1435–1465.

[2] R. Bryant, S. S. Chern, R. Gardner, H. Goldschmidt, P. Griffiths, Exterior Differential Systems, MSRI Publ. vol. 18, Springer Verlag, Berlin (1991).

[3] R. Bryant, P. Griffiths, Characteristic cohomology of differential systems, I: General theory, J. Amer. Math. Soc. vol. 8, no. 3 (1995), pp. 507–596.

[4] E. Cartan, Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre, Ann. École Normale, 27 (1910), pp. 109–192.

[5] E. Cartan, Sur les systèmes en involution déquations aux dérivées partielles du second ordre à une fonction inconnue de trois variables independantes, Bull. Soc. Math. France, 39 (1911), pp. 352–443.

[6] K. Kakie, On involutive systems of partial differential equations in two independent variables, J. Fac. Sci. Univ. Tokyo. Sect IA Math. 21 (1974), pp. 405–433.

[7] K. Kakie, The Cauchy problem for an involutive system of partial differential equations in two independent variables, J. Math. Soc. Japan. 27 (1975), No. 4, pp. 517–532.

[8] B. Kruglikov, V. Lychagin, Geometry of differential equations, Handbook of global analysis, 1214, Elsevier Sci. B. V. Amsterdam, (2008), pp. 725–771.

[9] B. Kruglikov, V. Lychagin, Mayer brackets and solvability of PDEs–I, Diff. Geom. and its Appl. 17 (2002), pp. 251–272.

[10] B. Kruglikov, V. Lychagin, Mayer brackets and solvability of PDEs–II, Trans. Amer. Math. Soc. Vol. 358, Number 3 (2005), pp. 1077–1103.

[11] S. Lie, Zur allegemeinen Theorie der partiellen Differetialgleichungen beliebiger Ordnung, Leipz. Ber. 1895, Heft 1, abgeliefert 7, 5, 1895, pp. 53–128, vorgelegt in der Sitzung vom 4. 2. 1895.

[12] V. Lychagin, Geometric theory of singularities of solutions of nonlinear differential equations. (Russian), Translated in J. Soviet Math. 51 (1990), no 6, pp. 2735–2757.

[13] V. Lychagin, A. Prastaro, Singularities of Cauchy data, characteristics, cocharacteristics and integral cobordism, Diff. Geom. and its Appl. 4 (1994), pp. 283–300.

[14] T. Morimoto, Geometric structures on filtered manifolds, Hokkaido Math. J. 22 (1993), pp. 263–347.

[15] R. Montgomery, M. Zhitomirskii, Geometric approach to Goursat flags, Ann. Inst. H. Poincaré-AN 18 (2001), pp. 459–493.

[16] T. Noda, K. Shibuya, Second order type-changing PDE for a scalar function on a plane, Osaka J. Math. Vol. 49, No. 1, (2012), pp. 101–124.

[17] T. Noda, K. Shibuya, Rank 2 prolongations of second order PDE and geometric singular solutions, Tokyo J. Math. Vol. 37, No. 1, (2014), pp. 73–110.

[18] T. Noda, K. Shibuya, K. Yamaguchi, Contact geometry of regular overdetermined systems of second order, in preparation.

[19] H. Sato, Contact geometry of second order partial differential equations: from Darboux and Goursat, through Cartan to modern mathematics, Suugaku Exposition 20 (2007), no. 2, pp. 137–148.

[20] K. Shibuya, On the prolongation of 2-jet space of 2 independent and 1 dependent variables, Hokkaido Math. J. 38 (2009), pp. 587–626.

[21] K. Shibuya, K. Yamaguchi, Drapeau theorem for differential systems, Diff. Geom. and its Appl. 27 (2009), pp. 793–808.

[22] N. Tanaka, On differential systems, graded Lie algebras and pseudo-groups, J. Math. Kyoto. Univ. 10 (1970), pp. 1–82.
[23] N. Tanaka, On generalized graded Lie algebras and geometric structures I, J. Math. Soc. Japan, 19 (1967), pp. 215–254.
[24] K. Yamaguchi, Contact geometry of higher order, Japan. J. Math. 8 (1) (1982), pp. 109–176.
[25] K. Yamaguchi, On involutive systems of second order of codimension 2, Proc. Japan. Acad. 58, Ser. A (1982), pp. 302–305.
[26] K. Yamaguchi, Geometrizations of Jet bundles, Hokkaido Math. J. 12 (1983), pp. 27–40.
[27] K. Yamaguchi, Typical classes in involutive systems of second order, Japan. J. Math. 11 (2) (1985), pp. 109–176.
[28] K. Yamaguchi, Differential systems associated with simple graded Lie algebras, Advanced Studies in Pure Math. 22 (1993), pp. 413–494.
[29] K. Yamaguchi, Contact geometry of second order I, Differential Equations—Geometry, Symmetries and Integrability—The Abel symposium 2008, Abel symposia 5, 2009, pp. 335–386.

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