Hiroshima Math. J.
43 (2013), 107–127

Composition operators on the Bergman spaces of a minimal bounded homogeneous domain

Satoshi Yamaji

(Received July 28, 2011)
(Revised January 17, 2012)

Abstract. Using an integral formula on a homogeneous Siegel domain, we give a necessary and sufficient condition for composition operators on the weighted Bergman space of a minimal bounded homogeneous domain to be compact in terms of a boundary behavior of the Bergman kernel.

1. Introduction

Composition operators have been studied on various function spaces of a complex domain, for example, Hardy spaces, Bergman spaces and Bloch spaces. In particular, the operators on Bergman spaces have been analysed by making use of the Bergman kernel. Actually, estimates of the Bergman kernel enable us to characterize the boundedness and compactness of composition operators, as well as Toeplitz operators and Hankel operators, on the Bergman space of the unit disk (for example, see [15]). In this paper, we consider composition operators on weighted Bergman spaces of a bounded homogeneous domain.

In 2007, Zhu [16] considered composition operators on the weighted Bergman space of the unit ball. His results are extended to the case where the domain is the Harish-Chandra realization of an irreducible bounded symmetric domain by Lv and Hu [9]. In this paper, we generalize their works further to weighted Bergman spaces of a minimal bounded homogeneous domain (for minimal domains, see [8], [10]). Indeed, the unit ball, the polydisk and a bounded symmetric domain in its Harish-Chandra realization are all minimal domains.

Let $\mathscr{U}$ be a minimal bounded homogeneous domain in $\mathbf{C}^d$, dV the Lebesgue measure on $\mathbf{C}^d$ and $\mathscr{O}(\mathscr{U})$ the space of all holomorphic functions on $\mathscr{U}$. The Bergman kernel $K_{\mathscr{U}} : \mathscr{U} \times \mathscr{U} \to \mathbf{C}$ is the reproducing kernel of the Bergman space $L_a^2(\mathscr{U}, dV) := L^2(\mathscr{U}, dV) \cap \mathscr{O}(\mathscr{U})$. For $\beta \in \mathbf{R}$, let dV_β denote

2010 *Mathematics Subject Classification*. Primary 47B33; Secondary 47B35, 32A25.
Key words and phrases. composition operator, Bergman space, bounded homogeneous domain, minimal domain, Carleson measure.

the measure on $\mathscr{U}$ given by $dV_\beta(z) := K_\mathscr{U}(z,z)^{-\beta}dV(z)$. We consider the weighted Bergman space $L_a^p(\mathscr{U}, dV_\beta) := L^p(\mathscr{U}, dV_\beta) \cap \mathcal{O}(\mathscr{U})$ for $0 < p < \infty$. It is known that there exists a constant $\varepsilon_{\min}$ such that $L_a^p(\mathscr{U}, dV_\beta)$ is non-trivial for all p if $\beta > \varepsilon_{\min}$. Throughout this paper, we always assume this non-trivializing condition. Every holomorphic map $\varphi : \mathscr{U} \to \mathscr{U}$ defines a composition operator C_φ on $\mathcal{O}(\mathscr{U})$, in particular on $L_a^p(\mathscr{U}, dV_\beta)$, by $C_\varphi f := f \circ \varphi$. Using Zhu's technique (see [16]) together with an integral formula established in Lemma 5.2, we obtain the following theorem, which is the main theorem of this paper.

THEOREM A (Theorem 6.1). *Assume that C_φ is bounded on $L_a^q(\mathscr{U}, dV_{\beta_0})$ for some $q > 0$ and $\beta_0 > \varepsilon_{\min}$. Then C_φ is compact on $L_a^p(\mathscr{U}, dV_\beta)$ for any $p > 0$ and $\beta > \beta_0 + \varepsilon_\mathscr{U}$ if and only if*

$$\lim_{z \to \partial\mathscr{U}} \frac{K_\mathscr{U}(\varphi(z), \varphi(z))}{K_\mathscr{U}(z,z)} = 0.$$

Here, $\varepsilon_\mathscr{U}$ is the non-negative constant given by (5.1). Similarly to [9] and [16], the assumption that C_φ is a bounded operator on $L_a^p(\mathscr{U}, dV_{\beta_0})$ for some $\beta_0 > \varepsilon_{\min}$ is needed only for the "if" part of Theorem A. The constant $\varepsilon_\mathscr{U}$ is equal to 0 for the case of unit ball, and coincides with the constant $\frac{a(r-1)}{N}$ that appears in the paper [9] for the case of the Harish-Chandra realization of an irreducible bounded symmetric domain (see section 7).

Zhu proved Theorem A for the case of the unit ball. His method is to apply Schur's theorem, in which a key is to find a positive function satisfying a certain inequality. In our case, Lemma 5.2 will replace Forelli-Rudin inequality in order to get such a positive function. In fact, let Φ be a biholomorphic map from $\mathscr{U}$ onto a homogeneous Siegel domain $\mathscr{D}$ (see [12]), and $J(\Phi, z')$ the complex Jacobi matrix of Φ at $z' \in \mathscr{U}$. Then, we show in Lemma 5.2 that the integral

$$\int_\mathscr{U} |K_\mathscr{U}(z,z')|^{1+\alpha} |\det J(\Phi, z')|^{1+2\beta-\alpha} dV_\beta(z')$$

is evaluated, under a convergence condition, as $K_\mathscr{U}(z,z)^{\alpha-\beta} |\det J(\Phi, z)|^{1+2\beta-\alpha}$ up to a positive constant not depending on z.

Before proving Theorem A, we show that the boundedness of C_φ on $L_a^p(\mathscr{U}, dV_\beta)$ is described in terms of Carleson measures. It is easy to see that C_φ is a bounded operator on $L_a^p(\mathscr{U}, dV_\beta)$ if and only if the pull-back measure $d\mu_{\varphi,\beta}$ of dV_β induced by φ is a Carleson measure for $L_a^p(\mathscr{U}, dV_\beta)$ (see section 4.1). Thanks to properties of Carleson measures, we obtain the following theorem.

Theorem B (Theorems 4.3 and 4.5). *If C_φ is a bounded (resp. compact) operator on $L^q_a(\mathcal{U}, dV_{\beta_0})$ for some $q > 0$ and $\beta_0 > \varepsilon_{\min}$, then C_φ is a bounded (resp. compact) operator on $L^p_a(\mathcal{U}, dV_\beta)$ for any $p > 0$ and $\beta \geq \beta_0$.*

By Theorem B, the assumption of Theorem A implies that C_φ is bounded on $L^q_a(\mathcal{U}, dV_{\beta_0})$ for any $q > 0$ and $\beta \geq \beta_0$. Therefore, we may use the boundedness of C_φ on both $L^2_a(\mathcal{U}, dV_\beta)$ and $L^2_a(\mathcal{U}, dV_{\beta_0})$ in section 6.

Let us explain the organization of this paper. In section 2, we review properties of the weighted Bergman space of a minimal bounded homogeneous domain and composition operators on the space. The estimate of the Bergman kernel given in Theorem 2.1 plays an important role in this section. In section 3, we get a characterization of Carleson measures and vanishing Carleson measures for the weighted Bergman space of a minimal bounded homogeneous domain (Theorems 3.2 and 3.3). By using these theorems, we prove properties of the boundedness and compactness of C_φ in section 4 (Theorems 4.3 and 4.5). In section 5, we show a key equality (Lemma 5.2). This equality enables us to pursue Zhu's method developed in [16], and we characterize the compactness of the composition operator (Theorem A) in section 6. In section 7, Theorem A is applied to the cases where $\mathcal{U}$ is the unit ball, the Harish-Chandra realization of an irreducible bounded symmetric domain, the polydisk and the representative domain of the tube domain over Vinberg cone. The last case is an example of non-symmetric bounded homogeneous domain. These domains are minimal domains with center 0, and we see that Theorem A actually covers the previously known first two and the polydisk cases.

2. Preliminaries

2.1. Weighted Bergman spaces of a minimal bounded homogeneous domain. Let D be a bounded domain in $\mathbf{C}^d$. We say that D is a minimal domain with center $t \in D$ if the following condition is satisfied: for every biholomorphism $\psi : D \to D'$ with $\det J(\psi, t) = 1$, we have

$$\mathrm{Vol}(D') \geq \mathrm{Vol}(D).$$

We know that D is a minimal domain with center t if and only if

$$K_D(z, t) = \frac{1}{\mathrm{Vol}(D)}$$

for any $z \in D$ (see [10, Theorem 3.1]). For example, the unit disk $\mathbf{D}$ and the unit ball $\mathbf{B}^d$ are minimal domains with center 0.

We fix a minimal bounded homogeneous domain $\mathcal{U}$ with center t. We denote by $K^{(\beta)}_{\mathcal{U}}$ the reproducing kernel of $L^2_a(\mathcal{U}, dV_\beta)$. It is known that

$K_{\mathcal{U}}^{(\beta)}(z,w) = C_\beta K_{\mathcal{U}}(z,w)^{1+\beta}$ for some positive constant C_β. For $z \in \mathcal{U}$, we denote by $k_z^{(\beta)}$ the normalized reproducing kernel of $L_a^2(\mathcal{U}, dV_\beta)$, that is,

$$k_z^{(\beta)}(w) := \frac{K_{\mathcal{U}}^{(\beta)}(w,z)}{K_{\mathcal{U}}^{(\beta)}(z,z)^{1/2}} = \sqrt{C_\beta}\left(\frac{K_{\mathcal{U}}(w,z)}{K_{\mathcal{U}}(z,z)^{1/2}}\right)^{1+\beta}. \tag{2.1}$$

For any Borel set E in $\mathcal{U}$, we define

$$\mathrm{Vol}_\beta(E) := \int_E dV_\beta(w).$$

Let $d_{\mathcal{U}}(\cdot,\cdot)$ be the Bergman distance on $\mathcal{U}$. For any $z \in \mathcal{U}$ and $r > 0$, let

$$B(z,r) := \{w \in \mathcal{U} \mid d_{\mathcal{U}}(z,w) \le r\}$$

be the Bergman metric disk with center z and radius r.

In [8], we proved the following theorem.

Theorem 2.1 ([8, Theorem A]). *For any $\rho > 0$, there exists $C_\rho > 0$ such that*

$$C_\rho^{-1} \le \left|\frac{K_{\mathcal{U}}(z,a)}{K_{\mathcal{U}}(a,a)}\right| \le C_\rho$$

for all $z, a \in \mathcal{U}$ with $d_{\mathcal{U}}(z,a) \le \rho$.

From Theorem 2.1, we see that $K_{\mathcal{U}}(\cdot,w)$ is a bounded function on $\mathcal{U}$ for each $w \in \mathcal{U}$ (see [8, Proposition 6.1]). Since the span of $\{K_{\mathcal{U}}^{(\beta)}(\cdot,w) \mid w \in \mathcal{U}\}$ is dense in $L_a^2(\mathcal{U}, dV_\beta)$, so is the space $H^\infty(\mathcal{U})$ of all bounded holomorphic functions on $\mathcal{U}$.

Moreover, Theorem 2.1 gives useful estimates. Indeed, we first deduce

$$C_\rho^{-2} \le \frac{K_{\mathcal{U}}(z,z)}{K_{\mathcal{U}}(a,a)} \le C_\rho^2 \tag{2.2}$$

for all $z, a \in \mathcal{U}$ with $d_{\mathcal{U}}(z,a) \le \rho$. On the other hand, we have

$$C^{-1}K_{\mathcal{U}}(a,a)^{-1} \le \left|\frac{K_{\mathcal{U}}(z,a)}{K_{\mathcal{U}}(a,a)}\right|^2 \mathrm{Vol}(B(a,\rho)) \le CK_{\mathcal{U}}(a,a)^{-1}$$

by [13, Lemma 3.3], so that Theorem 2.1 yields

$$C^{-1}K_{\mathcal{U}}(a,a)^{-1} \le \mathrm{Vol}(B(a,\rho)) \le CK_{\mathcal{U}}(a,a)^{-1}. \tag{2.3}$$

Lemma 2.2. *There exists a positive constant C such that*

$$C^{-1}K_{\mathscr{U}}(a,a)^{-(1+\beta)} \le \mathrm{Vol}_\beta(B(a,\rho)) \le CK_{\mathscr{U}}(a,a)^{-(1+\beta)} \tag{2.4}$$

for all $a \in \mathscr{U}$.

Proof. Since

$$\mathrm{Vol}_\beta(B(a,\rho)) = \int_{B(a,\rho)} K_{\mathscr{U}}(w,w)^{-\beta} dV(w),$$

we have

$$C^{-1}K_{\mathscr{U}}(a,a)^{-\beta}\,\mathrm{Vol}(B(a,\rho)) \le \mathrm{Vol}_\beta(B(a,\rho)) \le CK_{\mathscr{U}}(a,a)^{-\beta}\,\mathrm{Vol}(B(a,\rho))$$

by (2.2). This together with (2.3) yields (2.4). □

We now collect some estimates needed later. These are generalizations to minimal bounded homogeneous domains of Lemmas 1, 2 and 5 respectively in [14] stated for bounded symmetric domains. First, by (2.1), Lemma 2.2 and Theorem 2.1, we have the following lemma.

Lemma 2.3. *There exists a positive constant C such that*

$$C^{-1} \le |k_a^{(\beta)}(z)|^2\,\mathrm{Vol}_\beta(B(a,\rho)) \le C$$

for all $a \in \mathscr{U}$ and $z \in B(a,\rho)$.

Lemma 2.2 and (2.3) yield the following;

Lemma 2.4. *There exists a positive constant C such that*

$$C^{-1}\,\mathrm{Vol}_\beta(B(a,\rho)) \le \mathrm{Vol}_\beta(B(z,\rho)) \le C\,\mathrm{Vol}_\beta(B(a,\rho))$$

for all $a \in \mathscr{U}$ and $z \in B(a,\rho)$.

Lemma 2.5. *There exists a positive constant C such that*

$$|f(z)|^p \le \frac{C}{\mathrm{Vol}_\beta(B(z,\rho))}\int_{B(z,\rho)} |f(w)|^p dV_\beta(w) \tag{2.5}$$

for all $f \in \mathcal{O}(\mathscr{U})$, $p > 0$ and $z \in \mathscr{U}$.

Proof. By [13, Lemma 3.5], there exists a constant $C > 0$ such that

$$\begin{aligned} |f(z)|^p &\le \frac{C}{\mathrm{Vol}(B(z,\rho))}\int_{B(z,\rho)} |f(w)|^p dV(w) \\ &\le \frac{CK_{\mathscr{U}}(z,z)^\beta}{\mathrm{Vol}(B(z,\rho))}\int_{B(z,\rho)} |f(w)|^p dV_\beta(w), \end{aligned}$$

where the second inequality follows from (2.2). Since

$$\frac{K_{\mathscr{U}}(z,z)^{\beta}}{\mathrm{Vol}(B(z,\rho))} \le \frac{C}{\mathrm{Vol}_{\beta}(B(z,\rho))},$$

we obtain (2.5). □

2.2. Composition operators. In this section, we summarize some properties of the composition operator. Our reference is the book [15] and the paper [16]. Let φ be a holomorphic map from $\mathscr{U}$ to $\mathscr{U}$. We define a linear operator C_{φ} on $\mathcal{O}(\mathscr{U})$ by $C_{\varphi}f := f \circ \varphi$ $(f \in \mathcal{O}(\mathscr{U}))$. It is known that C_{φ} is always bounded on $L_a^p(\mathscr{U}, dV_{\beta})$ if $\mathscr{U}$ is the unit disk $\mathbf{D}$. However, it is not necessarily bounded for general domains (see for example [16]).

Let $\mu_{\varphi,\beta}$ be the pull-back measure of dV_{β} induced by φ, that is,

$$\mu_{\varphi,\beta}(E) := \mathrm{Vol}_{\beta}(\varphi^{-1}(E))$$

for any Borel set E in $\mathscr{U}$. Then, C_{φ} is a bounded operator on $L_a^p(\mathscr{U}, dV_{\beta})$ if and only if there exists a constant $C > 0$ such that the estimate

$$\int_{\mathscr{U}} |f(w)|^p d\mu_{\varphi,\beta}(w) \le C \int_{\mathscr{U}} |f(w)|^p dV_{\beta}(w) \tag{2.6}$$

holds for any $f \in L_a^p(\mathscr{U}, dV_{\beta})$.

Assume that C_{φ} is a bounded operator on $L_a^2(\mathscr{U}, dV_{\beta})$. Then, we have

$$C_{\varphi}^* f(w) = \langle C_{\varphi}^* f, K_w^{(\beta)} \rangle_{L^2(dV_{\beta})} = \langle f, C_{\varphi} K_w^{(\beta)} \rangle_{L^2(dV_{\beta})} \tag{2.7}$$

for any $f \in L_a^2(\mathscr{U}, dV_{\beta})$. Therefore, we have

$$\begin{aligned} C_{\varphi} C_{\varphi}^* f(w) &= \langle f, C_{\varphi} K_{\varphi(w)}^{(\beta)} \rangle_{L^2(dV_{\beta})} \\ &= \int_{\mathscr{U}} K_{\mathscr{U}}^{(\beta)}(\varphi(w), \varphi(u)) f(u) dV_{\beta}(u). \end{aligned} \tag{2.8}$$

We use (2.8) to characterize the compactness of C_{φ} in Theorem 6.1 below. Moreover, we have

$$\begin{aligned} C_{\varphi}^* C_{\varphi} f(w) &= \langle C_{\varphi} f, C_{\varphi} K_w^{(\beta)} \rangle_{L^2(dV_{\beta})} \\ &= \int_{\mathscr{U}} f(\varphi(u)) K_{\mathscr{U}}^{(\beta)}(w, \varphi(u)) dV_{\beta}(u) \\ &= \int_{\mathscr{U}} K_{\mathscr{U}}^{(\beta)}(w, u) f(u) d\mu_{\varphi,\beta}(u) \end{aligned}$$

by (2.7). The last integral represents the Toeplitz operator $T_{\mu_{\varphi,\beta}}$ with symbol $\mu_{\varphi,\beta}$, so that we obtain $C_\varphi^* C_\varphi = T_{\mu_{\varphi,\beta}}$. The boundedness of Toeplitz operators are discussed in [13], [15, section 7] and [16].

3. Carleson measures and vanishing Carleson measures

3.1. The Berezin symbol and the averaging function. For a Borel measure μ on $\mathscr{U}$, we define a function $\tilde{\mu}$ on $\mathscr{U}$ by

$$\tilde{\mu}(z) := \int_{\mathscr{U}} |k_z^{(\beta)}(w)|^2 d\mu(w).$$

The function $\tilde{\mu}$ is called the Berezin symbol of the measure μ. Fixing $\rho > 0$ once and for all, we set

$$\hat{\mu}(z) := \frac{\mu(B(z,\rho))}{\mathrm{Vol}_\beta(B(z,\rho))} \qquad (z \in \mathscr{U}).$$

We call $\hat{\mu}$ the averaging function of the Borel measure μ. The dependence of $\hat{\mu}$ on ρ will not be considered in this paper.

Lemma 3.1. *There exists a positive constant C such that*

$$\int_{\mathscr{U}} |f(z)|^p d\mu(z) \le C \int_{\mathscr{U}} \hat{\mu}(z) |f(z)|^p dV_\beta(z)$$

for any $p > 0$ and $f \in \mathcal{O}(\mathscr{U})$.

Proof. By Lemma 2.5, we have

$$\int_{\mathscr{U}} |f(z)|^p d\mu(z) \le \int_{\mathscr{U}} \left(\frac{C}{\mathrm{Vol}_\beta(B(z,\rho))} \int_{B(z,\rho)} |f(w)|^p dV_\beta(w) \right) d\mu(z) \tag{3.1}$$

for any $p > 0$ and $f \in \mathcal{O}(\mathscr{U})$. The right hand side of (3.1) is equal to

$$C \int_{\mathscr{U}} \int_{\mathscr{U}} \frac{\chi_{B(z,\rho)}(w)}{\mathrm{Vol}_\beta(B(z,\rho))} |f(w)|^p dV_\beta(w) d\mu(z). \tag{3.2}$$

Here, we interchange the order of the integrations in (3.2) by using Fubini's theorem. Then Lemma 2.4 shows that (3.2) is less than or equal to

$$C \int_{\mathscr{U}} \frac{\mu(B(w,\rho))}{\mathrm{Vol}_\beta(B(w,\rho))} |f(w)|^p dV_\beta(w).$$

Now the proof of Lemma 3.1 is complete. $\square$

3.2. Carleson measures. Let μ be a positive Borel measure on $\mathscr{U}$. We say that μ is a Carleson measure for $L_a^p(\mathscr{U}, dV_\beta)$ if there exists a constant $M > 0$ such that

$$\int_{\mathscr{U}} |f(z)|^p d\mu(z) \leq M \int_{\mathscr{U}} |f(z)|^p dV_\beta(z)$$

for all $f \in L_a^p(\mathscr{U}, dV_\beta)$. It is easy to see that μ is a Carleson measure for $L_a^p(\mathscr{U}, dV_\beta)$ if and only if $L_a^p(\mathscr{U}, dV_\beta) \subset L_a^p(\mathscr{U}, d\mu)$ and the inclusion map

$$i_p : L_a^p(\mathscr{U}, dV_\beta) \to L_a^p(\mathscr{U}, d\mu)$$

is bounded. The following theorem is a generalization of [14, Theorem 7] to minimal bounded homogeneous domains.

THEOREM 3.2. *Let μ be a positive Borel measure on $\mathscr{U}$. Then, the following conditions are all equivalent.*
(i) *μ is a Carleson measure for $L_a^p(\mathscr{U}, dV_\beta)$.*
(ii) *The Berezin symbol $\tilde{\mu}$ of μ is a bounded function on $\mathscr{U}$.*
(iii) *The averaging function $\hat{\mu}$ of μ is a bounded function on $\mathscr{U}$.*

PROOF. First, we prove (i) $\Rightarrow$ (ii). It is known that the Bergman kernel of a bounded homogeneous domain is zero-free (see [6] or [7, Proposition 3.1]). Therefore, we can define the single-valued holomorphic function function $K^{(\beta)}(\cdot, z)^{2/p}$ on the simply connected domain $\mathscr{U}$. Since $k_z^{(\beta)}(\cdot)^{2/p} \in L_a^p(\mathscr{U}, dV_\beta)$ for any $z \in \mathscr{U}$ and since μ is a Carleson measure for $L_a^p(\mathscr{U}, dV_\beta)$, we have

$$\int_{\mathscr{U}} |k_z^{(\beta)}(w)|^2 d\mu(w) \leq M \int_{\mathscr{U}} |k_z^{(\beta)}(w)|^2 dV_\beta(w) = M.$$

Therefore, $\tilde{\mu}$ is bounded. Next, we prove (ii) $\Rightarrow$ (iii). Take any $w \in \mathscr{U}$. By Lemma 2.3, there exists a positive constant C such that

$$C^{-1} \leq |k_z^{(\beta)}(w)|^2 \operatorname{Vol}_\beta(B(z, \rho)) \tag{3.3}$$

holds for any $w \in B(z, \rho)$. Integration of (3.3) on $B(z, \rho)$ by $d\mu$ gives

$$\frac{\mu(B(z, \rho))}{\operatorname{Vol}_\beta(B(z, \rho))} \leq C \int_{B(z, \rho)} |k_z^{(\beta)}(w)|^2 d\mu(w). \tag{3.4}$$

Therefore, we obtain

$$\hat{\mu}(z) \leq C \tilde{\mu}(z), \tag{3.5}$$

whence (ii) $\Rightarrow$ (iii) follows. The implication (iii) $\Rightarrow$ (i) holds by Lemma 3.1. $\square$

Similarly to [13, Theorem 4.1], we can prove that these conditions are equivalent to the following condition: (iv) The Toeplitz operator T_μ is bounded on $L_a^2(\mathscr{U}, dV_\beta)$.

3.3. Vanishing Carleson measures. Suppose that μ is a Carleson measure for $L_a^p(\mathscr{U}, dV_\beta)$. We say that μ is a vanishing Carleson measure for $L_a^p(\mathscr{U}, dV_\beta)$ if we have

$$\lim_{k\to\infty} \int_{\mathscr{U}} |f_k(w)|^p d\mu(w) = 0$$

whenever $\{f_k\}$ is a bounded sequence in $L_a^p(\mathscr{U}, dV_\beta)$ that converges to 0 uniformly on each compact subset of $\mathscr{U}$.

The following theorem generalizes [14, Theorem 11] to minimal bounded homogeneous domains.

Theorem 3.3. *Let μ be a finite positive Borel measure on $\mathscr{U}$. Then, the following conditions are all equivalent.*

(i) *μ is a vanishing Carleson measure for $L_a^p(\mathscr{U}, dV_\beta)$.*

(ii) *$\tilde{\mu}(z) \to 0$ as $z \to \partial\mathscr{U}$.*

(iii) *$\hat{\mu}(z) \to 0$ as $z \to \partial\mathscr{U}$.*

Proof. First, we prove (i) $\Rightarrow$ (ii). Proceeding in the same way as in [4, Lemma 1] and [4, Lemma 5], we see that $\{k_z^{(\beta)}\}$ converges to 0 uniformly on compact subsets of $\mathscr{U}$ as $z \to \partial\mathscr{U}$. Therefore, $\{k_z^{(\beta)}(\cdot)^{2/p}\}$ is a bounded sequence in $L_a^p(\mathscr{U}, dV_\beta)$ that converges to 0 uniformly on each compact subset of $\mathscr{U}$. Hence, (ii) holds. The implication (ii) $\Rightarrow$ (iii) follows from (3.5). Finally, we prove (iii) $\Rightarrow$ (i). Take any bounded sequence $\{f_n\}$ in $L_a^p(\mathscr{U}, dV_\beta)$ that converges to 0 uniformly on each compact subset of $\mathscr{U}$. Take any $\varepsilon > 0$. Then, there exists a constant $\delta > 0$ such that

$$\sup_{\operatorname{dist}(z, \partial\mathscr{U}) < \delta} |\hat{\mu}(z)| < \varepsilon$$

by (iii). Let $\mathscr{U}_\delta := \{z \in \mathscr{U} \mid \operatorname{dist}(z, \partial\mathscr{U}) < \delta\}$. Since $\mathscr{U} \backslash \mathscr{U}_\delta$ is a compact set, there exists an integer N such that

$$\sup_{z \in \mathscr{U}\backslash\mathscr{U}_\delta} |f_n(z)|^p < \varepsilon$$

for any $n \geq N$. Now, Lemma 3.1 yields

$$\begin{aligned} \int_{\mathscr{U}} |f_n(z)|^p d\mu(z) &\leq C \int_{\mathscr{U}} \hat{\mu}(z) |f_n(z)|^p dV_\beta(z) \\ &= C\left(\int_{\mathscr{U}\backslash\mathscr{U}_\delta} \hat{\mu}(z)|f_n(z)|^p dV_\beta(z) + \int_{\mathscr{U}_\delta} \hat{\mu}(z)|f_n(z)|^p dV_\beta(z) \right). \end{aligned} \tag{3.6}$$

Since the function $\hat{\mu}$ is continuous on the compact set $\mathscr{U}\backslash\mathscr{U}_\delta$, there exists a constant $M_\delta > 0$ such that

$$\sup_{z\in\mathscr{U}\backslash\mathscr{U}_\delta} \hat{\mu}(z) \le M_\delta.$$

Therefore, the first term of (3.6) is bounded by $CM_\delta\varepsilon$ if $n \ge N$. On the other hand, since $\{f_n\}$ is a bounded sequence in $L_a^p(\mathscr{U}, dV_\beta)$, there exists a constant $M > 0$ such that

$$\int_{\mathscr{U}} |f_n(z)|^p dV_\beta(z) \le M$$

for all $n \in \mathbf{N}$. Therefore, the second term of (3.6) is less than or equal to $CM\varepsilon$. Hence, we obtain

$$\int_{\mathscr{U}} |f_n(z)|^p d\mu(z) \le C(M + M_\delta)\varepsilon$$

for any $n \ge N$. Clearly, this shows that μ is a vanishing Carleson measure for $L_a^p(\mathscr{U}, dV_\beta)$. □

We can show that these conditions are also equivalent to the following condition (cf. [13, Theorem 5.1]): (iv) The Toeplitz operator T_μ is compact on $L_a^2(\mathscr{U}, dV_\beta)$.

4. Relation between Carleson measures and composition operators

4.1. Criterion of boundedness. From (2.6), we see that C_φ is a bounded operator on $L_a^p(\mathscr{U}, dV_\beta)$ if and only if the pull-back measure $\mu_{\varphi,\beta}$ is a Carleson measure for $L_a^p(\mathscr{U}, dV_\beta)$. By Theorem 3.2, the property of being a Carleson measure is independent of p. Hence, the boundedness of C_φ on $L_a^p(\mathscr{U}, dV_\beta)$ is also independent of p. We gather here boundedness conditions of C_φ on $L_a^p(\mathscr{U}, dV_\beta)$ as follows.

Lemma 4.1. *Let $\beta > \varepsilon_{\min}$. Then, the following conditions are all equivalent.*

(i) *C_φ is a bounded operator on $L_a^p(\mathscr{U}, dV_\beta)$.*
(ii) *The pull-back measure $\mu_{\varphi,\beta}$ is a Carleson measure for $L_a^p(\mathscr{U}, dV_\beta)$.*
(iii) *$\widetilde{\mu_{\varphi,\beta}}$ is a bounded function on $\mathscr{U}$.*
(iv) *$\widehat{\mu_{\varphi,\beta}}$ is a bounded function on $\mathscr{U}$.*
(v) *The function*

$$F_{\rho,\beta}(z) := \int_{B(z,\rho)} |k_z^{(\beta)}(w)|^2 d\mu_{\varphi,\beta}(w)$$

is bounded on $\mathscr{U}$.

Proof. For (v), we just note: (iii) $\Rightarrow$ (v) is trivial and (v) $\Rightarrow$ (iv) follows from (3.4). $\square$

If C_φ is bounded, we have the following estimate.

Lemma 4.2. *Assume that C_φ is bounded on $L_a^p(\mathcal{U}, dV_\beta)$ for some $p > 0$ and $\beta > \beta_0$. Then there exists a positive constant C such that*

$$K_{\mathcal{U}}(\varphi(z), \varphi(z)) \leq C K_{\mathcal{U}}(z, z)$$

for any $z \in \mathcal{U}$.

Proof. By Lemma 4.1, it is enough to consider $p = 2$. By (2.7), we have

$$C_\varphi^* k_z^{(\beta)}(w) = \langle k_z^{(\beta)}, C_\varphi K_w^{(\beta)} \rangle_{L^2(dV_\beta)} = K_{\mathcal{U}}^{(\beta)}(z,z)^{-1/2} \overline{\langle C_\varphi K_w^{(\beta)}, K_z^{(\beta)} \rangle}_{L^2(dV_\beta)}$$

$$= K_{\mathcal{U}}^{(\beta)}(z,z)^{-1/2} \overline{C_\varphi K_w^{(\beta)}(z)} = \frac{K_{\mathcal{U}}^{(\beta)}(w, \varphi(z))}{K_{\mathcal{U}}^{(\beta)}(z,z)^{1/2}}.$$

Therefore, we get

$$\|C_\varphi^* k_z^{(\beta)}\|_{L^2(dV_\beta)}^2 = \frac{K_{\mathcal{U}}^{(\beta)}(\varphi(z), \varphi(z))}{K_{\mathcal{U}}^{(\beta)}(z,z)} = \left(\frac{K_{\mathcal{U}}(\varphi(z), \varphi(z))}{K_{\mathcal{U}}(z,z)}\right)^{1+\beta}. \tag{4.1}$$

Since C_φ is a bounded operator on $L_a^2(\mathcal{U}, dV_\beta)$ and $\|k_z^{(\beta)}\|_{L^2(dV_\beta)} = 1$, the left hand side of (4.1) is bounded by a positive constant C. $\square$

Theorem 4.3. *If C_φ is a bounded operator on $L_a^q(\mathcal{U}, dV_{\beta_0})$ for some $q > 0$ and $\beta_0 > \varepsilon_{\min}$, then C_φ is a bounded operator on $L_a^p(\mathcal{U}, dV_\beta)$ for any $p > 0$ and $\beta \geq \beta_0$.*

Proof. The boundedness of C_φ on $L_a^q(\mathcal{U}, dV_{\beta_0})$ (resp. $L_a^p(\mathcal{U}, dV_\beta)$) is equivalent to the boundedness of $\widetilde{\mu_{\varphi,\beta_0}}$ (resp. $F_{\rho,\beta}$) by Lemma 4.1. Therefore, it is sufficient to prove

$$\widetilde{\mu_{\varphi,\beta_0}}(z) \geq C F_{\rho,\beta}(z). \tag{4.2}$$

Since $K_{\mathcal{U}}(\varphi(w), \varphi(w)) \leq C K_{\mathcal{U}}(w,w)$ by Lemma 4.2, we have

$$dV_{\beta_0}(w) = K_{\mathcal{U}}(w,w)^{\beta-\beta_0} dV_\beta(w) \geq C K_{\mathcal{U}}(\varphi(w), \varphi(w))^{\beta-\beta_0} dV_\beta(w).$$

Hence, we obtain

$$\begin{aligned}
&\widetilde{\mu_{\varphi,\beta_0}}(z) \\
&\quad = K_{\mathcal{U}}(z,z)^{-(1+\beta_0)} \int_{\mathcal{U}} |K_{\mathcal{U}}(z, \varphi(w))|^{2(1+\beta_0)} dV_{\beta_0}(w) \\
&\quad \geq C K_{\mathcal{U}}(z,z)^{-(1+\beta_0)} \int_{\mathcal{U}} |K_{\mathcal{U}}(z, \varphi(w))|^{2(1+\beta_0)} K_{\mathcal{U}}(\varphi(w), \varphi(w))^{\beta-\beta_0} dV_\beta(w). \qquad (4.3)
\end{aligned}$$

By the definition of the pull-back measure, the last term of (4.3) is rewritten and estimated as

$$CK_{\mathscr{U}}(z,z)^{-(1+\beta)}\int_{\mathscr{U}}|K_{\mathscr{U}}(z,w)|^{2(1+\beta)}\left(\frac{K_{\mathscr{U}}(w,w)K_{\mathscr{U}}(z,z)}{|K_{\mathscr{U}}(z,w)|^2}\right)^{\beta-\beta_0}d\mu_{\varphi,\beta}(w)$$

$$\geq CK_{\mathscr{U}}(z,z)^{-(1+\beta)}\int_{B(z,\rho)}|K_{\mathscr{U}}(z,w)|^{2(1+\beta)}$$

$$\times\left(\frac{K_{\mathscr{U}}(w,w)K_{\mathscr{U}}(z,z)}{|K_{\mathscr{U}}(z,w)|^2}\right)^{\beta-\beta_0}d\mu_{\varphi,\beta}(w). \tag{4.4}$$

Since $w \in B(z,\rho)$, Theorem 2.1 gives

$$\frac{K_{\mathscr{U}}(w,w)K_{\mathscr{U}}(z,z)}{|K_{\mathscr{U}}(z,w)|^2}=\left|\frac{K_{\mathscr{U}}(w,w)}{K_{\mathscr{U}}(z,w)}\frac{K_{\mathscr{U}}(z,z)}{K_{\mathscr{U}}(z,w)}\right|\geq C_\rho^{-2},$$

which makes the right hand side of (4.4) greater than or equal to

$$CK_{\mathscr{U}}(z,z)^{-(1+\beta)}\int_{B(z,\rho)}|K_{\mathscr{U}}(z,w)|^{2(1+\beta)}d\mu_{\varphi,\beta}(w)=CF_{\rho,\beta}(z).$$

Hence, (4.2) holds. □

4.2. Criterion of compactness. We say that C_φ is compact on $L_a^p(\mathscr{U},dV_\beta)$ if the image under C_φ of any bounded subset of $L_a^p(\mathscr{U},dV_\beta)$ is a relatively compact subset. We can show that C_φ is compact on $L_a^p(\mathscr{U},dV_\beta)$ if and only if

$$\lim_{k\to\infty}\int_{\mathscr{U}}|C_\varphi f_k(w)|^p dV_\beta(w)=0 \tag{4.5}$$

holds whenever $\{f_k\}$ is a bounded sequence in $L_a^p(\mathscr{U},dV_\beta)$ that converges to 0 uniformly on each compact subset of $\mathscr{U}$ (for the case $\mathscr{U}=\mathbf{D}$, see [3, Proposition 3.1]). Since (4.5) is equivalent to

$$\lim_{k\to\infty}\int_{\mathscr{U}}|f_k(w)|^p d\mu_{\varphi,\beta}(w)=0,$$

C_φ is a compact operator on $L_a^p(\mathscr{U},dV_\beta)$ if and only if $\mu_{\varphi,\beta}$ is a vanishing Carleson measure for $L_a^p(\mathscr{U},dV_\beta)$. This observation together with Theorem 3.3 gives the following compactness conditions of C_φ on $L_a^p(\mathscr{U},dV_\beta)$.

Lemma 4.4. *Let $\beta > \varepsilon_{\min}$. Then, the following conditions are all equivalent.*

(i) *C_φ is a compact operator on $L_a^p(\mathscr{U},dV_\beta)$.*

(ii) *$\mu_{\varphi,\beta}$ is a vanishing Carleson measure for $L_a^p(\mathscr{U},dV_\beta)$.*

(iii) $\lim_{z \to \partial \mathscr{U}} \widetilde{\mu_{\varphi,\beta}}(z) = 0.$

(iv) $\lim_{z \to \partial \mathscr{U}} \widehat{\mu_{\varphi,\beta}}(z) = 0.$

(v) $\lim_{z \to \partial \mathscr{U}} F_{\rho,\beta}(z) = 0.$

THEOREM 4.5. *If C_φ is a compact operator on $L_a^q(\mathscr{U}, dV_{\beta_0})$ for some $q > 0$ and $\beta_0 > \varepsilon_{\min}$, then C_φ is a compact operator on $L_a^p(\mathscr{U}, dV_\beta)$ for any $p > 0$ and $\beta \geq \beta_0$.*

PROOF. By Lemma 4.4, it is enough to prove

$$\lim_{z \to \partial \mathscr{U}} \widetilde{\mu_{\varphi,\beta_0}}(z) = 0 \Rightarrow \lim_{z \to \partial \mathscr{U}} F_{\rho,\beta}(z) = 0.$$

This follows from (4.2). □

5. Some equalities

5.1. Equality for a homogeneous Siegel domain. In order to give a compactness condition of the composition operators on $L_a^p(\mathscr{U}, dV_\beta)$ in terms of φ and $K_{\mathscr{U}}$, we use an integral formula on a homogeneous Siegel domain. First, we recall notation and properties of homogeneous Siegel domains following [1] and [6]. Let $\Omega \subset \mathbf{R}^n$ be an open convex cone not containing any straight lines and $F : \mathbf{C}^m \times \mathbf{C}^m \to \mathbf{C}^n$ a Hermitian map such that $F(u,u) \in \mathrm{Cl}(\Omega) \backslash \{0\}$, where $\mathrm{Cl}(\Omega)$ stands for the closure of Ω. Then, the Siegel domain $\mathscr{D}$ is defined by

$$\mathscr{D} = \left\{ (\xi, \eta) \in \mathbf{C}^n \times \mathbf{C}^m \,\middle|\, \frac{\xi - \bar{\xi}}{2i} - F(\eta, \eta) \in \Omega \right\}.$$

It is known that every bounded homogeneous domain is holomorphically equivalent to a homogeneous Siegel domain [12].

Let l be the rank of Ω. For $1 \leq j \leq l$, let $n_j \geq 0, q_j \geq 0$ and $d_j \leq 0$ be real numbers defined in [6] (These notations are also used in [1]. Note that d_j in [8] is $-d_j$ in the present notation). We write $\underline{n}$ for the vector of $\mathbf{R}^l$ whose components are n_j. The symbols $\underline{q}$ and $\underline{d}$ are used similarly. We see from [1, Proposition II.1] that the Bergman kernel of $\mathscr{D}$ is given by

$$K_{\mathscr{D}}(\zeta, \zeta') = C\left(\frac{\xi - \overline{\xi'}}{2i} - F(\eta, \eta') \right)^{2\underline{d} - \underline{q}} \qquad (\zeta = (\xi, \eta), \zeta' = (\xi', \eta')),$$

where $(\cdot)^{2\underline{d}-\underline{q}}$ is the compound power function defined in [6, (2.3)]. We put

$$\varepsilon_{\mathscr{U}} := \max\left\{ \frac{n_j}{2(-2d_j + q_j)} \,\middle|\, 1 \leq j \leq l \right\}. \tag{5.1}$$

Békollé and Kagou showed the following integral formula.

Lemma 5.1 ([1, Corollary II.4]). *For $\beta > \varepsilon_{\min}$ and $\alpha > \beta + \varepsilon_{\mathscr{U}}$, one has*

$$\int_{\mathscr{D}} |K_{\mathscr{D}}(\zeta, \zeta')|^{1+\alpha} K_{\mathscr{D}}(\zeta', \zeta')^{-\beta} dV(\zeta') = C_{\mathscr{D}}(\alpha, \beta) K_{\mathscr{D}}(\zeta, \zeta)^{\alpha-\beta},$$

where $C_{\mathscr{D}}(\alpha, \beta)$ does not depend on ζ.

5.2. Equality for a minimal bounded homogeneous domain. Let $\mathscr{D}$ be a Siegel domain biholomorphic to $\mathscr{U}$ and Φ a biholomorphic map from $\mathscr{U}$ onto $\mathscr{D}$. To get a formula for $\mathscr{U}$ analogous to Lemma 5.1, we consider the weighted Bergman space

$$L_a^p(\mathscr{D}, K_{\mathscr{D}}(\zeta, \zeta)^{-\beta} dV(\zeta)) := L^p(\mathscr{D}, K_{\mathscr{D}}(\zeta, \zeta)^{-\beta} dV(\zeta)) \cap \mathscr{O}(\mathscr{D}).$$

Actually, we only need the space for $p = 2$. Then we have an isometry

$$L_a^2(\mathscr{D}, K_{\mathscr{D}}(\zeta, \zeta)^{-\beta} dV(\zeta)) \ni f \mapsto \det J(\Phi, \cdot)^{1+\beta} f \circ \Phi \in L_a^2(\mathscr{U}, dV_\beta). \tag{5.2}$$

We note that the function $\det J(\Phi, \cdot)$ does not vanish on the simply connected domain $\mathscr{U}$, so that we have a single-valued function $\det J(\Phi, \cdot)^{1+\beta}$. The following lemma plays a key role to prove our main theorem.

Lemma 5.2. *Let $\beta > \varepsilon_{\min}$ and $\alpha > \beta + \varepsilon_{\mathscr{U}}$. Then, one has*

$$\begin{aligned}\int_{\mathscr{U}} &|K_{\mathscr{U}}(z, z')|^{1+\alpha} |\det J(\Phi, z')|^{1+2\beta-\alpha} dV_\beta(z') \\ &= C_{\mathscr{D}}(\alpha, \beta) K_{\mathscr{U}}(z, z)^{\alpha-\beta} |\det J(\Phi, z)|^{1+2\beta-\alpha}\end{aligned}$$

for any $z \in \mathscr{U}$.

Proof. Let $\zeta' = \Phi(z')$. Since

$$\begin{aligned}dV_\beta(z') &= K_{\mathscr{U}}(\Phi^{-1}(\zeta'), \Phi^{-1}(\zeta'))^{-\beta} |\det J(\Phi^{-1}, \zeta')|^2 dV(\zeta') \\ &= |\det J(\Phi, \Phi^{-1}(\zeta'))|^{-2(1+\beta)} K_{\mathscr{D}}(\zeta', \zeta')^{-\beta} dV(\zeta'),\end{aligned}$$

we have

$$\begin{aligned}\int_{\mathscr{U}} &|K_{\mathscr{U}}(z, z')|^{1+\alpha} |\det J(\Phi, z')|^{1+2\beta-\alpha} dV_\beta(z') \\ &= \int_{\mathscr{D}} |K_{\mathscr{U}}(z, \Phi^{-1}(\zeta'))|^{1+\alpha} |\det J(\Phi, \Phi^{-1}(\zeta'))|^{-(1+\alpha)} K_{\mathscr{D}}(\zeta', \zeta')^{-\beta} dV(\zeta').\end{aligned} \tag{5.3}$$

By the transformation formula of the Bergman kernel, we have

$$K_{\mathscr{U}}(z, \Phi^{-1}(\zeta')) = K_{\mathscr{D}}(\Phi(z), \zeta') \det J(\Phi, z) \overline{\det J(\Phi, \Phi^{-1}(\zeta'))}.$$

Therefore, the right hand side of (5.3) is equal to

$$\int_{\mathscr{D}} |K_{\mathscr{D}}(\Phi(z),\zeta')|^{1+\alpha}|\det J(\Phi,z)|^{1+\alpha}K_{\mathscr{D}}(\zeta',\zeta')^{-\beta}dV(\zeta'). \tag{5.4}$$

By Lemma 5.1, we rewrite (5.4) as

$$C_{\mathscr{D}}(\alpha,\beta)|\det J(\Phi,z)|^{1+2\beta-\alpha}K_{\mathscr{U}}(z,z)^{\alpha-\beta}.$$

This completes the proof. □

Corollary 5.3. *Let $\beta > \varepsilon_{\min}$ and $\alpha > \beta + \varepsilon_{\mathscr{U}}$. For any $z \in \mathscr{U}$, the function g_z defined by*

$$g_z(w) := K_{\mathscr{U}}(w,z)^{(1+\alpha)/2} \det J(\Phi,w)^{(1+2\beta-\alpha)/2} \qquad (w \in \mathscr{U})$$

is in $L^2_a(\mathscr{U}, dV_\beta)$. The norm is given by

$$\|g_z\|^2_{L^2(dV_\beta)} = C_{\mathscr{D}}(\alpha,\beta)K_{\mathscr{U}}(z,z)^{\alpha-\beta}|\det J(\Phi,z)|^{1+2\beta-\alpha}.$$

Proof. Note that g_z is the product of single-valued functions $K_{\mathscr{U}}(\cdot,z)^{(1+\alpha)/2}$ and $\det J(\Phi,\cdot)^{(1+2\beta-\alpha)/2}$. We have

$$\|g_z\|^2_{L^2(dV_\beta)} = \int_{\mathscr{U}} |K_{\mathscr{U}}(z,w)|^{1+\alpha}|\det J(\Phi,w)|^{1+2\beta-\alpha}K_{\mathscr{U}}(w,w)^{-\beta}dV(w).$$

By Lemma 5.2, this is equal to $C_{\mathscr{D}}(\alpha,\beta)K_{\mathscr{U}}(z,z)^{\alpha-\beta}|\det J(\Phi,z)|^{1+2\beta-\alpha}$. □

Corollary 5.3 enables us to find a positive function that satisfies the condition of Schur's theorem (see [15, Theorem 3.6]) in section 6.

6. The Main theorem

We are now able to prove our main theorem.

Theorem 6.1. *If C_φ is bounded on $L^q_a(\mathscr{U}, dV_{\beta_0})$ for some $q > 0$ and $\beta_0 > \varepsilon_{\min}$, then the following conditions are equivalent for any $p > 0$ and $\beta > \beta_0 + \varepsilon_{\mathscr{U}}$.*
(i) *C_φ is a compact operator on $L^p_a(\mathscr{U}, dV_\beta)$.*
(ii) $\displaystyle\lim_{z\to\partial\mathscr{U}} \frac{K_{\mathscr{U}}(\varphi(z),\varphi(z))}{K_{\mathscr{U}}(z,z)} = 0.$

Proof. By Theorem 4.5, we may assume that $p = q = 2$. First, we prove that (i) implies (ii). Assume that C_φ is a compact operator on $L^2_a(\mathscr{U}, dV_\beta)$. Then, C^*_φ is also compact. Since $\{k^{(\beta)}_z\}$ converges to 0 uniformly on compact subsets of $\mathscr{U}$ as $z \to \partial\mathscr{U}$, we have $\|C^*_\varphi k^{(\beta)}_z\|_{L^2(dV_\beta)} \to 0$ as $z \to \partial\mathscr{U}$. From (4.1), we obtain (ii).

Next, we prove that (ii) implies (i). Let S be the operator on $L_a^2(\mathscr{U}, dV_\beta)$ defined by

$$Sf(z) := \int_{\mathscr{U}} K_{\mathscr{U}}^{(\beta)}(\varphi(z), \varphi(w)) f(w) dV_\beta(w) \qquad (f \in L_a^2(\mathscr{U}, dV_\beta)).$$

Since C_φ is a bounded operator on $L_a^2(\mathscr{U}, dV_\beta)$, we have $C_\varphi C_\varphi^* = S$ by (2.8). Therefore the compactness of C_φ is equivalent to the compactness of S. Hence, it is sufficient to prove that S^+ is a compact operator on $L^2(\mathscr{U}, dV_\beta)$, where

$$S^+ f(z) := \int_{\mathscr{U}} |K_{\mathscr{U}}^{(\beta)}(\varphi(z), \varphi(w))| f(w) dV_\beta(w)$$

for $f \in L^2(\mathscr{U}, dV_\beta)$. For $r > 0$, let $\mathscr{U}_r := \{z \in \mathscr{U} \mid \mathrm{dist}(z, \partial\mathscr{U}) < r\}$. We define

$$K_{1,r}^+(z, w) := \chi_{\mathscr{U} \setminus \mathscr{U}_r}(z) |K_{\mathscr{U}}^{(\beta)}(\varphi(z), \varphi(w))|,$$

$$K_{2,r}^+(z, w) := \chi_{\mathscr{U}_r}(z) \chi_{\mathscr{U} \setminus \mathscr{U}_r}(w) |K_{\mathscr{U}}^{(\beta)}(\varphi(z), \varphi(w))|,$$

$$K_{3,r}^+(z, w) := \chi_{\mathscr{U}_r}(z) \chi_{\mathscr{U}_r}(w) |K_{\mathscr{U}}^{(\beta)}(\varphi(z), \varphi(w))|.$$

We denote by $S_{j,r}^+$ the integral operator on $L^2(\mathscr{U}, dV_\beta)$ with kernel $K_{j,r}^+$ $(j = 1, 2, 3)$. Then, we have

$$S^+ - S_{3,r}^+ = S_{1,r}^+ + S_{2,r}^+. \tag{6.1}$$

We will prove that $S_{1,r}^+$ and $S_{2,r}^+$ are compact operators on $L^2(\mathscr{U}, dV_\beta)$ for any $r > 0$ and that $\|S_{3,r}^+\| \to 0$ as $r \to 0$ through the subsequent three lemmas. Then letting $r \to 0$ in (6.1), we see that S^+ is a compact operator on $L^2(\mathscr{U}, dV_\beta)$. □

LEMMA 6.2. *The operators* $S_{1,r}^+$ *and* $S_{2,r}^+$ *are compact on* $L^2(\mathscr{U}, dV_\beta)$.

PROOF. It is enough to prove $K_{1,r}^+$ and $K_{2,r}^+$ are in $L^2(\mathscr{U} \times \mathscr{U}, dV_\beta \times dV_\beta)$ (for example, see [15, Theorem 3.5]). For $w \in \mathscr{U}$, let

$$K_{\varphi(z)}^{(\beta)}(w) := K_{\mathscr{U}}^{(\beta)}(w, \varphi(z)).$$

Then, $K_{\varphi(z)}^{(\beta)} \in L_a^2(\mathscr{U}, dV_\beta)$ and we have

$$\begin{aligned} \|K_{1,r}^+\|_{L^2(\mathscr{U} \times \mathscr{U})}^2 &= \int_{\mathscr{U} \setminus \mathscr{U}_r} \left\{ \int_{\mathscr{U}} |K_{\mathscr{U}}^{(\beta)}(\varphi(z), \varphi(w))|^2 dV_\beta(w) \right\} dV_\beta(z) \\ &= \int_{\mathscr{U} \setminus \mathscr{U}_r} \|C_\varphi K_{\varphi(z)}^{(\beta)}\|_{L^2(dV_\beta)}^2 dV_\beta(z). \end{aligned} \tag{6.2}$$

Since C_φ is a bounded operator on $L_a^p(\mathscr{U}, dV_{\beta_0})$, C_φ is bounded on $L_a^2(\mathscr{U}, dV_\beta)$ by Theorem 4.3. Hence, we have

$$\begin{aligned}\|C_\varphi K_{\varphi(z)}^{(\beta)}\|_{L^2(dV_\beta)}^2 &\leq C\|K_{\varphi(z)}^{(\beta)}\|_{L^2(dV_\beta)}^2 \\ &= CK_{\mathscr{U}}^{(\beta)}(\varphi(z), \varphi(z)) \leq CK_{\mathscr{U}}(z,z)^{1+\beta}, \qquad (6.3)\end{aligned}$$

where the last inequality follows from Lemma 4.2. Substituting (6.3) to (6.2), we obtain

$$\begin{aligned}\|K_{1,r}^+\|_{L^2(\mathscr{U}\times\mathscr{U})}^2 &\leq C\int_{\mathscr{U}\setminus\mathscr{U}_r} K_{\mathscr{U}}(z,z)^{1+\beta} dV_\beta(z) \\ &= C\int_{\mathscr{U}\setminus\mathscr{U}_r} K_{\mathscr{U}}(z,z) dV(z) \\ &< \infty.\end{aligned}$$

Similarly, we have $\|K_{2,r}^+\|_{L^2(\mathscr{U}\times\mathscr{U})} < \infty$. $\square$

Lemma 6.3. *For $z \in \mathscr{U}$, let*

$$h(z) := K_{\mathscr{U}}(z,z)^{\beta-\beta_0}|\det J(\Phi, \varphi(z))|^{1+2\beta_0-\beta}.$$

Then, one has

$$\int_{\mathscr{U}} K_{3,r}^+(z,w)h(w)dV_\beta(w) \leq C\chi_{\mathscr{U}_r}(z)\left(\frac{K_{\mathscr{U}}(\varphi(z), \varphi(z))}{K_{\mathscr{U}}(z,z)}\right)^{\beta-\beta_0} h(z). \qquad (6.4)$$

Proof. For $z \in \mathscr{U}$, we have

$$\begin{aligned}&\int_{\mathscr{U}} K_{3,r}^+(z,w)h(w)dV_\beta(w) \\ &\quad = \int_{\mathscr{U}} \chi_{\mathscr{U}_r}(z)\chi_{\mathscr{U}_r}(w)|K_{\mathscr{U}}(\varphi(z), \varphi(w))|^{1+\beta}|\det J(\Phi, \varphi(w))|^{1+2\beta_0-\beta} dV_{\beta_0}(w). \qquad (6.5)\end{aligned}$$

Let us define a holomorphic function g_z by

$$g_z(w) := \{K_{\mathscr{U}}(w, \varphi(z))^{1+\beta} \det J(\Phi, w)^{1+2\beta_0-\beta}\}^{1/2}.$$

Then, the right hand side of (6.5) is equal to

$$\chi_{\mathscr{U}_r}(z)\int_{\mathscr{U}_r} |g_z(\varphi(w))|^2 dV_{\beta_0}(w) \leq \chi_{\mathscr{U}_r}(z)\int_{\mathscr{U}} |C_\varphi g_z(w)|^2 dV_{\beta_0}(w).$$

Since $\beta_0 > \varepsilon_{\min}$ and $\beta > \beta_0 + \varepsilon_{\mathscr{U}}$, the function g_z is in $L_a^2(\mathscr{U}, dV_{\beta_0})$ by Corollary 5.3. Moreover, since C_φ is a bounded operator on $L_a^p(\mathscr{U}, dV_{\beta_0})$ by assumption,

C_φ is bounded on $L_a^2(\mathscr{U}, dV_{\beta_0})$ by Theorem 4.3. Therefore, we have

$$\int_{\mathscr{U}} K_{3,r}^+(z,w)h(w)dV_\beta(w) \le \chi_{\mathscr{U}_r}(z)\|C_\varphi g_z\|^2_{L^2(dV_{\beta_0})}$$
$$\le C\chi_{\mathscr{U}_r}(z)\|g_z\|^2_{L^2(dV_{\beta_0})}. \tag{6.6}$$

On the other hand, we have

$$\|g_z\|^2_{L^2(dV_{\beta_0})} = K_{\mathscr{U}}(\varphi(z), \varphi(z))^{\beta-\beta_0} \det J(\Phi, \varphi(z))^{1+2\beta_0-\beta} \tag{6.7}$$

by Corollary 5.3. Substituting (6.7) to (6.6), we obtain (6.4). □

Finally, the following lemma completes the proof of Theorem 6.1.

Lemma 6.4. *One has* $\|S_{3,r}^+\| \to 0$ *as* $r \to 0$.

Proof. Put

$$M(r) := \sup_{z \in \mathscr{U}_r} \left\{ \frac{K_{\mathscr{U}}(\varphi(z), \varphi(z))}{K_{\mathscr{U}}(z,z)} \right\}^{\beta-\beta_0}.$$

By Lemma 6.3, we have

$$\int_{\mathscr{U}} K_{3,r}^+(z,w)h(w)dV_\beta(w) \le CM(r)h(z).$$

Thanks to Schur's theorem, $S_{3,r}^+$ is a bounded operator on $L^2(\mathscr{U}, dV_\beta)$ with norm not exceeding $CM(r)$. Since we are assuming (ii) of Theorem 6.1, we obtain $M(r) \to 0$ as $r \to 0$. Hence we have $\|S_{3,r}^+\| \to 0$ as $r \to 0$. □

7. Examples

7.1. The Harish-Chandra realization of irreducible bounded symmetric domain. Let Ω be an irreducible bounded symmetric domain in its Harish-Chandra realization, r the rank of Ω and a, b nonnegative integers defined in [5]. In this case, we have $\varepsilon_{\min} = -\frac{1}{N}$ and $\varepsilon_\Omega = \frac{a(r-1)}{2N}$, where $N := a(r-1) + b + 2$ is the genus of Ω.

Corollary 7.1 ([9, Theorem]). *Suppose* $\beta_0 > -\frac{1}{N}$. *If the composition operator* C_φ *is bounded on* $L_a^q(\Omega, dV_{\beta_0})$ *for some* $q > 0$ *and* $\beta_0 + \frac{a(r-1)}{2N} < \beta$, *then* C_φ *is compact on* $L_a^p(\Omega, dV_\beta)$ *if and only if*

$$\lim_{z \to \partial\Omega} \frac{K_\Omega(\varphi(z), \varphi(z))}{K_\Omega(z,z)} = 0. \tag{7.1}$$

In particular, for the case $\Omega = \mathbf{B}^d$, we have $\varepsilon_{\min} = -\frac{1}{d+1}$, $\varepsilon_{\mathbf{B}^d} = 0$ and

$$K_{\mathbf{B}^d}(z, w) = \frac{d!}{\pi^d} \frac{1}{(1 - \langle z, w \rangle)^{d+1}}.$$

Therefore, (7.1) is equivalent to

$$\lim_{|z| \to 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0,$$

so we obtain [16, Theorem 4.1].

7.2. A non-symmetric minimal homogeneous domain. Finally, we show an example of a non-symmetric minimal bounded homogeneous domain. For $x \in \mathbf{R}^5$ with $x_1 \neq 0$, let

$$Q_1(x) := x_1, \qquad Q_2(x) := x_2 - \frac{x_4^2}{x_1}, \qquad Q_3(x) := x_3 - \frac{x_5^2}{x_1}$$

and $\Omega := \{x \in \mathbf{R}^5 \mid Q_j(x) > 0 \text{ for } 1 \leq j \leq 3\}$, which is called the Vinberg cone ([11], [2]). We consider the tube domain $T_\Omega := \mathbf{R}^5 + i\Omega$ and its representative domain $\mathscr{U}$. Then $\mathscr{U}$ is a non-symmetric minimal bounded homogeneous domain with center 0 by [7, Proposition 3.8]. In this case, we have $l = 3$ and $\underline{n} = (2, 0, 0)$, $\underline{d} = \left(-2, -\frac{3}{2}, -\frac{3}{2}\right)$, $\underline{q} = (0, 0, 0)$. Hence the constants $\varepsilon_{\min}$ and $\varepsilon_{\mathscr{U}}$ are given by $\varepsilon_{\min} = -\frac{1}{3}$, $\varepsilon_{\mathscr{U}} = \frac{1}{4}$.

For $\zeta := (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5) \in T_\Omega$, let

$$\zeta_{[1]} := \begin{pmatrix} \zeta_1 & \zeta_4 \\ \zeta_4 & \zeta_2 \end{pmatrix}, \qquad \zeta_{[2]} := \begin{pmatrix} \zeta_1 & \zeta_5 \\ \zeta_5 & \zeta_3 \end{pmatrix} \in \mathrm{Sym}(2, \mathbf{C}).$$

Then, the Bergman kernel of T_Ω satisfies

$$K_{T_\Omega}(\zeta, \zeta') = C\left(\frac{\zeta_1 - \overline{\zeta_1'}}{2i}\right)^2 \prod_{j=1}^{2} \det\left(\frac{\zeta_{[j]} - \overline{\zeta_{[j]}'}}{2i}\right)^{-3}.$$

Let σ be the Bergman mapping from T_Ω to $\mathscr{U}$ at $p_0 := (i, i, i, 0, 0)$ (see [7, (2.4)]). It follows from [8, Theorem 5.3] that

$$K_{\mathscr{U}}(\sigma(\zeta), \sigma(\zeta')) = \frac{1}{\mathrm{Vol}(\mathscr{U})}(1 - \mathscr{C}_1(\zeta_1)\overline{\mathscr{C}_1(\zeta_1')})^2 \prod_{j=1}^{2} \{\det(I_2 - \mathscr{C}_2(\zeta_{[j]})\overline{\mathscr{C}_2(\zeta_{[j]}')})\}^{-3},$$

where $\mathscr{C}_m$ $(m = 1, 2)$ denotes the Cayley transform

$$\mathscr{C}_m(Z) := (Z - iI_m)(Z + iI_m)^{-1} \qquad (Z \in \mathrm{Mat}(m, \mathbf{C})).$$

Assume that the composition operator C_φ is bounded on $L_a^q(\mathscr{U}, dV_{\beta_0})$ for some $q > 0$ and $\beta_0 > -\frac{1}{3}$. Applying Theorem A, we have the following; For $p > 0$ and $\beta > \beta_0 + \frac{1}{4}$, C_φ is compact on $L_a^p(\mathscr{U}, dV_\beta)$ if and only if

$$\lim_{z \to \partial\mathscr{U}} \frac{K_{\mathscr{U}}(\varphi(z), \varphi(z))}{K_{\mathscr{U}}(z, z)} = 0.$$

Acknowledgement

I would like to express my gratitude to Professor H. Ishi for important advices and suggestions. I thanks to Professor T. Ohsawa for helpful discussions and Professor N. Suzuki for his interest in this work. I would also like to give thanks to the referee for his useful comments and suggestions.

References

[1] D. Békollé and A. T. Kagou, Reproducing properties and L^p-estimates for Bergman projections in Siegel domains of type II., Studia. Math. **115** (1995), 219–239.

[2] D. Békollé and C. Nana, L^p-boundedness of Bergman projections in the tube domain over Vinberg's cone, J. Lie Theory **17** (2007), 115–144.

[3] C. Cowen and B. MacCluer, Composition operators on spaces of analytic function, CRC Press, Boca Raton, 1994.

[4] M. Engliš, Compact Toeplitz operators via the Berezin transform on bounded symmetric domains, Integr. Equ. Oper. Theory **20** (1999), 426–455.

[5] J. Fraut and A. Koranyi, Function spaces and reproducing kernels on bounded symmetric domains, J. Funct. Anal. **88** (1990), no. 1, 64–89.

[6] S. G. Gindikin, Analysis in homogeneous domains, Russian Math. Surveys **19-4** (1964), 1–89.

[7] H. Ishi and C. Kai, The representative domain of a homogeneous bounded domain, Kyushu J. Math. **64** (2010), 35–47.

[8] H. Ishi and S. Yamaji, Some estimates of the Bergman kernel of minimal bounded homogeneous domains, J. Lie Theory **21** (2011), 755–769.

[9] X. Lv and Z. Hu, Compact composition operators on weighted Bergman spaces on bounded symmetric domains, Acta Math. Scientia **31B**(2) (2011), 468–476.

[10] M. Maschler, Minimal domains and their Bergman kernel function, Pacific J. Math. **6** (1956), 501–516.

[11] È. B. Vinberg, Homogeneous cones, Soviet Math. Dokl. **1** (1960), 787–790.

[12] È. B. Vinberg, S. G. Gindikin, I. I. Pjateckiĭ-Šapiro, Classification and canonical realization of complex bounded homogeneous domains, Trans. Moscow Math. Soc. **12** (1963), 404–437.

[13] S. Yamaji, Positive Toeplitz operators on the Bergman space of a minimal bounded homogeneous domain, Hokkaido Math. J. **41** (2012), 257–274.

[14] K. H. Zhu, Positive Toeplitz operators on weighted Bergman spaces of bounded symmetric domains, J. Oper. Theory **20** (1988), 329–357.

[15] K. H. Zhu, Operator theory in function spaces, second edition, Amer. Math. Soc., Mathematical Surveys and Monographs Vol. **138**, 2007.

[16] K. H. Zhu, Compact composition operators on weighted Bergman spaces of the unit ball, Houston J. Math. **33** (2007), 273–283.

Satoshi Yamaji
Graduate School of Mathematics
Nagoya University
Chikusa-ku, Nagoya, 464-8602*, Japan*
E-mail: satoshi.yamaji@math.nagoya-u.ac.jp