HIROSHIMA MATH. J.
41 (2011), 55–87

Function spaces of parabolic Bloch type

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(Received August 19, 2009)
(Revised June 10, 2010)

ABSTRACT. The $L^{(\alpha)}$-harmonic function is the solution of the parabolic operator $L^{(\alpha)} = \partial_t + (-\Delta_x)^\alpha$. We study a function space $\tilde{\mathscr{B}}_\alpha(\sigma)$ consisting of $L^{(\alpha)}$-harmonic functions of parabolic Bloch type. In particular, we give a reproducing formula for functions in $\tilde{\mathscr{B}}_\alpha(\sigma)$. Furthermore, we study the fractional calculus on $\tilde{\mathscr{B}}_\alpha(\sigma)$. As an application, we also give a reproducing formula with fractional orders for functions in $\tilde{\mathscr{B}}_\alpha(\sigma)$. Moreover, we investigate the dual and pre-dual spaces of function spaces of parabolic Bloch type.

1. Introduction

The harmonic Bloch space on the upper half-space of $\mathbf{R}^{n+1}$ $(n \geq 1)$ was studied by Ramey and Yi [7]. Nishio, Shimomura, and Suzuki [5] introduced the α-parabolic Bloch space on the upper half-space and studied important properties of the space. It was also shown in [5] that when $\alpha = 1/2$, the 1/2-parabolic Bloch space coincides with the harmonic Bloch space of Ramey and Yi. Hence, investigation of the α-parabolic Bloch space contains that of the harmonic Bloch space. In this paper, we generalize the α-parabolic Bloch space, and study properties of its space.

We begin with recalling basic notations. Let H be the upper half-space of $\mathbf{R}^{n+1}$, that is, $H := \{X = (x,t) \in \mathbf{R}^{n+1}; x = (x_1, \dots, x_n) \in \mathbf{R}^n, t > 0\}$, and let $\partial_j := \partial/\partial x_j$ $(1 \leq j \leq n)$ and $\partial_t := \partial/\partial t$. Let $C(\Omega)$ be the set of all real-valued continuous functions on a region Ω, and for a positive integer k, $C^k(\Omega) \subset C(\Omega)$ denotes the set of all k times continuously differentiable functions on Ω, and put $C^\infty(\Omega) = \bigcap_k C^k(\Omega)$. The harmonic Bloch space $\mathscr{B}$ in [7] is the set of all harmonic functions u on H with

$$\|u\|_{\mathscr{B}} = |u(0,1)| + \sup_{(x,t) \in H} t|\nabla_{(x,t)} u(x,t)| < \infty, \tag{1.1}$$

where $\nabla_{(x,t)} = (\partial_1, \dots, \partial_n, \partial_t)$ denotes the gradient operator on $\mathbf{R}^{n+1}$. We also recall the definition of the α-parabolic Bloch space in [5]. For $0 < \alpha \leq 1$, the

2000 *Mathematics Subject Classification*. Primary 35K05; Secondary 31B10, 32A18.
Key words and phrases. Bloch space, parabolic operator of fractional order, reproducing formula.

parabolic operator $L^{(\alpha)}$ is defined by

$$L^{(\alpha)} := \partial_t + (-\Delta_x)^{\alpha},$$

where $\Delta_x := \partial_1^2 + \cdots + \partial_n^2$ is the Laplacian on the x-space $\mathbf{R}^n$. A function $u \in C(H)$ is said to be $L^{(\alpha)}$-harmonic if u satisfies $L^{(\alpha)}u = 0$ in the sense of distributions. (For details, see section 2 of this paper.) The α-parabolic Bloch space $\mathscr{B}_\alpha$ is the set of all $L^{(\alpha)}$-harmonic functions $u \in C^1(H)$ with

$$\|u\|_{\mathscr{B}_\alpha} = |u(0,1)| + \sup_{(x,t)\in H} \{t^{1/2\alpha}|\nabla_x u(x,t)| + t|\partial_t u(x,t)|\} < \infty, \tag{1.2}$$

where ∇_x also denotes the gradient operator on the x-space $\mathbf{R}^n$. It is shown in Theorem 7.4 of [5] that $\mathscr{B}_\alpha$ is a Banach space under the norm $\|\cdot\|_{\mathscr{B}_\alpha}$. Furthermore, (2.4) and Theorem 7.4 of [5] imply $\mathscr{B}_{1/2} = \mathscr{B}$. In this paper, we introduce the following function space of parabolic Bloch type.

DEFINITION 1. Let $0 < \alpha \le 1$. And we put $m(\alpha) = \min\{1, \frac{1}{2\alpha}\}$. Then, for a real number $\sigma > -m(\alpha)$, let $\mathscr{B}_\alpha(\sigma)$ be the set of all $L^{(\alpha)}$-harmonic functions $u \in C^1(H)$ with the norm

$$\|u\|_{\mathscr{B}_\alpha(\sigma)} := |u(0,1)| + \sup_{(x,t)\in H} t^{\sigma}\{t^{1/2\alpha}|\nabla_x u(x,t)| + t|\partial_t u(x,t)|\} < \infty. \tag{1.3}$$

Furthermore, let $\tilde{\mathscr{B}}_\alpha(\sigma)$ be the set of all functions $u \in \mathscr{B}_\alpha(\sigma)$ with $u(0,1) = 0$. We note that $\tilde{\mathscr{B}}_\alpha(\sigma) \cong \mathscr{B}_\alpha(\sigma)/\mathbf{R}$.

We have an interest in analyses of function spaces $\mathscr{B}_\alpha(\sigma)$, and our aim of this paper is the investigation of properties of these spaces. We remark that the condition $\sigma > -m(\alpha)$ in Definition 1 requires that the orders of t in (1.3) are positive, that is, $\sigma + \frac{1}{2\alpha} > 0$ and $\sigma + 1 > 0$. Furthermore, our results of this paper can be applied to study conjugate functions on the α-parabolic Bloch space, whose applications will be described elsewhere. We present main results of this paper.

THEOREM 1. *Let $0 < \alpha \le 1$ and $\sigma > -m(\alpha)$. Then, there exists a constant $C = C(n, \alpha, \sigma) > 0$ such that*

$$|u(x,t)| \le C\|u\|_{\mathscr{B}_\alpha(\sigma)} F_{\alpha,\sigma}(x,t)$$

for all $u \in \mathscr{B}_\alpha(\sigma)$ and $(x,t) \in H$, where

$$F_{\alpha,\sigma}(x,t) := \begin{cases} 1 + |x|^{-2\alpha\sigma} + t^{-\sigma} & (0 > \sigma > -m(\alpha)) \\ 1 + \log(1+|x|) + |\log t| & (\sigma = 0) \\ 1 + t^{-\sigma} & (\sigma > 0). \end{cases}$$

Let dV be the Lebesgue volume measure on H and $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$. The following theorem is a reproducing formula for functions in $\tilde{\mathscr{B}}_\alpha(\sigma)$, which is given by Theorem 4.5 of this paper. (Actually, our result is more general, see also Theorem 5.7.)

THEOREM 2. *Let* $0 < \alpha \le 1$ *and* $\sigma > -m(\alpha)$. *If* $k, m \in \mathbf{N}_0$ *satisfy* $m > \sigma$ *and* $k + m > 0$, *then*

$$u(x,t) = \frac{2^{k+m}}{\Gamma(k+m)} \int_H \mathscr{D}_t^k u(y,s) \omega_\alpha^m(x,t;y,s) s^{k+m-1}\, dV(y,s)$$

for all $u \in \tilde{\mathscr{B}}_\alpha(\sigma)$ *and* $(x,t) \in H$, *where* Γ *is the gamma function,* $\mathscr{D}_t = -\partial_t$, *and the kernel function* ω_α^m *is defined in section* 4.

We also give the definitions of parabolic Bergman spaces, which are closely related to the function space of parabolic Bloch type. For $1 \le p < \infty$ and $\lambda > -1$, the Lebesgue space $L^p(\lambda) := L^p(H, t^\lambda\, dV)$ is defined to be the Banach space of all Lebesgue measurable functions u on H with

$$\|u\|_{L^p(\lambda)} := \left(\int_H |u(x,t)|^p t^\lambda\, dV(x,t) \right)^{1/p} < \infty.$$

The α-parabolic Bergman space $\boldsymbol{b}_\alpha^p(\lambda)$ is the set of all $L^{(\alpha)}$-harmonic functions u on H with $u \in L^p(\lambda)$. Furthermore, $L^\infty := L^\infty(H, dV)$ is defined to be the Banach space of all Lebesgue measurable functions u on H with

$$\|u\|_{L^\infty} := \operatorname*{ess\,sup}_H |u| < \infty,$$

and let $\boldsymbol{b}_\alpha^\infty$ be the set of all $L^{(\alpha)}$-harmonic functions u on H with $u \in L^\infty$. (For details, see section 2 of this paper and [5].) As an application of Theorem 2, we obtain the following result.

THEOREM 3. *Let* $0 < \alpha \le 1$, $\sigma > -m(\alpha)$, *and* $\lambda > -1$. *Then,* $(\boldsymbol{b}_\alpha^1(\lambda))^* \cong \tilde{\mathscr{B}}_\alpha(\sigma)$ *under the pairing* $\langle \cdot, \cdot \rangle_{\lambda,\sigma}$, *where*

$$(1.4) \qquad \langle u, v \rangle_{\lambda,\sigma} := \frac{2^{\lambda+\sigma+2}}{\Gamma(\lambda+\sigma+2)} \int_H u(y,s) \mathscr{D}_t v(y,s) s^{\lambda+\sigma+1}\, dV(y,s),$$
$$u \in \boldsymbol{b}_\alpha^1(\lambda),\ v \in \tilde{\mathscr{B}}_\alpha(\sigma).$$

We also discuss a pre-dual space of $\boldsymbol{b}_\alpha^1(\lambda)$. For $\sigma > -m(\alpha)$, a function space of parabolic little Bloch type $\mathscr{B}_{\alpha,0}(\sigma)$ is the set of all functions $u \in \mathscr{B}_\alpha(\sigma)$ with

$$(1.5) \qquad \lim_{(x,t)\to\partial H \cup \{\infty\}} t^\sigma \{ t^{1/2\alpha} |\nabla_x u(x,t)| + t|\partial_t u(x,t)| \} = 0.$$

Furthermore, let $\tilde{\mathscr{B}}_{\alpha,0}(\sigma)$ be the set of all functions $u \in \mathscr{B}_{\alpha,0}(\sigma)$ with $u(0,1) = 0$. We also give the following result.

THEOREM 4. *Let* $0 < \alpha \leq 1$, $\sigma > -m(\alpha)$, *and* $\lambda > -1$. *Then,* $\boldsymbol{b}_\alpha^1(\lambda) \cong (\tilde{\mathscr{B}}_{\alpha,0}(\sigma))^*$ *under the pairing* (1.4), *that is,* $\langle u, v\rangle_{\lambda,\sigma}$ *with* $u \in \boldsymbol{b}_\alpha^1(\lambda)$ *and* $v \in \tilde{\mathscr{B}}_{\alpha,0}(\sigma)$.

We remark that the pairing (1.4) is equal to a natural pairing on a dense subset of $\boldsymbol{b}_\alpha^1(\lambda)$. In fact, for a real number η, let

$$\mathcal{S}(\eta) := \{u \in \boldsymbol{b}_\alpha^\infty; (1 + t + |x|^{2\alpha})^{n/2\alpha+\eta} u(x,t) \text{ is bounded on } H\}.$$

Then, Proposition 6.2 of [3] shows that $\mathcal{S}(\eta)$ is a dense subspace of $\boldsymbol{b}_\alpha^1(\lambda)$ when $\lambda > -1$ and $\eta > \lambda + 1$. By the similar argument as in the proof of Theorem 6.5 of [3], it is not hard to see that

$$\langle u, v\rangle_{\lambda,\sigma} = \frac{2^{\lambda+\sigma+1}}{\Gamma(\lambda+\sigma+1)} \int_H u(y,s) v(y,s) s^{\lambda+\sigma}\, dV(y,s), \tag{1.6}$$

$$u \in \mathcal{S}(\eta),\ v \in \tilde{\mathscr{B}}_\alpha(\sigma),$$

when $\sigma \geq 0$ and $\eta > \lambda + \sigma + 1$ (since $\sigma \geq 0$, the condition $\eta > \lambda + \sigma + 1$ implies that $\mathcal{S}(\eta)$ is dense in $\boldsymbol{b}_\alpha^1(\lambda)$). Furthermore, when $0 > \sigma > -m(\alpha)$, the equation (1.6) also holds under the conditions $\lambda + \sigma > -1$ and $\eta > \lambda + 1$.

We describe the construction of this paper. In section 2, we present preliminary facts. In particular, we recall the explicit definition of the $L^{(\alpha)}$-harmonic functions and introduce some known results. In section 3, we study basic properties of $\tilde{\mathscr{B}}_\alpha(\sigma)$ and give the proof of Theorem 1. In section 4, we give the proof of Theorem 2. Consequently, we show a reproducing formula for functions in $\tilde{\mathscr{B}}_\alpha(\sigma)$. In section 5, we study fractional calculus on $\tilde{\mathscr{B}}_\alpha(\sigma)$. As an application, we give a generalization of Theorem 2, which is a reproducing formula with fractional orders for functions in $\tilde{\mathscr{B}}_\alpha(\sigma)$. In section 6, we give the proofs of Theorems 3 and 4.

Throughout this paper, C will denote a positive constant whose value is not necessary the same at each occurrence; it may vary even within a line.

2. Preliminaries

In this section, we recall basic properties concerning the $L^{(\alpha)}$-harmonic functions. (For details, see [5].) We begin with describing about the operator $(-\Delta_x)^\alpha$. Since the case $\alpha = 1$ is trivial, we only describe the case $0 < \alpha < 1$. Let $C_c^\infty(H) \subset C(H)$ be the set of all infinitely differentiable functions on H with compact support. Then, $(-\Delta_x)^\alpha$ is the convolution operator defined by

$$(-\Delta_x)^\alpha \psi(x,t) := -C_{n,\alpha} \lim_{\delta \downarrow 0} \int_{|y|>\delta} (\psi(x+y,t) - \psi(x,t)) |y|^{-n-2\alpha} dy \tag{2.1}$$

for all $\psi \in C_c^\infty(H)$ and $(x,t) \in H$, where $C_{n,\alpha} = -4^\alpha \pi^{-n/2}\Gamma((n+2\alpha)/2)/\Gamma(-\alpha) > 0$. Let $\tilde{L}^{(\alpha)} := -\partial_t + (-\Delta_x)^\alpha$ be the adjoint operator of $L^{(\alpha)}$. Then, a function $u \in C(H)$ is said to be $L^{(\alpha)}$-harmonic if u satisfies $L^{(\alpha)}u = 0$ in the sense of distributions, that is, $\int_H |u \cdot \tilde{L}^{(\alpha)}\psi| dV < \infty$ and $\int_H u \cdot \tilde{L}^{(\alpha)}\psi \, dV = 0$ for all $\psi \in C_c^\infty(H)$. By (2.1) and the compactness of $\mathrm{supp}(\psi)$ (the support of ψ), there exist $0 < t_1 < t_2 < \infty$ and a constant $C > 0$ such that

$$\mathrm{supp}(\tilde{L}^{(\alpha)}\psi) \subset S = \mathbf{R}^n \times [t_1, t_2] \tag{2.2}$$

and

$$|\tilde{L}^{(\alpha)}\psi(x,t)| \le C(1+|x|)^{-n-2\alpha} \qquad \text{for } (x,t) \in S. \tag{2.3}$$

Hence, the condition $\int_H |u \cdot \tilde{L}^{(\alpha)}\psi| dV < \infty$ for all $\psi \in C_c^\infty(H)$ is equivalent to the following: for any $0 < t_1 < t_2 < \infty$,

$$\int_{t_1}^{t_2} \int_{\mathbf{R}^n} |u(x,t)|(1+|x|)^{-n-2\alpha} dx dt < \infty. \tag{2.4}$$

We also note that

$$\partial_j(-\Delta_x)^\alpha \psi = (-\Delta_x)^\alpha \partial_j \psi \qquad \text{and} \qquad \partial_t(-\Delta_x)^\alpha \psi = (-\Delta_x)^\alpha \partial_t \psi \tag{2.5}$$

for all $\psi \in C_c^\infty(H)$.

We describe the fundamental solution of $L^{(\alpha)}$. For $x \in \mathbf{R}^n$, let

$$W^{(\alpha)}(x,t) := \begin{cases} \dfrac{1}{(2\pi)^n} \displaystyle\int_{\mathbf{R}^n} \exp(-t|\xi|^{2\alpha} + ix \cdot \xi) d\xi & (t > 0) \\ 0 & (t \le 0), \end{cases}$$

where $x \cdot \xi$ denotes the inner product on $\mathbf{R}^n$ and $|\xi| = (\xi \cdot \xi)^{1/2}$. The function $W^{(\alpha)}$ is the fundamental solution of $L^{(\alpha)}$ and it is $L^{(\alpha)}$-harmonic on H. We note that

$$W^{(\alpha)} > 0 \quad \text{on } H \qquad \text{and} \qquad \int_{\mathbf{R}^n} W^{(\alpha)}(x,t) dx = 1 \quad \text{for all } 0 < t < \infty. \tag{2.6}$$

Furthermore, $W^{(\alpha)} \in C^\infty(H)$. The following lemma is Lemma 2.4 of [5].

Lemma 2.1 ([5, Lemma 2.4]). *Let $0 < \alpha \le 1$ and $1 \le p \le \infty$. If $f \in C(\mathbf{R}^n) \cap L^p(\mathbf{R}^n)$, then for every $x \in \mathbf{R}^n$,*

$$\lim_{s \to +0} \int_{\mathbf{R}^n} f(x-y) W^{(\alpha)}(y,s) dy = f(x).$$

We also present the following lemma, which is Theorem 4.1 of [5] and Lemma 3.1 of [8].

Lemma 2.2 (Theorem 4.1 of [5] and Lemma 3.1 of [8]). *Let* $0 < \alpha \le 1$, $1 \le p < \infty$, *and* $\lambda > -1$. *Then, every* $u \in \boldsymbol{b}_\alpha^p(\lambda)$ *satisfies the following Huygens property, that is,*

$$
(2.7) \quad u(x, t+s) = \int_{\mathbf{R}^n} u(x-y, t) W^{(\alpha)}(y, s) dy = \int_{\mathbf{R}^n} u(y, t) W^{(\alpha)}(x-y, s) dy
$$

holds for all $x \in \mathbf{R}^n$, $0 < s < \infty$, *and* $0 < t < \infty$. *Furthermore, every* $u \in \boldsymbol{b}_\alpha^\infty$ *also satisfies* (2.7).

Since $W^{(\alpha)} \in C^\infty(H)$, the Huygens property implies that $\boldsymbol{b}_\alpha^p(\lambda) \subset C^\infty(H)$. We also remark that a function satisfying the Huygens property is $L^{(\alpha)}$-harmonic, because $W^{(\alpha)}$ is $L^{(\alpha)}$-harmonic on H. For a multi-index $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbf{N}_0^n$, let $\partial_x^\gamma := \partial_1^{\gamma_1} \dots \partial_n^{\gamma_n}$. The following estimate is Lemma 1 of [6]: For a multi-index $\gamma \in \mathbf{N}_0^n$ and an integer $k \in \mathbf{N}_0$, there exists a constant $C = C(n, \alpha, \gamma, k) > 0$ such that

$$
(2.8) \qquad |\partial_x^\gamma \partial_t^k W^{(\alpha)}(x, t)| \le C(t + |x|^{2\alpha})^{-((n+|\gamma|)/2\alpha + k)}
$$

for all $(x, t) \in H$. When $(\gamma, k) = (0, 0)$, Lemma 3.1 of [5] gives the following estimate: there exists a constant $C = C(n, \alpha) > 0$ such that

$$
(2.9) \qquad W^{(\alpha)}(x, t) \le Ct(t + |x|^{2\alpha})^{-(n/2\alpha + 1)}
$$

for all $(x, t) \in H$. Furthermore, the following estimate is Lemma 3.3 of [8] and Theorem 5.4 of [5]: For $1 \le p < \infty$ and $\lambda > -1$ there exists a constant $C = C(n, \alpha, p, \lambda, \gamma, k) > 0$ such that

$$
(2.10) \qquad |\partial_x^\gamma \partial_t^k u(x, t)| \le C \|u\|_{L^p(\lambda)} t^{-(|\gamma|/2\alpha + k) - (n/2\alpha + \lambda + 1)(1/p)}
$$

for all $u \in \boldsymbol{b}_\alpha^p(\lambda)$ and $(x, t) \in H$. Furthermore, there exists a constant $C = C(n, \alpha, \gamma, k) > 0$ such that

$$
(2.11) \qquad |\partial_x^\gamma \partial_t^k u(x, t)| \le C \|u\|_{L^\infty} t^{-(|\gamma|/2\alpha + k)}
$$

for all $u \in \boldsymbol{b}_\alpha^\infty$ and $(x, t) \in H$.

The following lemma is Lemma 5 of [6]. We use this in our later arguments.

Lemma 2.3 ([6, Lemma 5]). *Let* $\theta, c \in \mathbf{R}$. *If* $\theta > -1$ *and* $\theta - c + \frac{n}{2\alpha} + 1 < 0$, *then there exists a constant* $C = C(n, \alpha, \theta, c) > 0$ *such that*

$$
\int_H \frac{s^\theta}{(t + s + |x-y|^{2\alpha})^c} \, dV(y, s) = Ct^{\theta - c + n/2\alpha + 1}
$$

for all $(x, t) \in H$.

3. Basic properties of $\mathscr{B}_\alpha(\sigma)$

In this section, we study basic properties of $\mathscr{B}_\alpha(\sigma)$. We begin with showing the following lemma.

LEMMA 3.1. *Let $0 < \alpha < 1$ and suppose that a function $u \in C^1(H)$ is $L^{(\alpha)}$-harmonic. Then the following statements hold.*

(1) *If $\partial_j u$ satisfies the condition (2.4), then $\partial_j u$ is also $L^{(\alpha)}$-harmonic.*

(2) *If $\partial_t u$ satisfies the condition (2.4), then $\partial_t u$ is also $L^{(\alpha)}$-harmonic.*

PROOF. (1) If u satisfies the condition (2.4) and $\partial_j u$ also satisfies the condition (2.4), then by the Fubini theorem and integrating by parts with respect to the variable x_j, (2.3) and (2.5) imply that

$$\int_H \partial_j u \cdot \tilde{L}^{(\alpha)}\psi \, dV = -\int_H u \cdot \tilde{L}^{(\alpha)}(\partial_j \psi) dV = 0$$

for all $\psi \in C_c^\infty(H)$. Thus, $\partial_j u$ is $L^{(\alpha)}$-harmonic. (2) Similarly, if $\partial_t u$ satisfies the condition (2.4), then the $L^{(\alpha)}$-harmonicity of $\partial_t u$ follows from (2.2) and (2.5). □

For a real number $\delta \geq 0$ and a function u on H, let $u^\delta(x,t) = u(x, t+\delta)$ for $(x,t) \in H$. Basic properties of functions in $\mathscr{B}_\alpha(\sigma)$ are given in the following. In particular, (1) of Theorem 3.2 is Theorem 1 of section 1.

THEOREM 3.2. *Let $0 < \alpha \leq 1$ and $\sigma > -m(\alpha)$. Then, the following statements hold.*

(1) *There exists a constant $C = C(n, \alpha, \sigma) > 0$ such that*

$$|u(x,t)| \leq C\|u\|_{\mathscr{B}_\alpha(\sigma)} F_{\alpha,\sigma}(x,t) \tag{3.1}$$

for all $u \in \mathscr{B}_\alpha(\sigma)$ and $(x,t) \in H$, where

$$F_{\alpha,\sigma}(x,t) := \begin{cases} 1 + |x|^{-2\alpha\sigma} + t^{-\sigma} & (0 > \sigma > -m(\alpha)) \\ 1 + \log(1+|x|) + |\log t| & (\sigma = 0) \\ 1 + t^{-\sigma} & (\sigma > 0). \end{cases} \tag{3.2}$$

(2) *If $u \in \mathscr{B}_\alpha(\sigma)$, then*

$$\lim_{s \to +0} \int_{\mathbf{R}^n} u(x-y, t) W^{(\alpha)}(y, s) dy = u(x,t)$$

for all $(x,t) \in H$.

(3) *Every $u \in \mathscr{B}_\alpha(\sigma)$ satisfies the Huygens property (2.7).*

(4) *Let* $(\gamma, k) \in \mathbf{N}_0^n \times \mathbf{N}_0 \backslash \{(0,0)\}$. *If* $u \in \mathscr{B}_\alpha(\sigma)$, *then* u *belongs to* $C^\infty(H)$ *and* $\partial_x^\gamma \partial_t^k u$ *is* $L^{(\alpha)}$*-harmonic. Furthermore, there exists a constant* $C = C(n, \alpha, \sigma, \gamma, k) > 0$ *such that*

$$|\partial_x^\gamma \partial_t^k u(x,t)| \le C t^{-(|\gamma|/2\alpha + k + \sigma)} \|u\|_{\mathscr{B}_\alpha(\sigma)} \tag{3.3}$$

for all $u \in \mathscr{B}_\alpha(\sigma)$ *and* $(x,t) \in H$.

(5) *The space* $\mathscr{B}_\alpha(\sigma)$ *is a Banach space under the norm* (1.3).

Proof. (1) Let $c > 0$ be arbitrary real number. Then, for $u \in \mathscr{B}_\alpha(\sigma)$ and $(x,t) \in H$, we obtain

$$\begin{aligned} |u(x,t)| &\le |u(0,1)| + \left|\int_1^c |\partial_t u(0,s)| ds\right| + \int_0^1 |x| \cdot |\nabla_x u(rx, c)| dr + \left|\int_c^t |\partial_t u(x,s)| ds\right| \\ &\le \|u\|_{\mathscr{B}_\alpha(\sigma)} \left(1 + \left|\int_1^c s^{-\sigma-1}\, ds\right| + |x| c^{-\sigma - 1/2\alpha} + \left|\int_c^t s^{-\sigma-1}\, ds\right|\right) \\ &\le C\|u\|_{\mathscr{B}_\alpha(\sigma)}(1 + I_{x,\sigma}(c)), \end{aligned}$$

where

$$I_{x,\sigma}(c) := \begin{cases} |\log c| + |x| c^{-1/2\alpha} + |\log t| & (\sigma = 0) \\ c^{-\sigma}(1 + |x| c^{-1/2\alpha}) + t^{-\sigma} & (\sigma \ne 0). \end{cases}$$

Since $c > 0$ is arbitrary, we can put $c = (1 + |x|)^{2\alpha}$. Then there exists a constant $C > 0$ such that

$$I_{x,\sigma}(c) \le C \begin{cases} 1 + |x|^{-2\alpha\sigma} + t^{-\sigma} & (0 > \sigma > -m(\alpha)) \\ 1 + \log(1 + |x|) + |\log t| & (\sigma = 0) \\ 1 + t^{-\sigma} & (\sigma > 0). \end{cases}$$

Thus we obtain the estimate (3.1).

(2) Let $u \in \mathscr{B}_\alpha(\sigma)$. Also, let $(x,t) \in H$ and $\varepsilon > 0$ be fixed. Then, there exists a real number $\delta > 0$ such that $|u(x-y,t) - u(x,t)| < \varepsilon$ for all $y \in \mathbf{R}^n$ with $|y| < \delta$. Therefore, (2.6), (3.1), and (2.9) imply that

$$\begin{aligned} &\left|\int_{\mathbf{R}^n} u(x-y,t) W^{(\alpha)}(y,s) dy - u(x,t)\right| \\ &\quad \le \varepsilon \int_{|y|<\delta} W^{(\alpha)}(y,s) dy + C\|u\|_{\mathscr{B}_\alpha(\sigma)} \int_{|y| \ge \delta} (F_{\alpha,\sigma}(x-y,t) + 1) W^{(\alpha)}(y,s) dy \\ &\quad \le \varepsilon + Cs \int_{|y| \ge \delta} \frac{F_{\alpha,\sigma}(x-y,t) + 1}{|y|^{n+2\alpha}}\, dy. \end{aligned}$$

Suppose that $0 > \sigma > -m(\alpha)$. Then, (3.2) implies that $F_{\alpha,\sigma}(x-y,t) \le C(1+|y|^{-2\alpha\sigma})$ for all $y \in \mathbf{R}^n$. Therefore, we obtain

$$\lim_{s\to+0}\left|\int_{\mathbf{R}^n} u(x-y,t)W^{(\alpha)}(y,s)dy - u(x,t)\right| \le \varepsilon.$$

The proof of the case $\sigma \ge 0$ is similar to that of $0 > \sigma > -m(\alpha)$.

(3) Let $u \in \mathscr{B}_\alpha(\sigma)$ and $s > 0$ be fixed. Then, by the definition of the norm (1.3), we have $\partial_j u^{s/2} \in L^\infty$. Therefore, $\partial_j u^{s/2}$ satisfies the condition (2.4). Furthermore, (1) of Theorem 3.2 implies that $\lim_{|x|\to\infty} u^{s/2}(x,t)(1+|x|)^{-n-2\alpha} = 0$ for each $t > 0$. Thus, by (1) of Lemma 3.1, we have $\partial_j u^{s/2} \in \boldsymbol{b}_\alpha^\infty$. Since every element in $\boldsymbol{b}_\alpha^\infty$ satisfies the Huygens property by Lemma 2.2, we obtain

$$\begin{aligned}\partial_j u(x,t+s) = \partial_j u^{s/2}(x,t+s/2) &= \int_{\mathbf{R}^n} \partial_j u^{s/2}(x-y,s/2)W^{(\alpha)}(y,t)dy \\ &= \int_{\mathbf{R}^n} \partial_j u(x-y,s)W^{(\alpha)}(y,t)dy\end{aligned}$$

for all $(x,t) \in H$. Hence, for $x = (x_1,\dots,x_n) \in \mathbf{R}^n$ and $x_j' \in \mathbf{R}$, put

$$x' = (x_1,\dots,x_{j-1},x_j',x_{j+1},\dots,x_n),$$

then we have

$$u(x,t+s) - u(x',t+s) = \int_{\mathbf{R}^n} (u(x-y,s) - u(x'-y,s))W^{(\alpha)}(y,t)dy.$$

Therefore, the function

$$v(x,t,s) := u(x,t+s) - \int_{\mathbf{R}^n} u(x-y,s)W^{(\alpha)}(y,t)dy \tag{3.4}$$

is a constant with respect to the variable x_j $(1 \le j \le n)$. By a similar argument with respect to s, the function v is also a constant with respect to the variable s. Since for each fixed $s > 0$ the function $v(\cdot,\cdot,s)$ is $L^{(\alpha)}$-harmonic by (3.4), we have $\partial_t v = \partial_t v + (-\Delta_x)^\alpha v = 0$. Therefore, v is a constant, and which is equal to $\lim_{t\to+0} v(x,t,s) = 0$ by (2) of Theorem 3.2.

(4) Let $u \in \mathscr{B}_\alpha(\sigma)$ and $(\gamma,k) \in \mathbf{N}_0^n \times \mathbf{N}_0 \backslash \{(0,0)\}$. Then, by (3) of Theorem 3.2, u belongs to $C^\infty(H)$ and $\partial_x^\gamma \partial_t^k u$ is $L^{(\alpha)}$-harmonic. Let $(y,s) \in H$. Put $\gamma' = (\gamma_1,\dots,\gamma_{j-1},\gamma_j-1,\gamma_{j+1},\dots,\gamma_n)$, where $\gamma = (\gamma_1,\dots,\gamma_n)$ with $\gamma_j \ne 0$. Then, since $\partial_j u^{s/2} \in \boldsymbol{b}_\alpha^\infty$ by the definition of (1.3), the estimate (2.11) implies that

$$|\partial_x^\gamma \partial_t^k u(y,s)| = |\partial_x^{\gamma'} \partial_t^k (\partial_j u^{s/2})(y,s/2)| \le Cs^{-(|\gamma'|/2\alpha+k)} \sup_{(x,t)\in H} |\partial_j u^{s/2}(x,t)|$$

$$\le Cs^{-(|\gamma|/2\alpha+k+\sigma)} \sup_{(x,t)\in H} (t+s/2)^{\sigma+1/2\alpha} |\partial_j u(x,t+s/2)|$$

$$\le Cs^{-(|\gamma|/2\alpha+k+\sigma)} \|u\|_{\mathscr{B}_\alpha(\sigma)}.$$

By a similar argument with respect to t, we also have the estimate (3.3).

(5) Let $\{u_\ell\}$ be a Cauchy sequence in $\mathscr{B}_\alpha(\sigma)$. Then, by (1) of Theorem 3.2, $\{u_\ell(x,t)\}$ is a Cauchy sequence in $\mathbf{R}$. Thus, we define a function u on H such that $u(x,t) = \lim_{\ell\to\infty} u_\ell(x,t)$ for each $(x,t) \in H$. Moreover, by (4) of Theorem 3.2, $\{\partial_j u_\ell(x,t)\}$ and $\{\partial_t u_\ell(x,t)\}$ are Cauchy sequences with respect to the locally uniform topology on each domain $\mathbf{R}^n \times [t_0,\infty)$ with $t_0 > 0$. Hence, u belongs to $C^1(H)$. Let $x \in \mathbf{R}^n$, $0 < s < \infty$, and $0 < t < \infty$ be fixed. Then, by (3) of Theorem 3.2, we have

$$u_\ell(x,t+s) = \int_{\mathbf{R}^n} u_\ell(x-y,t) W^{(\alpha)}(y,s)dy \tag{3.5}$$

for each ℓ. Since $\{u_\ell\}$ is a Cauchy sequence in $\mathscr{B}_\alpha(\sigma)$, (3.1) implies that $|u_\ell(x-y,t)| \le CF_{\alpha,\sigma}(x-y,t)$ for all $\ell \in \mathbf{N}$ and $y \in \mathbf{R}^n$. Suppose that $0 > \sigma > -m(\alpha)$. Then, (2.9) and (3.2) show that

$$|u_\ell(x-y,t) W^{(\alpha)}(y,s)| \le C \frac{1+|y|^{-2\alpha\sigma}}{1+|y|^{n+2\alpha}} \tag{3.6}$$

for all $\ell \in \mathbf{N}$ and $y \in \mathbf{R}^n$. Since the right-hand side of (3.6) is integrable with respect to y, (3.5) and the Lebesgue dominated convergence theorem imply that u satisfies the Huygens property. Hence, u is $L^{(\alpha)}$-harmonic. Furthermore, we show $\|u_\ell - u\|_{\mathscr{B}_\alpha(\sigma)} \to 0$ and $u \in \mathscr{B}_\alpha(\sigma)$. In fact, since $\{u_\ell\}$ is a Cauchy sequence in $\mathscr{B}_\alpha(\sigma)$, for every $\varepsilon > 0$ there exists $\ell_0 \in \mathbf{N}$ such that $\|u_\ell - u_{\ell'}\|_{\mathscr{B}_\alpha(\sigma)} < \varepsilon$ for all $\ell, \ell' \ge \ell_0$. Therefore, if $\ell, \ell' \ge \ell_0$, then

$$|u_\ell(0,1) - u_{\ell'}(0,1)| \\ + t^\sigma\{t^{1/2\alpha}|\nabla_x u_\ell(x,t) - \nabla_x u_{\ell'}(x,t)| + t|\partial_t u_\ell(x,t) - \partial_t u_{\ell'}(x,t)|\} < \varepsilon$$

for all $(x,t) \in H$. Since $u_{\ell'}(0,1) \to u(0,1)$ and $\partial_j u_{\ell'}(x,t) \to \partial_j u(x,t)$, $\partial_t u_{\ell'}(x,t) \to \partial_t u(x,t)$ for each $(x,t) \in H$, we obtain

$$|u_\ell(0,1) - u(0,1)| + t^\sigma\{t^{1/2\alpha}|\nabla_x u_\ell(x,t) - \nabla_x u(x,t)| + t|\partial_t u_\ell(x,t) - \partial_t u(x,t)|\} \le \varepsilon$$

for all $(x,t) \in H$. Hence, it follows that $\|u_\ell - u\|_{\mathscr{B}_\alpha(\sigma)} \le \varepsilon$ for all $\ell \ge \ell_0$. Also, we have $u = u - u_{\ell_0} + u_{\ell_0} \in \mathscr{B}_\alpha(\sigma)$. The proof of the case $\sigma \ge 0$ is similar to that of $0 > \sigma > -m(\alpha)$. This completes the proof. □

Remark 3.3. It is well-known that $W^{(1/2)}$ is the Poisson kernel (see (2.4) of [5]). Hence, (3) of Theorem 3.2 implies that every $u \in \mathscr{B}_{1/2}(\sigma)$ is harmonic on H. Conversely, every harmonic functions which satisfy the condition (1.3) is $L^{(1/2)}$-harmonic on H.

4. Reproducing formulae on $\mathscr{B}_\alpha(\sigma)$

We study reproducing formulae on $\mathscr{B}_\alpha(\sigma)$. Let $\gamma \in \mathbf{N}_0^n$ and $m \in \mathbf{N}_0$. Then, a function $\omega_\alpha^{\gamma,m}$ on $H \times H$ is defined by

$$\omega_\alpha^{\gamma,m}(x,t;y,s) = \partial_x^\gamma \mathscr{D}_t^m W^{(\alpha)}(x-y,t+s) - \partial_x^\gamma \mathscr{D}_t^m W^{(\alpha)}(-y,1+s) \tag{4.1}$$

for $(x,t),(y,s) \in H$, where $\mathscr{D}_t = -\partial_t$. In particular, we shall write $\omega_\alpha^m = \omega_\alpha^{0,m}$. We shall give reproducing formulae on $\mathscr{B}_\alpha(\sigma)$ using the kernel function ω_α^m. First, we present estimates of the function $\omega_\alpha^{\gamma,m}$. The following lemma is (2) of Proposition 3.1 of [3].

Lemma 4.1 ([3, (2) of Proposition 3.1]). *Let $0 < \alpha \le 1$, $\gamma \in \mathbf{N}_0^n$, and $m \in \mathbf{N}_0$. Then, for every $(x,t) \in H$, there exists a constant $C = C(n,\alpha,\gamma,m,x,t) > 0$ such that*

$$|\omega_\alpha^{\gamma,m}(x,t;y,s)| \le C(1+s+|y|^{2\alpha})^{-(n+|\gamma|)/2\alpha-m-m(\alpha)}$$

for all $(y,s) \in H$.

We give the following estimates, which are Lipschitz type estimates of functions in $\mathscr{B}_\alpha(\sigma)$.

Lemma 4.2. *Let $0 < \alpha \le 1$, $\sigma > -m(\alpha)$, $\gamma \in \mathbf{N}_0^n$, and $k \in \mathbf{N}_0$. Then, the following statements hold.*

(1) *For every real number $M > 1$, there exists a constant $C = C(n,\alpha,\gamma,k,M,\sigma) > 0$ such that*

$$\begin{aligned}|\partial_x^\gamma \mathscr{D}_t^k u(x,t+s) &- \partial_x^\gamma \mathscr{D}_t^k u(0,1+s)| \\ &\le C\|u\|_{\mathscr{B}_\alpha(\sigma)}\left(\frac{|x|}{(1+s)^{(|\gamma|+1)/2\alpha+k+\sigma}} + \frac{|t-1|}{(1+s)^{|\gamma|/2\alpha+k+1+\sigma}}\right)\end{aligned}$$

for all $u \in \mathscr{B}_\alpha(\sigma)$, $(x,t) \in \mathbf{R}^n \times [M^{-1}, M]$, and $s \ge 0$.

(2) *For every $(x,t) \in H$, there exists a constant $C = C(n,\alpha,\gamma,k,x,t,\sigma) > 0$ such that*

$$|\partial_x^\gamma \mathscr{D}_t^k u(x,t+s) - \partial_x^\gamma \mathscr{D}_t^k u(0,1+s)| \le C\|u\|_{\mathscr{B}_\alpha(\sigma)}(1+s)^{-|\gamma|/2\alpha-k-m(\alpha)-\sigma}$$

for all $u \in \mathscr{B}_\alpha(\sigma)$ and $s \ge 0$.

PROOF. (1) By (4) of Theorem 3.2, we have

$$\begin{aligned}
&|\partial_x^\gamma \mathscr{D}_t^k u(x,t+s) - \partial_x^\gamma \mathscr{D}_t^k u(0,1+s)| \\
&\quad \le |\partial_x^\gamma \mathscr{D}_t^k u(x,t+s) - \partial_x^\gamma \mathscr{D}_t^k u(0,t+s)| + |\partial_x^\gamma \mathscr{D}_t^k u(0,t+s) - \partial_x^\gamma \mathscr{D}_t^k u(0,1+s)| \\
&\quad \le \int_0^1 |x| \cdot |\nabla_x \partial_x^\gamma \mathscr{D}_t^k u(rx,t+s)| dr + \left| \int_1^t |\partial_x^\gamma \mathscr{D}_t^{k+1} u(0,\tau+s)| d\tau \right| \\
&\quad \le C\|u\|_{\mathscr{B}_\alpha(\sigma)} \left(\frac{|x|}{(1+s)^{(|\gamma|+1)/2\alpha+k+\sigma}} + \frac{|t-1|}{(1+s)^{|\gamma|/2\alpha+k+1+\sigma}} \right)
\end{aligned}$$

for all $u \in \mathscr{B}_\alpha(\sigma)$, $(x,t) \in \mathbf{R}^n \times [M^{-1}, M]$, and $s \ge 0$.

(2) The desired estimate immediately follows from (1) of Lemma 4.2. □

The following lemma is important for the proof of our reproducing formulae on $\mathscr{B}_\alpha(\sigma)$.

LEMMA 4.3. *Let* $0 < \alpha \le 1$, $\sigma > -m(\alpha)$, $u \in \mathscr{B}_\alpha(\sigma)$, $(x,t) \in H$, *and let* $c_1, c_2 > 0$ *be real numbers. Then, the following statements hold.*

(1) *For any* $0 < \varepsilon < m(\alpha)$, *there exists a constant* $C = C(n, \alpha, \sigma, \varepsilon) > 0$ *such that*

$$|u(y,s)| \le C\|u\|_{\mathscr{B}_\alpha(\sigma)} M_{\alpha,\sigma,\varepsilon}(y,s)$$

for all $(y,s) \in H$, *where*

$$M_{\alpha,\sigma,\varepsilon}(y,s) := \begin{cases} (1+s+|y|^{2\alpha})^{-\sigma} & (0 > \sigma > -m(\alpha)) \\ (1+s+|y|^{2\alpha})^{\varepsilon} + s^{-\varepsilon} & (\sigma = 0) \\ 1 + s^{-\sigma} & (\sigma > 0). \end{cases} \tag{4.2}$$

(2) *If* $k, m \in \mathbf{N}_0$, *then for every* $\delta > 0$ *and every* $y \in \mathbf{R}^n$,

$$\lim_{s\to\infty} \mathscr{D}_t^k u^\delta(y, c_1 s) \omega_\alpha^m(x,t;y,c_2 s) s^{k+m} = 0. \tag{4.3}$$

Furthermore, if $k, m \in \mathbf{N}_0$ *satisfy* $k + m > 0$, *then*

$$\int_H |\mathscr{D}_t^k u^\delta(y,c_1 s) \omega_\alpha^m(x,t;y,c_2 s)| s^{k+m-1} \, dV(y,s) < \infty. \tag{4.4}$$

(3) *If* $k, m \in \mathbf{N}_0$ *satisfy* $m > \sigma$ *and* $k + m > 0$, *then there exist a constant* $C = C(n, \alpha, \sigma, k, m, c_1, c_2) > 0$ *and a function* $G_{\alpha,\sigma,k,m}$ *on* H *such that*

$$|\mathscr{D}_t^k u^\delta(y,c_1 s) \omega_\alpha^m(x,t;y,c_2 s)| \le C G_{\alpha,\sigma,k,m}(y,s) \tag{4.5}$$

for all $(y,s) \in H$ *and* $0 < \delta \leq 1$, *and such that*

$$\int_H G_{\alpha,\sigma,k,m}(y,s) s^{k+m-1}\, dV(y,s) < \infty. \tag{4.6}$$

Proof. (1) By (1) of Theorem 3.2, we have

$$|u(y,s)| \leq C\|u\|_{\mathscr{B}_\alpha(\sigma)} F_{\alpha,\sigma}(y,s)$$

for all $(y,s) \in H$, where $F_{\alpha,\sigma}$ is the function defined in (3.2). If $0 > \sigma > -m(\alpha)$, then we get

$$F_{\alpha,\sigma}(y,s) = 1 + |y|^{-2\alpha\sigma} + s^{-\sigma} \leq C(1 + s + |y|^{2\alpha})^{-\sigma}$$

for all $(y,s) \in H$. Next, let $\sigma = 0$. Then, taking a constant ε with $0 < \varepsilon < m(\alpha)$, we also get

$$\begin{aligned} F_{\alpha,\sigma}(y,s) &= 1 + \log(1 + |y|) + |\log s| \\ &\leq C(1 + |y|^{2\alpha\varepsilon} + s^{\varepsilon} + s^{-\varepsilon}) \leq C((1 + s + |y|^{2\alpha})^{\varepsilon} + s^{-\varepsilon}) \end{aligned}$$

for all $(y,s) \in H$. Since the case $\sigma > 0$ is trivial, we obtain the desired result.

(2) Let $\delta > 0$ be fixed. Suppose $k = 0$ and $m \in \mathbf{N}_0$. Then, by (1) of Lemma 4.3, we have

$$|u^{\delta}(y, c_1 s)| \leq C M_{\alpha,\sigma,\varepsilon}(y, c_1 s + \delta)$$

for all $(y,s) \in H$. If $0 > \sigma > -m(\alpha)$, then we have

$$M_{\alpha,\sigma,\varepsilon}(y, c_1 s + \delta) = (1 + c_1 s + \delta + |y|^{2\alpha})^{-\sigma} \leq C(1 + s + |y|^{2\alpha})^{-\sigma} \tag{4.7}$$

for all $(y,s) \in H$. Next, let $\sigma = 0$ and $0 < \varepsilon < m(\alpha)$. Then, we also have

$$\begin{aligned} M_{\alpha,\sigma,\varepsilon}(y, c_1 s + \delta) &= (1 + c_1 s + \delta + |y|^{2\alpha})^{\varepsilon} + (c_1 s + \delta)^{-\varepsilon} \\ &\leq C((1 + s + |y|^{2\alpha})^{\varepsilon} + s^{-\varepsilon}) \end{aligned} \tag{4.8}$$

for all $(y,s) \in H$. Thus, if we put

$$E_{\alpha,\sigma,\varepsilon}(y,s) := \begin{cases} (1 + s + |y|^{2\alpha})^{-\sigma} & (0 > \sigma > -m(\alpha)) \\ (1 + s + |y|^{2\alpha})^{\varepsilon} + s^{-\varepsilon} & (\sigma = 0) \\ 1 & (\sigma > 0), \end{cases}$$

then Lemma 4.1 implies that for every $\delta > 0$ there exists a constant $C > 0$ such that

$$\begin{aligned} |u^{\delta}(y, c_1 s) \omega_\alpha^m(x,t;y,c_2 s)| s^m &\leq C s^m E_{\alpha,\sigma,\varepsilon}(y,s)(1 + c_2 s + |y|^{2\alpha})^{-n/2\alpha - m - m(\alpha)} \\ &\leq C s^m E_{\alpha,\sigma,\varepsilon}(y,s)(1 + s + |y|^{2\alpha})^{-n/2\alpha - m - m(\alpha)} \end{aligned}$$

for all $(y,s) \in H$. Therefore, (4.3) is obtained. Furthermore, if $m \neq 0$, then (4.4) follows from Lemma 2.3.

Suppose $k \in \mathbf{N}$ and $m \in \mathbf{N}_0$. Then, (4) of Theorem 3.2 implies that

$$|\mathscr{D}_t^k u^\delta(y, c_1 s)| \leq C(c_1 s + \delta)^{-(k+\sigma)} \|u\|_{\mathscr{B}_\alpha(\sigma)} \tag{4.9}$$

for all $(y,s) \in H$. Since $-1 \leq -m(\alpha) < \sigma$, there exists a real number θ such that

$$0 \geq m(\alpha) - 1 > \theta > -\min\{0, \sigma\} - 1 \geq -1.$$

Therefore, by Lemma 4.1, we have

$$\begin{aligned}
&|\mathscr{D}_t^k u^\delta(y, c_1 s) \omega_\alpha^m(x, t; y, c_2 s)| s^{k+m} \\
&\quad \leq C(c_1 s + \delta)^{-(k-1)+\theta} (c_1 s + \delta)^{-(\sigma+1+\theta)} (1 + c_2 s + |y|^{2\alpha})^{-n/2\alpha - m - m(\alpha)} s^{k+m} \\
&\quad \leq C s^{\theta+1+m} (1 + s + |y|^{2\alpha})^{-n/2\alpha - m - m(\alpha)}
\end{aligned}$$

for all $(y,s) \in H$. Hence, (4.3) is obtained, and (4.4) also follows from Lemma 2.3.

(3) Suppose $k = 0$. Let $\sigma > 0$. Then, we have

$$M_{\alpha,\sigma,\varepsilon}(y, c_1 s + \delta) = 1 + (c_1 s + \delta)^{-\sigma} \leq C(1 + s^{-\sigma}) \tag{4.10}$$

for all $(y,s) \in H$ and $\delta > 0$. Thus, (4.7), (4.8), (4.10), and Lemma 4.1 imply that

$$|u^\delta(y, c_1 s) \omega_\alpha^m(x, t; y, c_2 s)| \leq C M_{\alpha,\sigma,\varepsilon}(y, s) (1 + s + |y|^{2\alpha})^{-n/2\alpha - m - m(\alpha)}$$

for all $(y,s) \in H$ and $0 < \delta \leq 1$, where $\sigma > -m(\alpha)$ and C is a constant independent of δ. Hence, by the conditions $m \in \mathbf{N}$ and $m > \sigma$, Lemma 2.3 implies that $G_{\alpha,\sigma,0,m}(y,s) := M_{\alpha,\sigma,\varepsilon}(y,s)(1 + s + |y|^{2\alpha})^{-n/2\alpha - m - m(\alpha)}$ satisfies (4.6).

Suppose $k \in \mathbf{N}$. Then, since $k + \sigma > 0$, (4.9) implies that

$$|\mathscr{D}_t^k u^\delta(y, c_1 s)| \leq C(c_1 s + \delta)^{-(k+\sigma)} \|u\|_{\mathscr{B}_\alpha(\sigma)} \leq C s^{-(k+\sigma)}$$

for all $(y,s) \in H$ and $\delta > 0$. Therefore, Lemma 4.1 also implies that $G_{\alpha,\sigma,k,m}(y,s) := s^{-(k+\sigma)}(1 + s + |y|^{2\alpha})^{-n/2\alpha - m - m(\alpha)}$ satisfies (4.5). Furthermore, by the conditions $m > \sigma$ and $\sigma > -m(\alpha)$, Lemma 2.3 implies that $G_{\alpha,\sigma,k,m}$ also satisfies (4.6). $\square$

We give a reproducing formula for u^δ with $u \in \mathscr{B}_\alpha(\sigma)$ and $\delta > 0$.

Proposition 4.4. *Let* $0 < \alpha \le 1$, $\sigma > -m(\alpha)$, *and* $\delta > 0$. *If* $k, m \in \mathbf{N}_0$ *satisfy* $k + m > 0$, *then*

$$(4.11)\qquad u^{\delta}(x,t) - u^{\delta}(0,1) = \frac{(c_1+c_2)^{k+m}}{\Gamma(k+m)} \int_H \mathscr{D}_t^k u^{\delta}(y, c_1 s)\omega_{\alpha}^m(x,t;y,c_2 s)s^{k+m-1}\,dV(y,s)$$

for all $u \in \mathscr{B}_{\alpha}(\sigma)$, $(x,t) \in H$, *and real numbers* $c_1, c_2 > 0$.

Proof. We remark that the integrand in the right-hand side of the equality (4.11) belongs to $L^1(H, dV)$ by (4.4).

First, we show (4.11) with $k \in \mathbf{N}$ and $m = 0$. Since $\mathscr{D}_t^k u^{\delta} \in \boldsymbol{b}_{\alpha}^{\infty}$ for every $k \in \mathbf{N}$, Lemma 2.2 implies that

$$(4.12)\qquad \begin{aligned} &\int_H \mathscr{D}_t^k u^{\delta}(y, c_1 s)\omega_{\alpha}^0(x,t;y,c_2 s)s^{k-1}\,dV(y,s) \\ &= \int_0^{\infty}\int_{\mathbf{R}^n} \mathscr{D}_t^k u^{\delta}(y, c_1 s) \\ &\quad \times (W^{(\alpha)}(x-y, t+c_2 s) - W^{(\alpha)}(-y, 1+c_2 s))dy s^{k-1}\,ds \\ &= \int_0^{\infty} (\mathscr{D}_t^k u^{\delta}(x, t+(c_1+c_2)s) - \mathscr{D}_t^k u^{\delta}(0, 1+(c_1+c_2)s))s^{k-1}\,ds. \end{aligned}$$

We prove that the right-hand side of (4.12) is equal to $\frac{\Gamma(k)}{(c_1+c_2)^k}(u^{\delta}(x,t) - u^{\delta}(0,1))$ by induction on k. Let $k = 1$. Then, (2) of Lemma 4.2 implies that the right-hand side of (4.12) with $k = 1$ is equal to $(c_1+c_2)^{-1}(u^{\delta}(x,t) - u^{\delta}(0,1))$. Assume that the right-hand side of (4.12) is equal to $\frac{\Gamma(k)}{(c_1+c_2)^k}(u^{\delta}(x,t) - u^{\delta}(0,1))$. Then, integrating by parts, we have

$$(4.13)\qquad \begin{aligned} &\int_0^{\infty} (\mathscr{D}_t^{k+1} u^{\delta}(x, t+(c_1+c_2)s) - \mathscr{D}_t^{k+1} u^{\delta}(0, 1+(c_1+c_2)s))s^k\,ds \\ &= -(c_1+c_2)^{-1}[(\mathscr{D}_t^k u^{\delta}(x, t+(c_1+c_2)s) \\ &\qquad - \mathscr{D}_t^k u^{\delta}(0, 1+(c_1+c_2)s))s^k]_0^{\infty} \\ &\quad + (c_1+c_2)^{-1}k\int_0^{\infty} (\mathscr{D}_t^k u^{\delta}(x, t+(c_1+c_2)s) \\ &\qquad - \mathscr{D}_t^k u^{\delta}(0, 1+(c_1+c_2)s))s^{k-1}\,ds. \end{aligned}$$

By (2) of Lemma 4.2 and the assumption of induction, the first term and the second term of the right-hand side of (4.13) are equal to 0 and $\frac{\Gamma(k+1)}{(c_1+c_2)^{k+1}}(u^{\delta}(x,t) - u^{\delta}(0,1))$, respectively.

Next, we show (4.11) with $k \in \mathbf{N}_0$ and $m \in \mathbf{N}$ by induction on m. Let $m = 1$. If $k = 0$, then integrating by parts, we have

$$\begin{aligned}
&\int_H u^\delta(y, c_1 s)\omega_\alpha^1(x, t; y, c_2 s)dV(y, s) \\
&\quad = \int_{\mathbf{R}^n}\int_0^\infty u^\delta(y, c_1 s)\omega_\alpha^1(x, t; y, c_2 s)dsdy \\
&\quad = -\frac{1}{c_2}\int_{\mathbf{R}^n}[u^\delta(y, c_1 s)\omega_\alpha^0(x, t; y, c_2 s)]_0^\infty dy \\
&\qquad - \frac{c_1}{c_2}\int_{\mathbf{R}^n}\int_0^\infty \mathscr{D}_t u^\delta(y, c_1 s)\omega_\alpha^0(x, t; y, c_2 s)dsdy \\
&\quad = -\frac{1}{c_2}\int_{\mathbf{R}^n}\lim_{s\to\infty} u^\delta(y, c_1 s)\omega_\alpha^0(x, t; y, c_2 s)dy \\
&\qquad + \frac{1}{c_2}\int_{\mathbf{R}^n} u(y, \delta)(W^{(\alpha)}(x - y, t) - W^{(\alpha)}(-y, 1))dy \\
&\qquad - \frac{c_1}{c_2}\int_H \mathscr{D}_t u^\delta(y, c_1 s)\omega_\alpha^0(x, t; y, c_2 s)dV(y, s).
\end{aligned}$$

Therefore, (4.3), (3) of Theorem 3.2, and (4.11) with $k = 1$ and $m = 0$ imply that

$$\begin{aligned}
&\int_H u^\delta(y, c_1 s)\omega_\alpha^1(x, t; y, c_2 s)dV(y, s) \\
&\quad = \frac{1}{c_2}(u^\delta(x, t) - u^\delta(0, 1)) - \frac{c_1}{c_2(c_1 + c_2)}(u^\delta(x, t) - u^\delta(0, 1)) \\
&\quad = \frac{1}{c_1 + c_2}(u^\delta(x, t) - u^\delta(0, 1)).
\end{aligned}$$

If $k \geq 1$, then (4.3) and (4.11) with $m = 0$ imply that

$$\begin{aligned}
&\int_H \mathscr{D}_t^k u^\delta(y, c_1 s)\omega_\alpha^1(x, t; y, c_2 s)s^k\, dV(y, s) \\
&\quad = \int_{\mathbf{R}^n}\int_0^\infty \mathscr{D}_t^k u^\delta(y, c_1 s)\omega_\alpha^1(x, t; y, c_2 s)s^k\, dsdy \\
&\quad = -\frac{1}{c_2}\int_{\mathbf{R}^n}[\mathscr{D}_t^k u^\delta(y, c_1 s)\omega_\alpha^0(x, t; y, c_2 s)s^k]_0^\infty dy \\
&\qquad - \frac{c_1}{c_2}\int_{\mathbf{R}^n}\int_0^\infty \mathscr{D}_t^{k+1} u^\delta(y, c_1 s)\omega_\alpha^0(x, t; y, c_2 s)s^k\, dsdy
\end{aligned}$$

$$
\begin{aligned}
&+\frac{k}{c_2}\int_{\mathbf{R}^n}\int_0^{\infty}\mathscr{D}_t^k u^{\delta}(y,c_1 s)\omega_{\alpha}^0(x,t;y,c_2 s)s^{k-1}\,dsdy\\
&=-\frac{c_1\Gamma(k+1)}{c_2(c_1+c_2)^{k+1}}(u^{\delta}(x,t)-u^{\delta}(0,1))+\frac{k\Gamma(k)}{c_2(c_1+c_2)^{k}}(u^{\delta}(x,t)-u^{\delta}(0,1))\\
&=\frac{\Gamma(k+1)}{(c_1+c_2)^{k+1}}(u^{\delta}(x,t)-u^{\delta}(0,1)).
\end{aligned}
$$

Let $m\in\mathbf{N}$ be fixed, and assume that the equality (4.11) holds for all $k\in\mathbf{N}_0$. Then, (4.3) and the assumption imply that

$$
\begin{aligned}
&\int_H \mathscr{D}_t^k u^{\delta}(y,c_1 s)\omega_{\alpha}^{m+1}(x,t;y,c_2 s)s^{k+m}\,dV(y,s)\\
&\quad=-\frac{1}{c_2}\int_{\mathbf{R}^n}[\mathscr{D}_t^k u^{\delta}(y,c_1 s)\omega_{\alpha}^m(x,t;y,c_2 s)s^{k+m}]_0^{\infty}\,dy\\
&\qquad-\frac{c_1}{c_2}\int_{\mathbf{R}^n}\int_0^{\infty}\mathscr{D}_t^{k+1}u^{\delta}(y,c_1 s)\omega_{\alpha}^m(x,t;y,c_2 s)s^{k+m}\,dsdy\\
&\qquad+\frac{k+m}{c_2}\int_{\mathbf{R}^n}\int_0^{\infty}\mathscr{D}_t^k u^{\delta}(y,c_1 s)\omega_{\alpha}^m(x,t;y,c_2 s)s^{k+m-1}\,dsdy\\
&\quad=-\frac{c_1\Gamma(k+m+1)}{c_2(c_1+c_2)^{k+m+1}}(u^{\delta}(x,t)-u^{\delta}(0,1))\\
&\qquad+\frac{(k+m)\Gamma(k+m)}{c_2(c_1+c_2)^{k+m}}(u^{\delta}(x,t)-u^{\delta}(0,1))\\
&\quad=\frac{\Gamma(k+m+1)}{(c_1+c_2)^{k+m+1}}(u^{\delta}(x,t)-u^{\delta}(0,1)).
\end{aligned}
$$

Hence, this completes the proof. □

We give a reproducing formula for $u\in\mathscr{B}_{\alpha}(\sigma)$. The following theorem is the main result of this section, which gives Theorem 2.

THEOREM 4.5. *Let $0<\alpha\le 1$ and $\sigma>-m(\alpha)$. If $k,m\in\mathbf{N}_0$ satisfy $m>\sigma$ and $k+m>0$, then*

$$
u(x,t)-u(0,1)=\frac{(c_1+c_2)^{k+m}}{\Gamma(k+m)}\int_H \mathscr{D}_t^k u(y,c_1 s)\omega_{\alpha}^m(x,t;y,c_2 s)s^{k+m-1}\,dV(y,s)
$$

for all $u\in\mathscr{B}_{\alpha}(\sigma)$, $(x,t)\in H$, and real numbers $c_1,c_2>0$.

PROOF. By (3) of Lemma 4.3 and Proposition 4.4, the theorem immediately follows from the Lebesgue dominated convergence theorem. □

5. Reproducing formulae by fractional derivatives on $\mathscr{B}_\alpha(\sigma)$

In this section, we give reproducing formulae by fractional derivatives on $\tilde{\mathscr{B}}_\alpha(\sigma)$. First, we recall the definitions of the fractional integral and differential operators for functions on $\mathbf{R}_+ = (0, \infty)$. (For details, see [2].) For a real number $\kappa > 0$, let

$$\mathscr{F}\mathscr{C}^{-\kappa} := \{\varphi \in C(\mathbf{R}_+); \exists \kappa' > \kappa \text{ s.t. } \varphi(t) = O(t^{-\kappa'})(t \to \infty)\}. \tag{5.1}$$

For a function $\varphi \in \mathscr{F}\mathscr{C}^{-\kappa}$, we can define the fractional integral $\mathscr{D}_t^{-\kappa}\varphi$ of φ by

$$\mathscr{D}_t^{-\kappa}\varphi(t) := \frac{1}{\Gamma(\kappa)}\int_0^\infty \tau^{\kappa-1}\varphi(\tau + t)d\tau, \qquad t \in \mathbf{R}_+. \tag{5.2}$$

In particular, put $\mathscr{F}\mathscr{C}^0 := C(\mathbf{R}_+)$ and $\mathscr{D}_t^0\varphi := \varphi$. Moreover, let

$$\mathscr{F}\mathscr{C}^\kappa := \{\varphi; d_t^{\lceil\kappa\rceil}\varphi \in \mathscr{F}\mathscr{C}^{-(\lceil\kappa\rceil-\kappa)}\}, \tag{5.3}$$

where $d_t = d/dt$ and $\lceil\kappa\rceil$ is the smallest integer greater than or equal to κ. Then, we can also define the fractional derivative $\mathscr{D}_t^\kappa\varphi$ of $\varphi \in \mathscr{F}\mathscr{C}^\kappa$ by

$$\mathscr{D}_t^\kappa\varphi(t) := \mathscr{D}_t^{-(\lceil\kappa\rceil-\kappa)}((-d_t)^{\lceil\kappa\rceil}\varphi)(t), \qquad t \in \mathbf{R}_+. \tag{5.4}$$

Clearly, when $\kappa \in \mathbf{N}_0$, the operator $\mathscr{D}_t^\kappa$ coincides with the ordinary differential operator $(-d_t)^\kappa$. Some basic properties of the fractional differential operators are the following.

LEMMA 5.1 (Proposition 2.1 of [2] and Proposition 2.2 of [3]). *For real numbers $\kappa, \nu > 0$, the following statements hold.*

(1) *If $\varphi \in \mathscr{F}\mathscr{C}^{-\kappa}$, then $\mathscr{D}_t^{-\kappa}\varphi \in C(\mathbf{R}_+)$.*

(2) *If $\varphi \in \mathscr{F}\mathscr{C}^{-\kappa-\nu}$, then $\mathscr{D}_t^{-\kappa}\mathscr{D}_t^{-\nu}\varphi = \mathscr{D}_t^{-\kappa-\nu}\varphi$.*

(3) *If $d_t^k\varphi \in \mathscr{F}\mathscr{C}^{-\nu}$ for all integers $0 \le k \le \lceil\kappa\rceil - 1$ and $d_t^{\lceil\kappa\rceil}\varphi \in \mathscr{F}\mathscr{C}^{-(\lceil\kappa\rceil-\kappa)-\nu}$, then $\mathscr{D}_t^\kappa\mathscr{D}_t^{-\nu}\varphi = \mathscr{D}_t^{-\nu}\mathscr{D}_t^\kappa\varphi = \mathscr{D}_t^{\kappa-\nu}\varphi$.*

(4) *If $d_t^{k+\lceil\nu\rceil}\varphi \in \mathscr{F}\mathscr{C}^{-(\lceil\nu\rceil-\nu)}$ for all integers $0 \le k \le \lceil\kappa\rceil - 1$, $d_t^{\lceil\kappa\rceil+\ell}\varphi \in \mathscr{F}\mathscr{C}^{-(\lceil\kappa\rceil-\kappa)}$ for all integers $0 \le \ell \le \lceil\nu\rceil - 1$, and $d_t^{\lceil\kappa\rceil+\lceil\nu\rceil}\varphi \in \mathscr{F}\mathscr{C}^{-(\lceil\kappa\rceil-\kappa)-(\lceil\nu\rceil-\nu)}$, then $\mathscr{D}_t^\kappa\mathscr{D}_t^\nu\varphi = \mathscr{D}_t^{\kappa+\nu}\varphi$.*

(5) *If $d_t^{\lceil\kappa\rceil}\varphi \in \mathscr{F}\mathscr{C}^{-\lceil\kappa\rceil}$ and $\lim_{t\to\infty} d_t^k\varphi(t) = 0$ for all integers $0 \le k \le \lceil\kappa\rceil - 1$, then $\mathscr{D}_t^{-\kappa}\mathscr{D}_t^\kappa\varphi = \varphi$.*

Here, we give some examples of fractional derivatives of elementary functions.

EXAMPLE 5.2. *Let $\kappa > 0$ be a real number. Then, the following statements hold.*

(1) *For every real number ν, we have $\mathscr{D}_t^\nu e^{-\kappa t} = \kappa^\nu e^{-\kappa t}$ for all $t \in \mathbf{R}_+$.*

(2) *For every real number $\nu > -\kappa$, we have $\mathscr{D}_t^\nu t^{-\kappa} = \frac{\Gamma(\kappa+\nu)}{\Gamma(\kappa)} t^{-\kappa-\nu}$ for all $t \in \mathbf{R}_+$.*

We present some properties of fractional derivatives of fundamental solution $W^{(\alpha)}$. By (2.8), we note that for each $x \in \mathbf{R}^n$, the function $W^{(\alpha)}(x,\cdot)$ belongs to $\mathscr{F}\mathscr{C}^{\kappa}$ for $\kappa > -\frac{n}{2\alpha}$. The following lemma is Theorem 3.1 of [2].

Lemma 5.3 (Theorem 3.1 of [2]). *Let $0 < \alpha \le 1$, and let $\gamma \in \mathbf{N}_0^n$ be a multi-index and ν a real number such that $\nu > -\frac{n}{2\alpha}$. Then, the following statements hold.*

(1) *The derivatives $\partial_x^{\gamma}\mathscr{D}_t^{\kappa}W^{(\alpha)}(x,t)$ and $\mathscr{D}_t^{\kappa}\partial_x^{\gamma}W^{(\alpha)}(x,t)$ can be defined, and the equation $\partial_x^{\gamma}\mathscr{D}_t^{\nu}W^{(\alpha)}(x,t) = \mathscr{D}_t^{\nu}\partial_x^{\gamma}W^{(\alpha)}(x,t)$ holds. Furthermore, there exists a constant $C = C(n,\alpha,\gamma,\nu) > 0$ such that*

$$|\partial_x^{\gamma}\mathscr{D}_t^{\nu}W^{(\alpha)}(x,t)| \le C(t+|x|^{2\alpha})^{-((n+|\gamma|)/2\alpha+\nu)}$$

for all $(x,t) \in H$.

(2) *If a real number κ satisfies the condition $\kappa+\nu > -\frac{n}{2\alpha}$, then the derivative $\mathscr{D}_t^{\kappa}\partial_x^{\gamma}\mathscr{D}_t^{\nu}W^{(\alpha)}(x,t)$ is well-defined, and*

$$\mathscr{D}_t^{\kappa}\partial_x^{\gamma}\mathscr{D}_t^{\nu}W^{(\alpha)}(x,t) = \partial_x^{\gamma}\mathscr{D}_t^{\kappa+\nu}W^{(\alpha)}(x,t).$$

(3) *The derivative $\partial_x^{\gamma}\mathscr{D}_t^{\nu}W^{(\alpha)}(x,t)$ is $L^{(\alpha)}$-harmonic on H.*

We also give basic properties of fractional derivatives of functions in $\mathscr{B}_\alpha(\sigma)$.

Proposition 5.4. *Let $0 < \alpha \le 1$, $\sigma > -m(\alpha)$, and let $\gamma \in \mathbf{N}_0^n$ be a multi-index and κ a real number such that $\kappa = 0$ or $\kappa > \max\{0,-\sigma\}$. If $u \in \mathscr{B}_\alpha(\sigma)$, then the following statements hold.*

(1) *The derivatives $\partial_x^{\gamma}\mathscr{D}_t^{\kappa}u(x,t)$ and $\mathscr{D}_t^{\kappa}\partial_x^{\gamma}u(x,t)$ can be defined, and the equation $\partial_x^{\gamma}\mathscr{D}_t^{\kappa}u(x,t) = \mathscr{D}_t^{\kappa}\partial_x^{\gamma}u(x,t)$ holds. Furthermore, if $(\gamma,\kappa) \ne (0,0)$, then there exists a constant $C = C(n,\alpha,\sigma,\gamma,\kappa) > 0$ such that*

$$|\partial_x^{\gamma}\mathscr{D}_t^{\kappa}u(x,t)| \le Ct^{-(|\gamma|/2\alpha+\kappa+\sigma)}\|u\|_{\mathscr{B}_\alpha(\sigma)}$$

for all $(x,t) \in H$.

(2) *If $\nu = 0$ or $\nu > \max\{0,-\sigma\}$, then*

$$\mathscr{D}_t^{\nu}\partial_x^{\gamma}\mathscr{D}_t^{\kappa}u(x,t) = \partial_x^{\gamma}\mathscr{D}_t^{\nu+\kappa}u(x,t) \tag{5.5}$$

Furthermore, if $\nu < 0$, then (5.5) also holds whenever $\nu < \sigma$ and $\nu+\kappa > \max\{0,-\sigma\}$.

(3) *The derivative $\partial_x^{\gamma}\mathscr{D}_t^{\kappa}u$ is $L^{(\alpha)}$-harmonic on H.*

Proof. (1) Let $\kappa > \max\{0,-\sigma\}$. Then, by (4) of Theorem 3.2, we have $|\mathscr{D}_t^{\lceil\kappa\rceil}u(x,t)| \le Ct^{-(\lceil\kappa\rceil+\sigma)}$, because $\lceil\kappa\rceil \in \mathbf{N}$. Since $\kappa > -\sigma$, $\mathscr{D}_t^{\lceil\kappa\rceil}u(x,\cdot)$ belongs to $\mathscr{F}\mathscr{C}^{-(\lceil\kappa\rceil-\kappa)}$ for every $x \in \mathbf{R}^n$. Thus, $\mathscr{D}_t^{\kappa}u(x,t)$ is well-defined. Similarly, $\mathscr{D}_t^{\kappa}\partial_x^{\gamma}u(x,t)$ is well-defined, and differentiating through the integral, we obtain

$$\partial_x^{\gamma}\mathscr{D}_t^{\kappa}u = \mathscr{D}_t^{-(\lceil\kappa\rceil-\kappa)}\partial_x^{\gamma}\mathscr{D}_t^{\lceil\kappa\rceil}u = \mathscr{D}_t^{-(\lceil\kappa\rceil-\kappa)}\mathscr{D}_t^{\lceil\kappa\rceil}\partial_x^{\gamma}u = \mathscr{D}_t^{\kappa}\partial_x^{\gamma}u.$$

Therefore, $\partial_x^\gamma \mathscr{D}_t^\kappa u$ is well-defined and $\partial_x^\gamma \mathscr{D}_t^\kappa u = \mathscr{D}_t^\kappa \partial_x^\gamma u$. Furthermore, (4) of Theorem 3.2 and (2) of Example 5.2 imply that

$$\begin{aligned} |\partial_x^\gamma \partial_t^\kappa u(x,t)| &= |\mathscr{D}_t^{-(\lceil\kappa\rceil-\kappa)} \partial_x^\gamma \mathscr{D}_t^{\lceil\kappa\rceil} u(x,t)| \\ &\le C(\mathscr{D}_t^{-(\lceil\kappa\rceil-\kappa)} t^{-(|\gamma|/2\alpha+\lceil\kappa\rceil+\sigma)})\|u\|_{\mathscr{B}_\alpha(\sigma)} = Ct^{-(|\gamma|/2\alpha+\kappa+\sigma)}\|u\|_{\mathscr{B}_\alpha(\sigma)}. \end{aligned}$$

(2) By (1) of Proposition 5.4, it suffices to show that $\mathscr{D}_t^\nu \mathscr{D}_t^\kappa \partial_x^\gamma u = \mathscr{D}_t^{\nu+\kappa} \partial_x^\gamma u$. We may suppose $\kappa, \nu \ne 0$. Assume that the real number $\nu > 0$ satisfies the condition $\nu > -\sigma$. We claim that (4) of Lemma 5.1 can be applied to $\partial_x^\gamma u$. In fact, $|\mathscr{D}_t^m \partial_x^\gamma u(x,t)| \le Ct^{-(|\gamma|/2\alpha+m+\sigma)}$ for all integers $m \ge 1$ by (1) of Proposition 5.4. Thus, the condition $\kappa > -\sigma$ implies that $\mathscr{D}_t^{\ell+\lceil\kappa\rceil} \partial_x^\gamma u(x,\cdot) \in \mathscr{F}\mathscr{C}^{-(\lceil\kappa\rceil-\kappa)}$ for all integers $\ell \ge 0$, and the assumption $\nu > -\sigma$ implies that $\mathscr{D}_t^{\lceil\nu\rceil+k} \partial_x^\gamma u(x,\cdot) \in \mathscr{F}\mathscr{C}^{-(\lceil\nu\rceil-\nu)}$ for all integers $k \ge 0$. Also, the condition $\nu+\kappa > -\sigma$ implies that $\mathscr{D}_t^{\lceil\nu\rceil+\lceil\kappa\rceil} \partial_x^\gamma u(x,\cdot) \in \mathscr{F}\mathscr{C}^{-(\lceil\nu\rceil-\nu)-(\lceil\kappa\rceil-\kappa)}$. Hence, we can apply (4) of Lemma 5.1 to $\partial_x^\gamma u$, and we obtain $\mathscr{D}_t^\nu \mathscr{D}_t^\kappa \partial_x^\gamma u = \mathscr{D}_t^{\nu+\kappa} \partial_x^\gamma u$.

Assume $\nu < 0$. If $\nu < \sigma$ and $\nu+\kappa > \max\{0,-\sigma\}$, then $\nu_1 := -\nu > 0$ and $\kappa_1 := \nu+\kappa > 0$. Also, we have $\nu_1 > -\sigma$, $\kappa_1 > -\sigma$, and $\nu_1+\kappa_1 > -\sigma$. Therefore, the above argument implies that

$$\mathscr{D}_t^\nu \mathscr{D}_t^\kappa \partial_x^\gamma u = \mathscr{D}_t^\nu \mathscr{D}_t^{\nu_1+\kappa_1} \partial_x^\gamma u = \mathscr{D}_t^\nu \mathscr{D}_t^{\nu_1} \mathscr{D}_t^{\kappa_1} \partial_x^\gamma u = \mathscr{D}_t^\nu \mathscr{D}_t^{-\nu} \mathscr{D}_t^{\nu+\kappa} \partial_x^\gamma u.$$

Since (5) of Lemma 5.1 can be applied to $\mathscr{D}_t^{\nu+\kappa} \partial_x^\gamma u$ by the condition $\nu+\kappa > \max\{0,-\sigma\}$, we obtain $\mathscr{D}_t^\nu \mathscr{D}_t^{-\nu} \mathscr{D}_t^{\nu+\kappa} \partial_x^\gamma u = \mathscr{D}_t^{\nu+\kappa} \partial_x^\gamma u$.

(3) Since when $\kappa \in \mathbf{N}_0$, the assertion was already obtained by (4) of Theorem 3.2, we assume that $\kappa \notin \mathbf{N}_0$. Let $(\gamma,\kappa) \ne (0,0)$. And let $\psi \in C_c^\infty(H)$. Then, by (2.2) and (2.3), there exist $0 < t_1 < t_2 < \infty$ and $C > 0$ such that

$$|\tilde{L}^{(\alpha)}\psi(x,t)| \le C(1+|x|)^{-n-2\alpha} \cdot \chi_{[t_1,t_2]}(t)$$

for all $(x,t) \in H$, where $\chi_{[t_1,t_2]}$ is the characteristic function of the interval $[t_1,t_2]$. Therefore, by (4) of Theorem 3.2, we have

$$\begin{aligned} &\int_0^\infty \tau^{\lceil\kappa\rceil-\kappa-1} \int_H |\mathscr{D}_t^{\lceil\kappa\rceil} \partial_x^\gamma u(x,t+\tau)\tilde{L}^{(\alpha)}\psi(x,t)|dV(x,t)d\tau \\ &\quad \le C\int_0^\infty \tau^{\lceil\kappa\rceil-\kappa-1} \int_{t_1}^{t_2}\int_{\mathbf{R}^n} (t+\tau)^{-(|\gamma|/2\alpha+\lceil\kappa\rceil+\sigma)}(1+|x|)^{-n-2\alpha}dxdtd\tau \\ &\quad \le C\int_0^\infty \tau^{\lceil\kappa\rceil-\kappa-1}(1+\tau)^{-(|\gamma|/2\alpha+\lceil\kappa\rceil+\sigma)}d\tau < \infty. \end{aligned}$$

Since $\partial_x^\gamma \mathscr{D}_t^\kappa u = \mathscr{D}_t^\kappa \partial_x^\gamma u$, the Fubini theorem implies $\partial_x^\gamma \mathscr{D}_t^\kappa u$ is $L^{(\alpha)}$-harmonic. $\square$

It is known that the parabolic Bergman functions satisfy the following reproducing formulae, which are shown in Theorem 5.2 of [2].

Lemma 5.5 (Theorem 5.2 of [2]). *Let* $0 < \alpha \le 1$, $1 \le p < \infty$, *and* $\lambda > -1$. *If real numbers* κ *and* ν *satisfy* $\kappa > -\frac{\lambda+1}{p}$ *and* $\nu > \frac{\lambda+1}{p}$, *then*

$$(5.6)\quad u(x,t) = \frac{2^{\kappa+\nu}}{\Gamma(\kappa+\nu)} \int_H \mathscr{D}_t^{\kappa} u(y,s) \mathscr{D}_t^{\nu} W^{(\alpha)}(x-y, t+s) s^{\kappa+\nu-1}\, dV(y,s)$$

for all $u \in \boldsymbol{b}_\alpha^p(\lambda)$ *and* $(x,t) \in H$. *Furthermore,* (5.6) *also holds for* $\nu = \lambda + 1$ *when* $p = 1$.

We shall give reproducing formulae by fractional derivatives on $\tilde{\mathscr{B}}_\alpha(\sigma)$, which are generalizations of Theorem 2 in section 1. First, we generalize the function defined in (4.1) as follows. For a multi-index $\gamma \in \mathbf{N}_0^n$ and a real number $\nu > -\frac{n}{2\alpha}$, Lemma 5.3 implies that a function $\omega_\alpha^{\gamma,\nu}$ on $H \times H$ can be defined by

$$\omega_\alpha^{\gamma,\nu}(x,t;y,s) = \partial_x^{\gamma} \mathscr{D}_t^{\nu} W^{(\alpha)}(x-y, t+s) - \partial_x^{\gamma} \mathscr{D}_t^{\nu} W^{(\alpha)}(-y, 1+s)$$

for all $(x,t), (y,s) \in H$. We shall also write $\omega_\alpha^{\nu} = \omega_\alpha^{0,\nu}$. We give basic properties of the function $\omega_\alpha^{\gamma,\nu}$.

Lemma 5.6. *Let* $0 < \alpha \le 1$, $\sigma > -m(\alpha)$, $\gamma \in \mathbf{N}_0^n$, *and* $\nu > -\frac{n}{2\alpha}$. *Then, the following statements hold.*

(1) *For every* $(x,t) \in H$, *there exists a constant* $C = C(n,\alpha,\gamma,\nu,x,t) > 0$ *such that*

$$|\omega_\alpha^{\gamma,\nu}(x,t;y,s)| \le C(1 + s + |y|^{2\alpha})^{-(n+|\gamma|)/2\alpha - \nu - m(\alpha)}$$

for all $(y,s) \in H$.

(2) *If* $\rho > -1$ *and* $\eta := \frac{|\gamma|}{2\alpha} + \nu - \rho - 1 > -m(\alpha)$, *then there exists a constant* $C = C(n,\alpha,\gamma,\nu,\rho) > 0$ *such that*

$$\int_H |\omega_\alpha^{\gamma,\nu}(x,t;y,s)| s^{\rho}\, dV(y,s) \le C F_{\alpha,\eta}(x,t)$$

for all $(x,t) \in H$, *where the function* $F_{\alpha,\eta}$ *is defined in* (3.2).

(3) *If* $\frac{n+|\gamma|}{2\alpha} + \nu + m(\alpha) > \sigma$, *then for every* $(x,t) \in H$, *the function* $\omega_\alpha^{\gamma,\nu}(x,t;\cdot,\cdot)$ *belongs to* $\tilde{\mathscr{B}}_{\alpha,0}(\sigma)$.

Proof. The assertion (1) is (2) of Proposition 3.1 of [3]. (2) Let $c > 0$ be an arbitrary real number. Then, (1) of Lemma 5.3 and Lemma 2.3 imply that

$$(5.7)\qquad \int_H |\omega_\alpha^{\gamma,\nu}(x,t;y,s)| s^\rho \, dV(y,s)$$

$$\leq \int_H |\partial_x^\gamma \mathscr{D}_t^\nu W^{(\alpha)}(x-y,t+s) - \partial_x^\gamma \mathscr{D}_t^\nu W^{(\alpha)}(x-y,c+s)| s^\rho \, dV(y,s)$$

$$+ \int_H |\partial_x^\gamma \mathscr{D}_t^\nu W^{(\alpha)}(x-y,c+s) - \partial_x^\gamma \mathscr{D}_t^\nu W^{(\alpha)}(-y,c+s)| s^\rho \, dV(y,s)$$

$$+ \int_H |\partial_x^\gamma \mathscr{D}_t^\nu W^{(\alpha)}(-y,c+s) - \partial_x^\gamma \mathscr{D}_t^\nu W^{(\alpha)}(-y,1+s)| s^\rho \, dV(y,s)$$

$$\leq \left| \int_c^t \int_H |\partial_x^\gamma \mathscr{D}_t^{\nu+1} W^{(\alpha)}(x-y,\tau+s)| s^\rho \, dV(y,s) d\tau \right|$$

$$+ \int_0^1 |x| \int_H |\nabla_x \partial_x^\gamma \mathscr{D}_t^\nu W^{(\alpha)}(rx-y,c+s)| s^\rho \, dV(y,s) dr$$

$$+ \left| \int_1^c \int_H |\partial_x^\gamma \mathscr{D}_t^{\nu+1} W^{(\alpha)}(-y,\tau+s)| s^\rho \, dV(y,s) d\tau \right|$$

$$\leq C\left(\left| \int_c^t \tau^{\rho-|\gamma|/2\alpha-\nu} \, d\tau \right| + |x| c^{\rho-(|\gamma|+1)/2\alpha-\nu+1} + \left| \int_1^c \tau^{\rho-|\gamma|/2\alpha-\nu} \, d\tau \right| \right)$$

$$= C\left(\left| \int_c^t \tau^{-\eta-1} \, d\tau \right| + |x| c^{-\eta-1/2\alpha} + \left| \int_1^c \tau^{-\eta-1} \, d\tau \right| \right).$$

Assume $\eta = 0$, then

$$\int_H |\omega_\alpha^{\gamma,\nu}(x,t;y,s)| s^\rho \, dV(y,s) \leq C(I_x(c) + |\log t|),$$

where $I_x(c) = |\log c| + |x| c^{-1/2\alpha}$. Thus, as in the proof of (1) of Theorem 3.2, putting $c = (1+|x|)^{2\alpha}$, we obtain

$$\int_H |\omega_\alpha^{\gamma,\nu}(x,t;y,s)| s^\rho \, dV(y,s) \leq C(1 + \log(1+|x|) + |\log t|).$$

Assume $\eta \neq 0$, then (5.7) implies

$$\int_H |\omega_\alpha^{\gamma,\nu}(x,t;y,s)| s^\rho \, dV(y,s) \leq C(1 + J_{x,\eta}(c) + t^{-\eta}),$$

where $J_{x,\eta}(c) = c^{-\eta} + |x| c^{-\eta-1/2\alpha}$. Therefore, the same argument as in the proof of (1) of Theorem 3.2 shows the desired estimates.

(3) Let $(x,t) \in H$ be fixed. Then, by (3) of Lemma 5.3, the function $\omega_\alpha^{\gamma,\nu}(x,t;\cdot,\cdot)$ is $L^{(\alpha)}$-harmonic. Furthermore, (1) of Lemma 5.6 implies that for $j = 1,\dots,n$,

$$|\partial_{y_j} \omega_\alpha^{\gamma,\nu}(x,t;y,s)| \leq C(1 + s + |y|^{2\alpha})^{-(n+|\gamma|+1)/2\alpha-\nu-m(\alpha)}$$

and

$$|\mathscr{D}_s \omega_\alpha^{\gamma,\nu}(x,t;y,s)| \le C(1+s+|y|^{2\alpha})^{-(n+|\gamma|)/2\alpha-\nu-1-m(\alpha)}$$

for all $(y,s) \in H$. Hence, we obtain the function $\omega_\alpha^{\gamma,\nu}(x,t;\cdot,\cdot)$ belongs to $\tilde{\mathscr{B}}_{\alpha,0}(\sigma)$. $\square$

We define an auxiliary function on $\mathbf{R}$, which is used later. For $\nu \in \mathbf{R}$, let

$$\mathscr{N}(\nu) = \begin{cases} \lceil \nu \rceil & (\nu \ge 0) \\ 0 & (\nu < 0). \end{cases}$$

Now, we give reproducing formulae by fractional derivatives on $\mathscr{B}_\alpha(\sigma)$.

THEOREM 5.7. *Let $0 < \alpha \le 1$ and $\sigma > -m(\alpha)$. If real numbers $\kappa \in \mathbf{R}_+$ and $\nu \in \mathbf{R}$ satisfy $\kappa > -\sigma$ and $\nu > \sigma$, then*

$$(5.8) \quad u(x,t) - u(0,1) = \frac{2^{\kappa+\nu}}{\Gamma(\kappa+\nu)} \int_H \mathscr{D}_t^\kappa u(y,s) \omega_\alpha^\nu(x,t;y,s) s^{\kappa+\nu-1}\, dV(y,s)$$

for all $u \in \mathscr{B}_\alpha(\sigma)$ and $(x,t) \in H$. If $\kappa = 0$ and $\nu > \max\{0,\sigma\}$, then (5.8) also holds.

PROOF. Let $u \in \mathscr{B}_\alpha(\sigma)$ and $(x,t) \in H$. And, let $\kappa \in \mathbf{R}_+$ and $\nu \in \mathbf{R}$ be real numbers with $\kappa > -\sigma$ and $\nu > \sigma$.

Suppose first that $\kappa \notin \mathbf{N}$ and $\nu \notin \mathbf{N}_0$. Then, the definitions of the fractional derivative (5.2) and (5.4) imply that

$$\begin{aligned}
(5.9) \quad & \int_H \mathscr{D}_t^\kappa u(y,s) \omega_\alpha^\nu(x,t;y,s) s^{\kappa+\nu-1}\, dV(y,s) \\
&= \int_H \frac{1}{\Gamma(\lceil\kappa\rceil-\kappa)} \int_0^\infty \tau_1^{\lceil\kappa\rceil-\kappa-1} \mathscr{D}_t^{\lceil\kappa\rceil} u(y,s+\tau_1) d\tau_1 \\
&\quad \times \frac{1}{\Gamma(\mathscr{N}(\nu)-\nu)} \int_0^\infty \tau_2^{\mathscr{N}(\nu)-\nu-1} \omega_\alpha^{\mathscr{N}(\nu)}(x,t;y,s+\tau_2) d\tau_2 s^{\kappa+\nu-1}\, dV(y,s) \\
&= \int_H \frac{1}{\Gamma(\lceil\kappa\rceil-\kappa)} \int_0^\infty \tau_1^{\lceil\kappa\rceil-\kappa-1} \mathscr{D}_t^{\lceil\kappa\rceil} u(y,(1+\tau_1)s) d\tau_1 \\
&\quad \times \frac{1}{\Gamma(\mathscr{N}(\nu)-\nu)} \int_0^\infty \tau_2^{\mathscr{N}(\nu)-\nu-1} \omega_\alpha^{\mathscr{N}(\nu)}(x,t;y,(1+\tau_2)s) d\tau_2 \\
&\quad \times s^{\lceil\kappa\rceil+\mathscr{N}(\nu)-1}\, dV(y,s).
\end{aligned}$$

Furthermore, (4) of Theorem 3.2 and Lemma 4.1 imply that

$$\int_H \int_0^\infty \tau_1^{\lceil\kappa\rceil-\kappa-1} |\mathscr{D}_t^{\lceil\kappa\rceil} u(y,(1+\tau_1)s)| d\tau_1$$
$$\times \int_0^\infty \tau_2^{\mathscr{N}(\nu)-\nu-1} |\omega_\alpha^{\mathscr{N}(\nu)}(x,t;y,(1+\tau_2)s)| d\tau_2 s^{\lceil\kappa\rceil+\mathscr{N}(\nu)-1}\, dV(y,s)$$
$$\leq C \int_H \int_0^\infty \frac{\tau_1^{\lceil\kappa\rceil-\kappa-1}}{((1+\tau_1)s)^{\lceil\kappa\rceil+\sigma}}\, d\tau_1$$
$$\times \int_0^\infty \frac{\tau_2^{\mathscr{N}(\nu)-\nu-1}}{(1+(1+\tau_2)s+|y|^{2\alpha})^{n/2\alpha+\mathscr{N}(\nu)+m(\alpha)}}\, d\tau_2 s^{\lceil\kappa\rceil+\mathscr{N}(\nu)-1}\, dV(y,s)$$
$$= C \int_0^\infty \frac{\tau_1^{\lceil\kappa\rceil-\kappa-1}}{(1+\tau_1)^{\lceil\kappa\rceil+\sigma}}\, d\tau_1 \int_0^\infty \frac{\tau_2^{\mathscr{N}(\nu)-\nu-1}}{(1+\tau_2)^{-\sigma+\mathscr{N}(\nu)}}\, d\tau_2$$
$$\times \int_H \frac{s^{-\sigma+\mathscr{N}(\nu)-1}}{(1+s+|y|^{2\alpha})^{n/2\alpha+\mathscr{N}(\nu)+m(\alpha)}}\, dV(y,s).$$

Since $\kappa > -\sigma$ and $\nu > \sigma$, we have

$$\int_0^\infty \frac{\tau_1^{\lceil\kappa\rceil-\kappa-1}}{(1+\tau_1)^{\lceil\kappa\rceil+\sigma}}\, d\tau_1 < \infty$$

and

$$\int_0^\infty \frac{\tau_2^{\mathscr{N}(\nu)-\nu-1}}{(1+\tau_2)^{-\sigma+\mathscr{N}(\nu)}}\, d\tau_2 < \infty,$$

respectively. Moreover, by the conditions $\nu > \sigma$ and $\sigma > -m(\alpha)$, Lemma 2.3 implies that

$$\int_H \frac{s^{-\sigma+\mathscr{N}(\nu)-1}}{(1+s+|y|^{2\alpha})^{n/2\alpha+\mathscr{N}(\nu)+m(\alpha)}}\, dV(y,s) < \infty.$$

Hence, by the Fubini theorem, (5.9) and Theorem 4.5 show that

$$\int_H \mathscr{D}_t^\kappa u(y,s) \omega_\alpha^\nu(x,t;y,s) s^{\kappa+\nu-1}\, dV(y,s)$$
$$= \frac{1}{\Gamma(\lceil\kappa\rceil-\kappa)\Gamma(\mathscr{N}(\nu)-\nu)} \int_0^\infty \tau_1^{\lceil\kappa\rceil-\kappa-1} \int_0^\infty \tau_2^{\mathscr{N}(\nu)-\nu-1}$$
$$\times \int_H \mathscr{D}_t^{\lceil\kappa\rceil} u(y,(1+\tau_1)s) \omega_\alpha^{\mathscr{N}(\nu)}(x,t;y,(1+\tau_2)s) s^{\lceil\kappa\rceil+\mathscr{N}(\nu)-1}\, dV(y,s) d\tau_1 d\tau_2$$

$$= \frac{1}{\Gamma(\lceil\kappa\rceil - \kappa)\Gamma(\mathcal{N}(\nu) - \nu)} \int_0^\infty \tau_1^{\lceil\kappa\rceil-\kappa-1} \int_0^\infty \tau_2^{\mathcal{N}(\nu)-\nu-1}$$

$$\times \frac{\Gamma(\lceil\kappa\rceil + \mathcal{N}(\nu))}{(2+\tau_1+\tau_2)^{\lceil\kappa\rceil+\mathcal{N}(\nu)}} (u(x,t) - u(0,1)) d\tau_1 d\tau_2$$

$$= \frac{\Gamma(\kappa+\nu)}{2^{\kappa+\nu}} (u(x,t) - u(0,1)).$$

Next, we remark that the proof can be done similarly when $\kappa \in \mathbf{N}$ or $\nu \in \mathbf{N}_0$. Thus, we omit it. (When $\kappa \in \mathbf{N}$ and $\nu \in \mathbf{N}_0$, the assertion of the theorem follows from Theorem 4.5.)

Finally, we assume that $\kappa = 0$ and $\nu > \max\{0, \sigma\}$. When $\kappa = 0$ and $\nu \in \mathbf{N}$, the assertion of the theorem follows from Theorem 4.5. Therefore, we suppose $\kappa = 0$ and $\nu \notin \mathbf{N}$. Since

$$\begin{aligned}(5.10)\qquad & \int_H u(y,s)\omega_\alpha^\nu(x,t;y,s)s^{\nu-1}\, dV(y,s) \\ &= \int_H u(y,s) \frac{1}{\Gamma(\lceil\nu\rceil - \nu)} \\ &\quad \times \int_0^\infty \tau^{\lceil\nu\rceil-\nu-1}\omega_\alpha^{\lceil\nu\rceil}(x,t;y,(1+\tau)s)d\tau s^{\lceil\nu\rceil-1}\, dV(y,s),\end{aligned}$$

it suffices to show that we can apply the Fubini theorem to the right-hand side of the equality (5.10). Since $\nu > 0$, we can choose a constant ε with $0 < \varepsilon < \min\{\nu, m(\alpha)\}$. Then, (1) of Lemma 4.3 implies that $|u(y,s)| \le CM_{\alpha,\sigma,\varepsilon}(y,s)$ for all $(y,s) \in H$, where $M_{\alpha,\sigma,\varepsilon}$ is the function defined in (4.2). Therefore, Lemma 4.1 shows that

$$\begin{aligned}(5.11)\qquad & \int_H |u(y,s)| \int_0^\infty \tau^{\lceil\nu\rceil-\nu-1}|\omega_\alpha^{\lceil\nu\rceil}(x,t;y,(1+\tau)s)|d\tau s^{\lceil\nu\rceil-1}\, dV(y,s) \\ &\le C\int_H M_{\alpha,\sigma,\varepsilon}(y,s) \\ &\quad \times \int_0^\infty \frac{\tau^{\lceil\nu\rceil-\nu-1}}{(1+(1+\tau)s+|y|^{2\alpha})^{n/2\alpha+\lceil\nu\rceil+m(\alpha)}}\, d\tau s^{\lceil\nu\rceil-1}\, dV(y,s) \\ &= C\int_0^\infty \frac{\tau^{\lceil\nu\rceil-\nu-1}}{(1+\tau)^{\lceil\nu\rceil}} \int_H \frac{M_{\alpha,\sigma,\varepsilon}(y,(1+\tau)^{-1}s)s^{\lceil\nu\rceil-1}}{(1+s+|y|^{2\alpha})^{n/2\alpha+\lceil\nu\rceil+m(\alpha)}}\, dV(y,s)d\tau.\end{aligned}$$

If $0 > \sigma > -m(\alpha)$, then the right-hand side of the equality (5.11) is less than or equal to

$$(5.12)\qquad C\int_0^\infty \frac{\tau^{\lceil\nu\rceil-\nu-1}}{(1+\tau)^{\lceil\nu\rceil}}\, d\tau \int_H \frac{s^{\lceil\nu\rceil-1}}{(1+s+|y|^{2\alpha})^{n/2\alpha+\lceil\nu\rceil+m(\alpha)+\sigma}}\, dV(y,s),$$

and by the conditions $\nu > 0$ and $-m(\alpha) - \sigma < 0$, Lemma 2.3 implies that (5.12) is finite. If $\sigma = 0$, then the right-hand side of the equality (5.11) is less than or equal to

$$(5.13)\qquad C\int_0^\infty \frac{\tau^{\lceil\nu\rceil-\nu-1}}{(1+\tau)^{\lceil\nu\rceil}}\,d\tau \int_H \frac{s^{\lceil\nu\rceil-1}}{(1+s+|y|^{2\alpha})^{n/2\alpha+\lceil\nu\rceil+m(\alpha)-\varepsilon}}\,dV(y,s) + C\int_0^\infty \frac{\tau^{\lceil\nu\rceil-\nu-1}}{(1+\tau)^{\lceil\nu\rceil-\varepsilon}}\,d\tau \int_H \frac{s^{\lceil\nu\rceil-1-\varepsilon}}{(1+s+|y|^{2\alpha})^{n/2\alpha+\lceil\nu\rceil+m(\alpha)}}\,dV(y,s).$$

Here, the first term of (5.13) is finite because $\nu > 0$ and $-m(\alpha) + \varepsilon < 0$, and the second term of (5.13) is finite because $\nu - \varepsilon > 0$ and $-\varepsilon - m(\alpha) < 0$, respectively. If $\sigma > 0$, then the right-hand side of the equality (5.11) is less than or equal to

$$(5.14)\qquad C\int_0^\infty \frac{\tau^{\lceil\nu\rceil-\nu-1}}{(1+\tau)^{\lceil\nu\rceil}}\,d\tau \int_H \frac{s^{\lceil\nu\rceil-1}}{(1+s+|y|^{2\alpha})^{n/2\alpha+\lceil\nu\rceil+m(\alpha)}}\,dV(y,s) + C\int_0^\infty \frac{\tau^{\lceil\nu\rceil-\nu-1}}{(1+\tau)^{\lceil\nu\rceil-\sigma}}\,d\tau \int_H \frac{s^{\lceil\nu\rceil-1-\sigma}}{(1+s+|y|^{2\alpha})^{n/2\alpha+\lceil\nu\rceil+m(\alpha)}}\,dV(y,s),$$

and thus the first term of (5.14) is finite by the conditions $\nu > 0$ and $-m(\alpha) < 0$, and the second term of (5.14) is finite by the conditions $\nu - \sigma > 0$ and $-\sigma - m(\alpha) < 0$, respectively. Hence, this completes the proof of the theorem. □

As an application of the reproducing formula, we give estimates of the normal derivative norms on $\tilde{\mathscr{B}}_\alpha(\sigma)$. The following operator is important for our estimates and is also used in the next section. For $0 < \alpha \le 1$, $\kappa > -\frac{n}{2\alpha}$, and $\rho \in \mathbf{R}$, the integral operator $\Pi_\alpha^{\kappa,\rho}$ is defined by

$$(5.15)\qquad \Pi_\alpha^{\kappa,\rho} f(x,t) := \int_H f(y,s)\omega_\alpha^\kappa(x,t;y,s)s^\rho\,dV(y,s)$$

for $(x,t) \in H$, whenever the integral is well-defined. We need the following.

THEOREM 5.8. *Let $0 < \alpha \le 1$ and $\sigma > -m(\alpha)$. Then, for every real number $\nu > 0$, $\Pi_\alpha^{\nu+\sigma,\nu-1}$ is a bounded linear operator from L^∞ onto $\tilde{\mathscr{B}}_\alpha(\sigma)$.*

PROOF. Let $f \in L^\infty$ and $(x,t) \in H$. Then, by (1) of Lemma 5.6 and Lemma 2.3, $\Pi_\alpha^{\nu+\sigma,\nu-1} f(x,t)$ is well-defined. Furthermore, we show $\Pi_\alpha^{\nu+\sigma,\nu-1} f \in \tilde{\mathscr{B}}_\alpha(\sigma)$ and there exists a constant $C > 0$ independent of f such that $\|\Pi_\alpha^{\nu+\sigma,\nu-1} f\|_{\mathscr{B}_\alpha(\sigma)} \le C\|f\|_{L^\infty}$. In fact, by (2) of Lemma 5.6, for every $0 < t_1 < t_2 < \infty$, we have

$$\int_{t_1}^{t_2}\int_{\mathbf{R}^n} |\Pi_\alpha^{\nu+\sigma,\nu-1} f(x,t)|(1+|x|)^{-n-2\alpha}dxdt$$
$$\le C\|f\|_{L^\infty}\int_{t_1}^{t_2}\int_{\mathbf{R}^n} F_{\alpha,\sigma}(x,t)(1+|x|)^{-n-2\alpha}dxdt < \infty,$$

where $F_{\alpha,\sigma}$ is the function defined in (3.2). Therefore, $\Pi_\alpha^{\nu+\sigma,\nu-1} f$ satisfies the condition (2.4). Thus, by the definition of $\omega_\alpha^\kappa(x,t;y,s)$, $\Pi_\alpha^{\nu+\sigma,\nu-1} f$ is $L^{(\alpha)}$-harmonic and $\Pi_\alpha^{\nu+\sigma,\nu-1} f(0,1) = 0$. Moreover, Lemma 2.3 implies

$$|\partial_j \Pi_\alpha^{\nu+\sigma,\nu-1} f(x,t)| \le Ct^{-(\sigma+1/2\alpha)}\|f\|_{L^\infty}$$

and

$$|\partial_t \Pi_\alpha^{\nu+\sigma,\nu-1} f(x,t)| \le Ct^{-(\sigma+1)}\|f\|_{L^\infty}.$$

Hence, we have $\Pi_\alpha^{\nu+\sigma,\nu-1} f \in \tilde{\mathscr{B}}_\alpha(\sigma)$ and $\|\Pi_\alpha^{\nu+\sigma,\nu-1} f\|_{\mathscr{B}_\alpha(\sigma)} \le C\|f\|_{L^\infty}$.

Let $u \in \tilde{\mathscr{B}}_\alpha(\sigma)$. Then, (4) of Theorem 3.2 implies $t^{1+\sigma}\mathscr{D}_t u \in L^\infty$. By Theorem 5.7 with $\kappa = 1$, we have $u = \frac{2^{1+\nu+\sigma}}{\Gamma(1+\nu+\sigma)}\Pi_\alpha^{\nu+\sigma,\nu-1}(t^{1+\sigma}\mathscr{D}_t u)$. Thus, $\Pi_\alpha^{\nu+\sigma,\nu-1}$ is onto. □

We give estimates of the normal derivative norms on $\tilde{\mathscr{B}}_\alpha(\sigma)$.

Theorem 5.9. *Let $0 < \alpha \le 1$ and $\sigma > -m(\alpha)$. Then, for every real number $\kappa > \max\{0, -\sigma\}$, there exists a constant $C = C(n,\alpha,\sigma,\kappa) > 0$ independent of u such that*

$$C^{-1}\|u\|_{\mathscr{B}_\alpha(\sigma)} \le \|t^{\kappa+\sigma}\mathscr{D}_t^\kappa u\|_{L^\infty} \le C\|u\|_{\mathscr{B}_\alpha(\sigma)}$$

for all $u \in \tilde{\mathscr{B}}_\alpha(\sigma)$.

Proof. Let $\kappa > \max\{0, -\sigma\}$ be a real number and $u \in \tilde{\mathscr{B}}_\alpha(\sigma)$. Then, (1) of Proposition 5.4 implies that

$$\|t^{\kappa+\sigma}\mathscr{D}_t^\kappa u\|_{L^\infty} \le C\|u\|_{\mathscr{B}_\alpha(\sigma)}.$$

Furthermore, by Theorem 5.7, we have $u = \frac{2^{1+\kappa+\sigma}}{\Gamma(1+\kappa+\sigma)}\Pi_\alpha^{1+\sigma,0}(t^{\kappa+\sigma}\mathscr{D}_t^\kappa u)$. Therefore, Theorem 5.8 with $\nu = 1$ implies that

$$\|u\|_{\mathscr{B}_\alpha(\sigma)} = C\|\Pi_\alpha^{1+\sigma,0}(t^{\kappa+\sigma}\mathscr{D}_t^\kappa u)\|_{\mathscr{B}_\alpha(\sigma)} \le C\|t^{\kappa+\sigma}\mathscr{D}_t^\kappa u\|_{L^\infty}.$$

This completes the proof. □

6. Dual spaces

In this section, we give the proofs of Theorems 3 and 4. We begin with recalling the definition of the integral pairing (1.4) on $\boldsymbol{b}_\alpha^1(\lambda) \times \tilde{\mathscr{B}}_\alpha(\sigma)$. For $u \in \boldsymbol{b}_\alpha^1(\lambda)$ and $v \in \tilde{\mathscr{B}}_\alpha(\sigma)$, the integral pairing $\langle u, v\rangle_{\lambda,\sigma}$ in (1.4) is defined by

$$\langle u, v\rangle_{\lambda,\sigma} = \frac{2^{\lambda+\sigma+2}}{\Gamma(\lambda+\sigma+2)} \int_H u(y,s)\mathscr{D}_t v(y,s) s^{\lambda+\sigma+1}\, dV(y,s).$$

By the definition, we clearly have there exists a constant $C > 0$ such that

$$|\langle u, v\rangle_{\lambda,\sigma}| \le C\|u\|_{L^1(\lambda)}\|v\|_{\mathscr{B}_\alpha(\sigma)} \tag{6.1}$$

for all $u \in \boldsymbol{b}_\alpha^1(\lambda)$ and $v \in \tilde{\mathscr{B}}_\alpha(\sigma)$.

THEOREM 6.1. *Let $0 < \alpha \le 1$, $\sigma > -m(\alpha)$, and $\lambda > -1$. Then, $(\boldsymbol{b}_\alpha^1(\lambda))^* \cong \tilde{\mathscr{B}}_\alpha(\sigma)$ under the pairing*

$$\Phi_v(u) := \langle u, v\rangle_{\lambda,\sigma}, \quad u \in \boldsymbol{b}_\alpha^1(\lambda),$$

where Φ_v is the linear functional on $\boldsymbol{b}_\alpha^1(\lambda)$ induced by $v \in \tilde{\mathscr{B}}_\alpha(\sigma)$. Furthermore, there exists a constant $C = C(n, \alpha, \sigma, \lambda) > 0$ independent of v such that

$$C^{-1}\|v\|_{\mathscr{B}_\alpha(\sigma)} \le \|\Phi_v\| \le C\|v\|_{\mathscr{B}_\alpha(\sigma)}$$

for all $v \in \tilde{\mathscr{B}}_\alpha(\sigma)$.

PROOF. For every $v \in \tilde{\mathscr{B}}_\alpha(\sigma)$, we define a mapping $\imath$ by $\imath(v) = \Phi_v$. Then, the inequality (6.1) implies that $\imath : \tilde{\mathscr{B}}_\alpha(\sigma) \to (\boldsymbol{b}_\alpha^1(\lambda))^*$ and $\|\Phi_v\| \le C\|v\|_{\mathscr{B}_\alpha(\sigma)}$.

We show that $\imath$ is injective. Thus, we assume that $v \in \tilde{\mathscr{B}}_\alpha(\sigma)$ and $\Phi_v = \imath(v) = 0$. Then, by (2) of Lemma 5.6, $\omega_\alpha^{\lambda+\sigma+1}(x,t;\cdot,\cdot)$ belongs to $\boldsymbol{b}_\alpha^1(\lambda)$ for each $(x,t) \in H$. Therefore, by Theorem 5.7, we obtain

$$\begin{aligned} v(x,t) &= \frac{2^{\lambda+\sigma+2}}{\Gamma(\lambda+\sigma+2)} \int_H \mathscr{D}_t v(y,s)\omega_\alpha^{\lambda+\sigma+1}(x,t;y,s) s^{\lambda+\sigma+1}\, dV(y,s) \\ &= \Phi_v(\omega_\alpha^{\lambda+\sigma+1}(x,t;\cdot,\cdot)) = 0 \end{aligned}$$

for each $(x,t) \in H$. Hence, $\imath$ is injective.

We show that for each $\Phi \in (\boldsymbol{b}_\alpha^1(\lambda))^*$, there exists $v \in \tilde{\mathscr{B}}_\alpha(\sigma)$ such that $\imath(v) = \Phi$ and $\|v\|_{\mathscr{B}_\alpha(\sigma)} \le C\|\Phi\|$. Therefore, let $\Phi \in (\boldsymbol{b}_\alpha^1(\lambda))^*$. Then, the Hahn-Banach theorem and the Riesz representation theorem imply that there exists a function $f \in L^\infty$ such that

$$\Phi(u) = \int_H u(y,s) f(y,s) s^\lambda\, dV(x,t)$$

for all $u \in \boldsymbol{b}_\alpha^1(\lambda)$ and $\|f\|_{L^\infty} = \|\Phi\|$. Put $v := \Pi_\alpha^{\lambda+\sigma+1,\lambda} f$. Then, Theorem 5.8 implies that $v \in \tilde{\mathscr{B}}_\alpha(\sigma)$ and $\|v\|_{\mathscr{B}_\alpha(\sigma)} \le C\|\Phi\|$. We claim $\iota(v) = \Phi$. Indeed, differentiating through the integral, we have

$$\mathscr{D}_t v(x,t) = \mathscr{D}_t \Pi_\alpha^{\lambda+\sigma+1,\lambda} f(x,t) = \int_H f(y,s) \mathscr{D}_t^{\lambda+\sigma+2} W^{(\alpha)}(x-y,t+s) s^\lambda \, dV(y,s).$$

Therefore, the Fubini theorem and Lemma 5.5 imply that

$$\begin{aligned}
\langle u, v \rangle_{\lambda,\sigma} &= \frac{2^{\lambda+\sigma+2}}{\Gamma(\lambda+\sigma+2)} \int_H u(x,t) \mathscr{D}_t v(x,t) t^{\lambda+\sigma+1} \, dV(x,t) \\
&= \frac{2^{\lambda+\sigma+2}}{\Gamma(\lambda+\sigma+2)} \int_H u(x,t) \int_H f(y,s) \mathscr{D}_t^{\lambda+\sigma+2} W^{(\alpha)}(x-y,t+s) \\
&\quad \times s^\lambda \, dV(y,s) t^{\lambda+\sigma+1} \, dV(x,t) \\
&= \int_H \frac{2^{\lambda+\sigma+2}}{\Gamma(\lambda+\sigma+2)} \int_H u(x,t) \mathscr{D}_t^{\lambda+\sigma+2} W^{(\alpha)}(x-y,t+s) \\
&\quad \times t^{\lambda+\sigma+1} \, dV(x,t) f(y,s) s^\lambda \, dV(y,s) \\
&= \int_H u(y,s) f(y,s) s^\lambda \, dV(y,s) = \Phi(u)
\end{aligned}$$

for all $u \in \boldsymbol{b}_\alpha^1(\lambda)$. This completes the proof. □

Next, we give the proof of Theorem 4. Let $C_0(H)$ be the set of all continuous functions which vanish continuously at $\partial H \cup \{\infty\}$. We need the following lemma.

Lemma 6.2. *Let $0 < \alpha \le 1$, $\sigma > -m(\alpha)$, and $\nu > 0$. Then,*

$$\tilde{\mathscr{B}}_{\alpha,0}(\sigma) = \{u \in \tilde{\mathscr{B}}_\alpha(\sigma); t^{\sigma+1} \mathscr{D}_t u \in C_0(H)\} = \{\Pi_\alpha^{\nu+\sigma,\nu-1} f; f \in C_0(H)\}.$$

Proof. We show the first equality. Take $u \in \tilde{\mathscr{B}}_\alpha(\sigma)$ with $t^{\sigma+1}\mathscr{D}_t u \in C_0(H)$. Then, differentiating through the integral (5.8) with $\kappa = 1$ and $\nu = \sigma + 1$, we have

$$\partial_j u(x,t) = \frac{2^{\sigma+2}}{\Gamma(\sigma+2)} \int_H \mathscr{D}_t u(y,s) \partial_j \mathscr{D}_t^{\sigma+1} W^{(\alpha)}(x-y,t+s) s^{\sigma+1} \, dV(y,s).$$

For given $\varepsilon > 0$, there exists a compact subset $K \subset H$ such that $|s^{\sigma+1}\mathscr{D}_t u(y,s)| < \varepsilon$ for all $(y,s) \in H \backslash K$. Therefore, we obtain

$$(6.2)\qquad t^{\sigma+1/2\alpha}|\partial_j u(x,t)|$$
$$\leq Ct^{\sigma+1/2\alpha}\varepsilon\int_{H\backslash K}|\partial_j\mathscr{D}_t^{\sigma+1}W^{(\alpha)}(x-y,t+s)|dV(y,s)$$
$$+ Ct^{\sigma+1/2\alpha}\|u\|_{\mathscr{B}_\alpha(\sigma)}\int_K|\partial_j\mathscr{D}_t^{\sigma+1}W^{(\alpha)}(x-y,t+s)|dV(y,s).$$

The first term of the right-hand side of (6.2) is less than $C\varepsilon$ by (1) of Lemma 5.3 and Lemma 2.3. Furthermore, (1) of Lemma 5.3 implies that the second term of the right-hand side of (6.2) tends to 0 as $(x,t)\to\partial H\cup\{\infty\}$. It follows that $u\in\tilde{\mathscr{B}}_{\alpha,0}(\sigma)$. The converse inclusion is trivial by the definition of $\tilde{\mathscr{B}}_{\alpha,0}(\sigma)$.

We show the second equality. Take $f\in C_0(H)$, and put $u=\Pi_\alpha^{\nu+\sigma,\nu-1}f$. Then, Theorem 5.8 implies $u\in\tilde{\mathscr{B}}_\alpha(\sigma)$. For given $\varepsilon>0$, there exists a compact subset $K\subset H$ such that $|f(y,s)|<\varepsilon$ for all $(y,s)\in H\backslash K$. Thus, differentiating through the integral, we have

$$t^{\sigma+1}|\mathscr{D}_t u(x,t)|\leq t^{\sigma+1}\varepsilon\int_{H\backslash K}|\mathscr{D}_t^{\nu+\sigma+1}W^{(\alpha)}(x-y,t+s)|s^{\nu-1}\,dV(y,s)$$
$$+ t^{\sigma+1}\|f\|_{L^\infty}\int_K|\mathscr{D}_t^{\nu+\sigma+1}W^{(\alpha)}(x-y,t+s)|s^{\nu-1}\,dV(y,s).$$

Therefore, by the similar argument as above, we obtain $t^{\sigma+1}\mathscr{D}_t u\in C_0(H)$. We can easily show the converse inclusion by Theorem 5.7. This completes the proof. $\square$

We shall show an extended version of Theorem 4.

Theorem 6.3. *Let $0<\alpha\leq 1$, $\sigma>-m(\alpha)$, and $\lambda>-1$. Then, $\boldsymbol{b}_\alpha^1(\lambda)\cong(\tilde{\mathscr{B}}_{\alpha,0}(\sigma))^*$ under the pairing*

$$\Psi_u(v)=\langle u,v\rangle_{\lambda,\sigma},\quad v\in\tilde{\mathscr{B}}_{\alpha,0}(\sigma),$$

where Ψ_u is the linear functional on $\tilde{\mathscr{B}}_{\alpha,0}(\sigma)$ induced by $u\in\boldsymbol{b}_\alpha^1(\lambda)$. Furthermore, there exists a constant $C=C(n,\alpha,\sigma,\lambda)>0$ independent of u such that

$$C^{-1}\|u\|_{L^1(\lambda)}\leq\|\Psi_u\|\leq C\|u\|_{L^1(\lambda)}$$

for all $u\in\boldsymbol{b}_\alpha^1(\lambda)$.

Proof. For every $u\in\boldsymbol{b}_\alpha^1(\lambda)$, we define a mapping π by $\pi(u)=\Psi_u$. Then, the inequality (6.1) implies that $|\Psi_u(v)|\leq C\|u\|_{L^1(\lambda)}\|v\|_{\mathscr{B}_\alpha(\sigma)}$ for all $v\in\tilde{\mathscr{B}}_{\alpha,0}(\sigma)$. Thus, we can consider $\pi:\boldsymbol{b}_\alpha^1(\lambda)\to(\tilde{\mathscr{B}}_{\alpha,0}(\sigma))^*$ and we also have $\|\Psi_u\|\leq C\|u\|_{L^1(\lambda)}$.

We show that π is injective. We assume that $u \in \boldsymbol{b}_\alpha^1(\lambda)$ and $\Psi_u = \pi(u) = 0$. Then, by (3) of Lemma 5.6, $\omega_\alpha^{\lambda+\sigma+1}(x,t;\cdot,\cdot)$ belongs to $\tilde{\mathscr{B}}_{\alpha,0}(\sigma)$ for each $(x,t) \in H$. Therefore, by Lemma 5.5, we obtain

$$\begin{aligned}
u(x,t) &= \frac{2^{\lambda+\sigma+2}}{\Gamma(\lambda+\sigma+2)} \int_H u(y,s) \mathscr{D}_t^{\lambda+\sigma+2} W^{(\alpha)}(x-y,t+s) s^{\lambda+\sigma+1}\, dV(y,s) \\
&= \frac{2^{\lambda+\sigma+2}}{\Gamma(\lambda+\sigma+2)} \int_H u(y,s) \mathscr{D}_t \omega_\alpha^{\lambda+\sigma+1}(x,t;y,s) s^{\lambda+\sigma+1}\, dV(y,s) \\
&= \Psi_u(\omega_\alpha^{\lambda+\sigma+1}(x,t;\cdot,\cdot)) = 0
\end{aligned}$$

for each $(x,t) \in H$. Hence, π is injective.

We show that for each $\Psi \in (\tilde{\mathscr{B}}_{\alpha,0}(\sigma))^*$, there exists $u \in \boldsymbol{b}_\alpha^1(\lambda)$ such that $\pi(u) = \Psi$ and $\|u\|_{L^1(\lambda)} \le C\|\Psi\|$. Let $\Psi \in (\tilde{\mathscr{B}}_{\alpha,0}(\sigma))^*$. We define a mapping Λ by

$$\Lambda(f) = \frac{2^{\lambda+\sigma+2}}{\Gamma(\lambda+\sigma+2)} \Psi(\Pi_\alpha^{\lambda+\sigma+1,\lambda} f), \qquad f \in C_0(H).$$

Then, Theorem 5.8 and Lemma 6.2 imply that Λ is a bounded linear functional on $C_0(H)$ and $\|\Lambda\| \le C\|\Psi\|$. Thus, the Riesz representation theorem shows that there exists a bounded signed measure μ on H such that

$$\Lambda(f) = \int_H f(x,t)\, d\mu(x,t), \qquad f \in C_0(H)$$

and $\|\mu\| = \|\Lambda\|$. We define a function u on H by

$$u(y,s) = \int_H \mathscr{D}_t^{\lambda+\sigma+2} W^{(\alpha)}(x-y,t+s) t^{\sigma+1}\, d\mu(x,t).$$

Then, (1) of Lemma 5.3 and Lemma 2.3 imply that

$$\begin{aligned}
\|u\|_{L^1(\lambda)} &\le \int_H \int_H |\mathscr{D}_t^{\lambda+\sigma+2} W^{(\alpha)}(x-y,t+s)| s^\lambda\, dV(y,s) t^{\sigma+1}\, d|\mu|(x,t) \\
&\le C \int_H t^{-(\sigma+1)} t^{\sigma+1}\, d|\mu|(x,t) = C\|\mu\|.
\end{aligned}$$

Hence, we have $\|u\|_{L^1(\lambda)} \le C\|\mu\| = C\|\Lambda\| \le C'\|\Psi\|$ and $u \in \boldsymbol{b}_\alpha^1(\lambda)$. We assert $\pi(u) = \Psi$. In fact, take $v \in \tilde{\mathscr{B}}_{\alpha,0}(\sigma)$. Then, since

$$v = \frac{2^{\lambda+\sigma+2}}{\Gamma(\lambda+\sigma+2)} \Pi_\alpha^{\lambda+\sigma+1,\lambda}(t^{\sigma+1}\mathscr{D}_t v)$$

by Theorem 5.7, the definition of Λ implies

$$\Psi(v) = \frac{2^{\lambda+\sigma+2}}{\Gamma(\lambda+\sigma+2)} \Psi(\Pi_\alpha^{\lambda+\sigma+1,\lambda}(t^{\sigma+1}\mathscr{D}_t v)) = \Lambda(t^{\sigma+1}\mathscr{D}_t v)$$
$$= \int_H t^{\sigma+1}\mathscr{D}_t v(x,t)d\mu(x,t).$$

On the other hand, the definition of u and the Fubini theorem show that

$$\langle u, v\rangle_{\lambda,\sigma} = \frac{2^{\lambda+\sigma+2}}{\Gamma(\lambda+\sigma+2)} \int_H u(y,s)\mathscr{D}_t v(y,s) s^{\lambda+\sigma+1}\,dV(y,s)$$
$$= \frac{2^{\lambda+\sigma+2}}{\Gamma(\lambda+\sigma+2)} \int_H \int_H \mathscr{D}_t v(y,s)\mathscr{D}_t^{\lambda+\sigma+2} W^{(\alpha)}(x-y,t+s)$$
$$\times s^{\lambda+\sigma+1}\,dV(y,s)t^{\sigma+1}\,d\mu(x,t).$$

Since Theorem 5.7 again implies

$$\frac{2^{\lambda+\sigma+2}}{\Gamma(\lambda+\sigma+2)} \int_H \mathscr{D}_t v(y,s)\mathscr{D}_t^{\lambda+\sigma+2} W^{(\alpha)}(x-y,t+s)s^{\lambda+\sigma+1}\,dV(y,s)$$
$$= \mathscr{D}_t\left(\frac{2^{\lambda+\sigma+2}}{\Gamma(\lambda+\sigma+2)} \int_H \mathscr{D}_t v(y,s)\omega_\alpha^{\lambda+\sigma+1}(x,t;y,s)s^{\lambda+\sigma+1}\,dV(y,s)\right)$$
$$= \mathscr{D}_t v(x,t),$$

we obtain $\langle u, v\rangle_{\lambda,\sigma} = \Psi(v)$. It follows that $\pi(u) = \Psi$. □

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