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A necessary condition for representatives of elements of Artin groups of dihedral type to be geodesic

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Abstract. In this paper, we consider representatives of each element of Artin groups of finite type in terms of the standard Artin generator system Σ and give a necessary condition for representatives of elements of Artin groups of finite type to be geodesic.

1. Introduction

For a finitely generated group G with a given generator system S, we have the notion of the spherical growth series

$$\mathscr{S}_{G,S}(t) := \sum_{n=0}^{\infty} \gamma_n t^n,$$

where γ_n for $n \in \mathbf{Z}_{\geq 0}$ is the number of elements in G whose lengths with respect to S are equal to n (cf. [19], [22]). The series $\mathscr{S}_{G,S}(t)$ provides a way to capture the combinatorial structure of (G, S). In many cases, the series are known to be rational (cf. [6], [7], [12], [13], [14]). The fact is usually proved by exhibiting a unique geodesic representative of each element of G with respect to S and showing that the set of all such geodesics is recognized by a deterministic finite state automaton. In fact, Charney [9] has constructed such an automaton for each Artin group of finite type over a generator system Λ and obtained an explicit rational function expression of its growth series with respect to Λ. Here we remark that the generator system Λ includes the standard generator system Σ (see §2 for the definition) as a proper subset. It is an interesting open question how to present a rational function expression for each Artin group of finite type with respect to Σ (see Chapter VI in [10]). Mairesse-Mathéus [18] successfully constructed such an automaton for each

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Artin group of dihedral type and obtained a concrete rational function expression for its growth series with respect to Σ. But it is still unknown how to construct an automaton and how to express the growth series with respect to Σ as a rational function for any Artin group of finite type (even for the braid group of n strands with $n \geq 4$ (that is, the Artin group of type A_{n-1})).

In order to construct the automaton that recognizes geodesic representatives of elements of an Artin group of dihedral type, Mairesse-Mathéus [18] described the necessary and sufficient condition that representatives of elements of Artin groups of dihedral type are geodesic with respect to the standard generator system Σ by extending a procedure for the braid group of three strands (that is, the Artin group of type A_2) given in [3]. But we are still far from characterizing geodesic representatives for general Artin groups. Under the above condition, in this paper we consider any Artin group of finite type and provide a necessary condition such that representatives of elements of such an Artin group are geodesic with respect to Σ. This is a generalization of a subset of the characterization of geodesic representatives for dihedral type given by Mairesse-Mathéus. We demonstrate the use of properties of the fundamental element Δ (see §2 for the definition and the properties).

2. Artin groups and Artin monoids

In this section, we summarize definitions and basic facts on Artin groups and Artin monoids from [5].

Let $M = (m_{i,j})_{i,j \in I}$ be a Coxeter matrix (see Chapter 3 in [4]) whose entries are indexed by a finite set $I = \{1, \dots, n\}$. That is, M is a symmetric matrix such that $m_{i,i} = 1$ for $i \in I$ and $m_{i,j} \in \mathbf{Z}_{\geq 2}$ or $m_{i,j} = \infty$ for $i, j \in I$ with $i \neq j$. Associated with a Coxeter matrix M, we introduce the Artin group G_M, the Artin monoid G_M^+ and the Coxeter group $\overline{G}_M$ as follows.

First, we fix a finite set, called an *alphabet*

$$\Sigma^+ = \{\sigma_i \mid i \in I\}$$

of letters indexed by I and set

$$\Sigma^- := \{\sigma_i^{-1} \mid i \in I\},$$

$$\Sigma := \Sigma^+ \cup \Sigma^-.$$

Let Σ^*, $(\Sigma^+)^*$ and $(\Sigma^-)^*$ be the free monoids generated by Σ, Σ^+ and Σ^-, respectively. We call an element of Σ^* (resp. $(\Sigma^+)^*$) a *word* (resp. a *positive word*). The length of a word w is the number of letters in w which is denoted by $|w|$. The length of the empty word is zero.

In order to define the Artin group G_M and the Artin monoid G_M^+, we introduce a notation for $i, j \in I$ and a non-negative integer $q \in \mathbf{Z}_{\geq 0}$:

$$\langle \sigma_i \sigma_j \rangle^q := \underbrace{\sigma_i \sigma_j \sigma_i \cdots}_{q \text{ letters}},$$

which is a positive word of length q starting with σ_i followed by alternating σ_j and σ_i.

Definition 1. The *Artin group* associated with a Coxeter matrix M is a group presented by

$$G_M := \langle \sigma_i \ \ (i \in I) \mid \langle \sigma_i \sigma_j \rangle^{m_{i,j}} = \langle \sigma_j \sigma_i \rangle^{m_{j,i}} \ \ (i, j \in I) \rangle.$$

(If $m_{i,j} = \infty$, then there is no relation between σ_i and σ_j.)

Denote the canonical monoid homomorphism by $\pi : \Sigma^* \to G_M$. A word $w \in \pi^{-1}(g)$ is called a representative of g. The length of a group element g is

$$\|g\| = \min\{k \mid g = \pi(s_1 \cdots s_k), s_i \in \Sigma\}.$$

A representative w of g is a *geodesic* if $|w| = \|g\|$.

Definition 2. The *Coxeter group* associated with a Coxeter matrix M is a group presented by

$$\bar{G}_M := \langle \sigma_i \ \ (i \in I) \mid \langle \sigma_i \sigma_j \rangle^{m_{i,j}} = \langle \sigma_j \sigma_i \rangle^{m_{j,i}} \ \ (i, j \in I), \ \ \sigma_i^2 = 1 \ \ (i \in I) \rangle.$$

M is a Coxeter matrix *of finite type* if $\bar{G}_M$ is a finite group. Indecomposable Coxeter matrices of finite type are classified into the following types: A_n $(n \geq 1)$, B_n $(n \geq 2)$, D_n $(n \geq 4)$, E_n $(6 \leq n \leq 8)$, F_4, G_2, H_n $(n = 3, 4)$ and $I_2(p)$ $(p \geq 5, p \neq 6)$ (for examples, see [4]). In the following discussion, M is always one of these types.

Definition 3. The *Artin monoid* associated with a Coxeter matrix M is a monoid presented by

$$G_M^+ := \langle \sigma_i \ \ (i \in I) \mid \langle \sigma_i \sigma_j \rangle^{m_{i,j}} = \langle \sigma_j \sigma_i \rangle^{m_{j,i}} \ \ (i, j \in I) \rangle^+,$$

where the right-hand side is a quotient of the free monoid $(\Sigma^+)^*$ by an equivalence relation on $(\Sigma^+)^*$ defined as follows: (i) two positive words $\omega, \omega' \in (\Sigma^+)^*$ are elementary equivalent if there are positive words $u, v \in (\Sigma^+)^*$ and indices $i, j \in I$ such that $\omega = u \langle \sigma_i \sigma_j \rangle^{m_{i,j}} v$ and $\omega' = u \langle \sigma_j \sigma_i \rangle^{m_{j,i}} v$, and (ii) two positive words $\omega, \omega' \in (\Sigma^+)^*$ are equivalent if there is a sequence $\omega_0 = \omega$, $\omega_1, \ldots, \omega_k = \omega'$ for some $k \in \mathbf{Z}_{\geq 0}$ such that ω_i is elementary equivalent to ω_{i+1} for $i = 0, \ldots, k-1$.

DEFINITION 4. We call the set Σ^+ the *standard generator system* of the Artin group G_M, of the Artin monoid G_M^+ and of the Coxeter group $\bar{G}_M$.

By Definitions 1, 2 and 3, there are natural homomorphisms $G_M^+ \to G_M$ and $G_M \to \bar{G}_M$. For the former homomorphism, the following injectivity is well known:

THEOREM 1 (see Section 5.5 in [5]). *Let M be a Coxeter matrix of finite type. Then the homomorphism $G_M^+ \to G_M$ is injective.*

Then we consider G_M^+ to be a subset of G_M. In order to understand the composite homomorphism $G_M^+ \to G_M \to \bar{G}_M$, let us recall the concepts of square-free elements.

DEFINITION 5. An element $g \in G_M^+$ is called a *square-free element* if no word ω in the equivalent class of g admits an expression $u\sigma_i\sigma_i v$ for some $u, v \in (\Sigma^+)^*$ and some $i \in I$. We regard the identity element of G_M^+ as a square-free element. Set

$$\mathrm{QFG}_M^+ := \{\mu \in G_M^+ \mid \mu \text{ is a square free element}\},$$

$$\mathrm{QFG}_M^- := \{\mu^{-1} \in G_M \mid \mu \in \mathrm{QFG}_M^+\}.$$

THEOREM 2 (see Section 5.6 in [5]). *Let M be a Coxeter matrix of finite type. Then the restriction of the canonical map $G_M^+ \to \bar{G}_M$ to the subset QFG_M^+ is bijective.*

Next we review basic facts about fundamental elements.

DEFINITION 6. We say that $\omega' \in G_M^+$ divides $\omega \in \mathrm{QFG}_M^+$ from the left (resp. right) and denote $\omega' |_l \omega$ (resp. $\omega' |_r \omega$), if there are words $u, v \in (\Sigma^+)^*$ such that u belongs to the equivalence class ω' and uv (resp. vu) belongs to the equivalence class ω. For an element $\omega \in G_M^+$, set

$$I_l(\omega) := \{i \in I \mid \sigma_i |_l \omega\} \qquad \text{and} \qquad I_r(\omega) := \{i \in I \mid \sigma_i |_r \omega\}.$$

For an element $\omega^{-1} \in \mathrm{QFG}_M^-$, we define $I_r(\omega^{-1})$ and $I_l(\omega^{-1})$ similarly.

LEMMA 1 (see Section 5 in [5]). *Let M be a Coxeter matrix of finite type. For any subset J of I, there exists an element $\Delta_J \in G_M^+$ with the following two properties:*

(1) *For any $i \in J$, we have $\sigma_i |_l \Delta_J$ and $\sigma_i |_r \Delta_J$.*

(2) *If an element $u \in G_M^+$ satisfies $\sigma_i |_l u$ (resp. $\sigma_i |_r u$) for any $i \in J$, then $\Delta_J |_l u$ (resp. $\Delta_J |_r u$).*

The element Δ_J is unique and is called the *fundamental element for* J. The fundamental element for I is simply denoted by Δ. The fundamental element Δ is the unique longest element of QFG_M^+.

We have the following table of the length of the fundamental elements.

	A_n	B_n	D_n	E_6	E_7	E_8	F_4	G_2	H_3	H_4	$I_2(p)$
$\|\Delta\|$	$n(n+1)/2$	n^2	$n(n-1)$	36	63	120	24	6	15	60	p

The next lemma is a key in this paper.

Lemma 2 (see Section 5 in [5], [8] and [16]). *Let M be a Coxeter matrix of finite type.*

(1) *For each square-free element $\mu \in \mathrm{QFG}_M^+$, there exists a positive word $\tilde{\mu} \in (\Sigma^+)^*$ such that $\pi(\tilde{\mu}) \in \mathrm{QFG}_M^+$ and $\pi(\tilde{\mu})\mu = \Delta$.*

(2) *For each word $w \in (\Sigma^+)^*$ (resp. $w \in (\Sigma^-)^*$), there exists a word $\hat{w} \in (\Sigma^+)^*$ (resp. $\hat{w} \in (\Sigma^-)^*$) such that $\Delta\pi(w) = \pi(\hat{w})\Delta$ and $|w| = |\hat{w}|$.*

Finally, we comment on the growth series for Artin monoids. Positive words in an equivalent class in G_M^+ have the same length. Hence, if a representative w of an element of G_M^+ is a positive word, w is geodesic. Fuchiwaki-Fujii-Saito-Tsuchioka [15] succeeded in constructing automata which recognize geodesic representatives of Artin monoids of finite type and concretely presented rational function expressions of growth series. Deligne [11], Xu [23], Albenque-Nadeau [1, 2], Krammer [17] and Saito [21] independently derived formulae for the rational function expressions of growth series for Artin monoids of finite type.

3. A necessary condition for representatives to be geodesic

In this section, we present the main theorem that describes a necessary condition such that representatives of Artin group elements are geodesic. This is a generalization of Proposition 4.3.(iii) in [18].

Let w be an element of Σ^*. Write w as

$$w = x^{(1)} \cdot Y^{(1)} \cdot x^{(2)} \cdot Y^{(2)} \cdot \cdots \cdot x^{(m)} \cdot Y^{(m)}, \tag{1}$$

where

$$x^{(a)} \in (\Sigma^+)^*, \quad Y^{(a)} \in (\Sigma^-)^*, \qquad (a \in \{1, \dots, m\})$$

and $x^{(1)}$ or $Y^{(m)}$ may be the empty word. The words $x^{(a)}$ and $Y^{(a)}$ can be decomposed as follows:

$$\begin{aligned} x^{(a)} &= x_1^{(a)} \cdot \cdots \cdot x_{k_a}^{(a)}, \\ Y^{(a)} &= Y_1^{(a)} \cdot \cdots \cdot Y_{K_a}^{(a)}, \end{aligned} \tag{2}$$

where

$$\pi(x_b^{(a)}) \in \mathrm{QFG}_M^+, \qquad (b \in \{1, \dots, k_a\}),$$
$$\pi(Y_b^{(a)}) \in \mathrm{QFG}_M^-, \qquad (b \in \{1, \dots, K_a\}).$$

We assume that

$$I_r(\pi(x_{k_a}^{(a)})) \cap I_l(\pi(Y_1^{(a)})) = \phi \qquad \text{and} \qquad I_r(\pi(Y_{K_a}^{(a)})) \cap I_l(\pi(x_1^{(a+1)})) = \phi.$$

Proposition 1. *Let M be a Coxeter matrix of finite type. Let g be an element of G_M and let $w \in \pi^{-1}(g)$. If w has a decomposition:*

$$w = x \cdot v \cdot Y$$

such that

$$x \in (\Sigma^+)^*, \qquad v \in \Sigma^*, \qquad Y \in (\Sigma^-)^*,$$
$$\pi(x) \in \mathrm{QFG}_M^+, \qquad \pi(Y) \in \mathrm{QFG}_M^-,$$

then there exists a representative $\hat{w} \in \pi^{-1}(g)$ with the following property:

$$^\exists \hat{Y} \in (\Sigma^-)^*, \qquad ^\exists \hat{v} \in \Sigma^*, \qquad ^\exists \hat{x} \in (\Sigma^+)^*$$

such that

$$\hat{w} = \hat{Y} \cdot \hat{v} \cdot \hat{x},$$
$$\pi(\hat{Y}) \in \mathrm{QFG}_M^-, \qquad \pi(\hat{x}) \in \mathrm{QFG}_M^+,$$
$$|x| + |Y| + |\hat{x}| + |\hat{Y}| = 2\|\Delta\|.$$

In particular, if $|x| + |Y| > \|\Delta\|$, then w is not a geodesic representative of g.

From Lemma 2, we have

Lemma 3. *Let M be a Coxeter matrix of finite type. Then, for each word $w \in \Sigma^*$, there exists a word $\hat{w} \in \Sigma^*$ such that $\Delta\pi(w) = \pi(\hat{w})\Delta$ and $|\hat{w}| = |w|$.*

Proof. Let us write w as in (1). Then, by (2) of Lemma 2, there exist words $\hat{x}^{(a)} \in (\Sigma^+)^*$ and $\hat{Y}^{(a)} \in (\Sigma^-)^*$ such that $\pi(\hat{x}^{(a)})\Delta = \Delta\pi(x^{(a)})$, $\pi(\hat{Y}^{(a)})\Delta =$

$\Delta\pi(Y^{(a)})$, $|\hat{x}^{(a)}| = |x^{(a)}|$ and $|\hat{Y}^{(a)}| = |Y^{(a)}|$. Set

$$\hat{w} := \hat{x}^{(1)}\hat{Y}^{(1)}\cdots\hat{x}^{(m)}\hat{Y}^{(m)}.$$

Then we have $\Delta\pi(w) = \pi(\hat{w})\Delta$ and $|\hat{w}| = |w|$. □

Proof (Proposition 1). By Lemma 3, we find a word $\hat{v} \in \Sigma^*$ such that $|\hat{v}| = |v|$ and $\Delta\pi(v) = \pi(\hat{v})\Delta$. By (1) of Lemma 2, there exists a word $x' \in (\Sigma^+)^*$ such that $\pi(x') \in \mathrm{QFG}_M^+$ and $\pi(x')\pi(x) = \Delta$. Set $\hat{Y} := (x')^{-1}$. Then we have $\pi(\hat{Y}) \in \mathrm{QFG}_M^-$ and $\pi(\hat{Y}) = \pi(x)\Delta^{-1}$. Applying (1) of Lemma 2 to the word $\pi(Y^{-1}) \in \mathrm{QFG}_M^+$, we find a positive word $\hat{x}$ with $\pi(\hat{x}) \in \mathrm{QFG}_M^+$ and $\pi(\hat{x})\pi(Y^{-1}) = \Delta$. Consequently,

$$\begin{aligned}\pi(xvY) &= \pi(x\cdot\Delta^{-1}\Delta\cdot vY) = \pi(x\Delta^{-1}\cdot\Delta v\cdot Y) = \pi(\hat{Y}\cdot\hat{v}\Delta\cdot Y)\\ &= \pi(\hat{Y}\hat{v}\cdot\Delta Y) = \pi(\hat{Y}\hat{v}\hat{x}),\end{aligned}$$

and

$$|x| + |\hat{Y}| + |Y| + |\hat{x}| = 2\|\Delta\|.$$

Thus, we have

$$|x| + |Y| > \|\Delta\| \quad \Rightarrow \quad |\hat{Y}| + |\hat{x}| < \|\Delta\|.$$

Hence, if $|x| + |Y| > \|\Delta\|$, we have $|w| > |\hat{w}|$. □

By Proposition 1, we can show the following theorem.

Theorem 3. *Let M be a Coxeter matrix of finite type. Let $g \in G_M$ and let $w \in \Sigma^*$ be a representative of g. Let w be written as in* (1) *and* (2)*. Then, if there exist $x_\beta^{(\alpha)}$ and $Y_\delta^{(\gamma)}$ such that $|x_\beta^{(\alpha)}| + |Y_\delta^{(\gamma)}| > \|\Delta\|$, the word w is not a geodesic representative of g.*

Proof. Case 1: $\alpha \le \gamma$. Consider the following part of the word w:

$$w' := x_\beta^{(\alpha)}\cdot v'\cdot Y_\delta^{(\gamma)}.$$

Then, by applying Proposition 1 to w', we have that w' is not geodesic. Thus, w is not geodesic neither.

Case 2: $\alpha \ge \gamma$. Reading words from right to left instead of reading from left to right, we can show that w is not geodesic as in Case 1. □

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