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On Support Theorems

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1. Introduction. Let R_n be the n dimensional Euclidean space and let Ξ_n be the dual of R_n. The elements of R_n and Ξ_n are sequences $x=(x_1, x_2, \cdots, x_n)$ and $\xi=(\xi_1, \xi_2, \cdots, \xi_n)$ of real numbers. We put

$$D=(D_1, D_2, \cdots, D_n) \qquad \text{with} \quad D_j=\frac{1}{i}\frac{\partial}{\partial x_j} \qquad (j=1, 2, \cdots, n).$$

For convenience' sake we use the notations:

$$x=(x', t), \qquad x'=(x_1, x_2, \cdots, x_{n-1}), \qquad t=x_n,$$

$$\xi=(\xi', \tau), \qquad \xi'=(\xi_1, \xi_2, \cdots, \xi_{n-1}), \qquad \tau=\xi_n,$$

$$D_{x'}=(D_1, D_2, \cdots, D_{n-1}), \qquad D_t=D_n.$$

We denote by Ξ_{n-1} the $n-1$ dimensional space consisting of elements ξ'.

Let $\mathcal{D}$, $\mathcal{S}$ and $\mathcal{O}_M$ be the spaces of all C^∞-functions with compact supports, all rapidly decreasing C^∞-functions and all slowly increasing C^∞-functions on R_n respectively. These spaces are provided with usual topologies of L. Schwartz [4]. Let $\mathcal{D}'$ and $\mathcal{S}'$ be the strong duals of $\mathcal{D}$ and $\mathcal{S}$ respectively and let $\mathcal{O}'_C$ be the space of all rapidly decreasing distributions. We shall denote by $\mathcal{O}_M(\Xi_{n-1})$ the space $\mathcal{O}_M$ considered on Ξ_{n-1}. By the partial Fourier transform of $T \in \mathcal{S}'$ we understand the Fourier transform of T with respect to the first $n-1$ variables which will be denoted by $\hat{T}(\xi', t)$.

For any $A(\xi') \in \mathcal{O}_M(\Xi_{n-1})$, we define the operator $A(D_{x'})$ on $\mathcal{S}'$ as follows: The partial Fourier transform of $A(D_{x'})\,T$, $T \in \mathcal{S}'$, is $A(\xi')\,\hat{T}(\xi', t)$. In this paper we are concerned with the operator of the following form:

$$F(D_{x'}, D_t)=D_t^m+A_1(D_{x'})D_t^{m-1}+\cdots+A_m(D_{x'})$$

$$\text{with} \quad A_j(\xi') \in \mathcal{O}_M(\Xi_{n-1}) \quad (j=1, 2, \cdots, m) \quad \text{and} \quad m \geq 1.$$

J. Peetre observed in [2, 3] that the operator

$$D_t-i(1+\sum_{j=1}^{n-1}D_j^2)^{1/2}$$

leaves invariant for every element T of a subspace of $\mathscr{S}'$ the infimum k_T of t-coordinates of points of its support. We shall show that if $F(\xi', \tau)=0$ has only roots τ whose imaginary parts are $> c$ (a positive constant) then $F(D_{x'}, D_t)$ leaves k_T invariant for every $T \in \mathscr{S}'$ (Theorem 1), and that in the general case $F(D_{x'}, D_t)$ leaves k_T invariant for every $T \in \mathscr{S}'$ such that $k_T > -\infty$ (Theorem 2). It is the purpose of this paper to give elementary proofs of these facts.

2. For any $T \in \mathscr{D}'$ we denote by k_T

$$\inf \{t\colon\ x \in \operatorname{supp} T\},$$

where we understand $k_T = +\infty$ if supp T is empty.

We use the notation $[t<\alpha]$ for the set of all elements x of R_n such that $t<\alpha$ and similarly for $[t\leq\alpha]$ etc.

We begin with

LEMMA 1. *Let $T \in \mathscr{D}'$.*

(1) *If a sequence $\{T_j\}$ of $\mathscr{D}'$ converges in $\mathscr{D}'$ to T, then $\overline{\lim\limits_{j\to\infty}}\, k_{T_j} \leq k_T$.*

(2) *Let $\{\rho_j\}$ be a sequence of regularization, let $\{\alpha_j\}$ be a sequence of multiplicators and put $T_j = \alpha_j T * \rho_j$. Then $\lim\limits_{j\to\infty} k_{T_j} = k_T$.*

PROOF. (1): We put $a = \overline{\lim\limits_{j\to\infty}}\, k_{T_j}$. If $a=-\infty$, the assertion is evident. Assume that $a>-\infty$ and let α be any real number such that $\alpha < a$. Then there exists an increasing sequence $\{j_k\}$ such that $\alpha < k_{T_{j_k}}$. If $\phi \in \mathscr{D}$ and supp $\phi \subset [t<\alpha]$, then we have

$$< T_{j_k}, \phi > = 0$$

since supp $T_{j_k} \subset [t\geq\alpha]$. Passing to the limit, we obtain $<T, \phi>=0$ and therefore $\alpha \leq k_T$. Thus $a \leq k_T$.

(2): Since

$$\operatorname{supp} T_j \subset \operatorname{supp}(\alpha_j T) + \operatorname{supp} \rho_j \subset \operatorname{supp} T + \operatorname{supp} \rho_j,$$

we have $k_{T_j} \geq k_T + k_{\rho_j}$. T_j converges in $\mathscr{D}'$ to T and supp ρ_j converges to the origin as $j\to\infty$. Hence by (1)

$$\underline{\lim_{j\to\infty}}\, k_{T_j} \geq k_T + \lim_{j\to\infty} k_{\rho_j} = k_T \geq \overline{\lim_{j\to\infty}}\, k_{T_j}.$$

Consequently, $\lim\limits_{j\to\infty} k_{T_j} = k_T$.

Thus the proof is complete.

In the sequel F denotes the operator $F(D_{x'}, D_t)$ stated in the introduction. We prove

LEMMA 2. *For any* $T \in \mathscr{S}'$ *we have* $k_T \leq k_{F(T)}$.

PROOF. Since $F(T) = F\delta * T$ with δ, the Dirac measure, we have

$$\begin{aligned}\operatorname{supp} F(T) &\subset \operatorname{supp} F\delta + \operatorname{supp} T \\ &\subset [t=0] + [t \geq k_T] = [t \geq k_T].\end{aligned}$$

Therefore the assertion is immediate.

LEMMA 3. *Assume that* $F(\xi', \tau) = 0$ *has only roots* τ *with positive imaginary parts and let* $\phi \in \mathscr{S}$. *If* $F(\phi) \in \mathscr{D}$, *then* $k_\phi = k_{F(\phi)}$.

PROOF. We put $\psi = F(\phi)$. Since $k_\phi \leq k_\psi$ by Lemma 2, we are only to prove that $k_\psi \leq k_\phi$. The partial Fourier transform of ϕ satisfies the differential equation

$$D_t^m \hat{\phi}(\xi', t) + A_1(\xi') D_t^{m-1} \hat{\phi}(\xi', t) + \cdots + A_m(\xi') \hat{\phi}(\xi', t) = \hat{\psi}(\xi', t).$$

If we consider the equation on $[t < k_\psi]$, this becomes

$$D_t^m \hat{\phi}(\xi', t) + A_1(\xi') D_t^{m-1} \hat{\phi}(\xi' t) + \cdots + A_m(\xi') \hat{\phi}(\xi', t) = 0.$$

We now fix ξ' and let $\tau_1, \tau_2, \cdots, \tau_k$ be the distinct roots of $F(\xi', \tau) = 0$ with respective multiplicities $m_1, m_2, \cdots, m_k$ where $\sum_{j=1}^{k} m_j = m$. Then we have

$$\hat{\phi}(\xi', t) = \sum_{j=1}^{k} P_j(t) e^{i\tau_j t} \qquad \text{for } t < k_\psi,$$

where $P_j(t)\,(j = 1, 2, \cdots, k)$ are polynomials in t of degree $m_j - 1$.

We assert that $P_j(t) \equiv 0$ $(j = 1, 2, \cdots, k)$. In fact, suppose that $P_j(t) \not\equiv 0$. If we put

$$\chi(t) = (D_t - \tau_1)^{m_1} \cdots (D_t - \tau_{j-1})^{m_{j-1}} (D_t - \tau_{j+1})^{m_{j+1}} \cdots (D_t - \tau_k)^{m_k} \hat{\phi}(\xi', t),$$

then $\chi(t)$ is rapidly decreasing since $\phi \in \mathscr{S}$. Now $\chi(t)$ can be written in the form

$$\chi(t) = e^{i\tau_j t} Q(D_t) P_j(t)$$

where

$$Q(D_t) = (D_t + \tau_j - \tau_1)^{m_1} \cdots (D_t + \tau_j - \tau_{j-1})^{m_{j-1}}$$

$$\times (D_t + \tau_j - \tau_{j+1})^{m_{j+1}} \cdots (D_t + \tau_j - \tau_k)^{m_k}.$$

Since $\tau_j \neq \tau_h$ for all $h \neq j$, $Q(D_t)P_j(t) \not\equiv 0$. The imaginary part of τ_j being positive,

$$e^{-(\mathrm{Im}\,\tau_j)t} Q(D_t) P_j(t)$$

is not rapidly decreasing for $t < k_\psi$ and therefore $\chi(t)$ is not also, which is a contradiction. Hence we have $P_j(t) \equiv 0$ $(j = 1, 2, \dots, k)$, as was asserted.

Thus we have

$$\hat{\phi}(\xi', t) = 0 \qquad \text{for } t < k_\psi,$$

which shows that $k_\psi \leq k_{\hat{\phi}(\xi', t)}$. By using the fact that

$$k_\phi = \inf_{\xi'} k_{\hat{\phi}(\xi', t)},$$

we conclude that $k_\psi \leq k_\phi$.

The proof of the lemma is complete.

3. We shall now show the two theorems stated in the introduction. We first prove the following

Theorem 1. *Assume that $F(\xi', \tau) = 0$ has only roots τ whose imaginary parts are $> c$ for a positive constant c. Then $k_T = k_{F(T)}$ for every $T \in \mathcal{S}'$.*

Proof. Denoting the roots of $F(\xi', \tau) = 0$ by $\tau_j(\xi')$ $(j = 1, 2, \dots, m)$, we have

$$F(\xi', \tau) = \prod_{j=1}^{m} \big(\tau - \tau_j(\xi')\big)$$

and therefore

$$|F(\xi', \tau)| \geq \prod_{j=1}^{m} \mathrm{Im}\,\tau_j(\xi') \geq c^m > 0.$$

Hence $1/F(\xi', \tau)$ is in $\mathcal{O}_M$. Let G be the inverse Fourier transform of $1/F(\xi', \tau)$. Then G is an element of $\mathcal{O}'_C$ and for every $T \in \mathcal{S}'$

$$G * F(T) = F(G * T) = T.$$

Now take a sequence $\{\rho_j\}$ of regularization and a sequence $\{\alpha_j\}$ of multiplicators and put $T_j=\alpha_j T*\rho_j$. Then $T_j \in \mathcal{D}$, $G*T_j \in \mathcal{S}$ and $F(G*T_j)=T_j$. Therefore it follows from Lemma 3 that $k_{T_j}=k_{G*T_j}$. Since T_j converges in $\mathcal{S}'$ to T, $G*T_j$ converges in $\mathcal{S}'$ to $G*T$. Therefore by using Lemma 1 we see that

$$k_T=\lim_{j\to\infty} k_{T_j}=\lim_{j\to\infty} k_{G*T_j}\leq k_{G*T}.$$

Since $F(G*T)=T$, Lemma 2 shows that $k_{G*T}\leq k_T$. Hence it follows that $k_T=k_{G*T}$. By replacing T by $F(T)$, we have

$$k_{F(T)}=k_{G*F(T)}=k_T.$$

Thus the proof is complete.

We next prove the following

THEOREM 2. *Let T be a distribution $\in \mathcal{S}'$ such that $k_T>-\infty$. Then $k_T=k_{F(T)}$.*

PROOF. By the assumption

$$A_j(\xi')\in \mathcal{O}_M(\Xi_{n-1}) \qquad (j=1, 2, \cdots, m).$$

Hence there exist a positive integer h and a positive constant c such that

$$|A_j(\xi')|^{1/j}\leq |\xi'|^{2h} \qquad \text{for } |\xi'|\geq c \qquad (j=1, 2, \cdots, m).$$

If we denote the roots of $F(\xi', \tau)=0$ by $\tau_j(\xi')$ $(j=1, 2, \cdots, m)$, it is easily shown that

$$|\tau_j(\xi')|<2|\xi'|^{2h} \qquad \text{for } |\xi'|\geq c \qquad (j=1, 2, \cdots, m).$$

On the other hand for $|\xi'|\leq c$ we have with a constant d

$$|\mathrm{Im}\,\tau_j(\xi')|<d \qquad (j=1, 2, \cdots, m).$$

We put

$$U(\xi', t)=e^{-2(|\xi'|^{2h}+d)t}\hat{T}(\xi', t).$$

Then we assert that $U\in \mathcal{S}'$. In fact, take a real number a such that $a<k_T$ and choose a bounded C^∞-function $\phi(t)$ in such a way that

$$\phi(t)=\begin{cases}1 & \text{for } t\geq a\\ 0 & \text{for } t\leq a-1.\end{cases}$$

Then we have

$$\phi(t)e^{-2(|\xi'|^{2h}+d)t}\in O_M$$

and $U=\phi U$. It follows that $U\in\mathscr{S}'$, as was asserted. We now define $S\in\mathscr{S}'$ as follows:

$$\hat{S}(\xi',t)=U(\xi',t).$$

Then it is evident that $k_S=k_T$.

Let F_1 be the operator of the same type as F, which is defined by

$$F_1(\xi',D_t)=(D_t-2i(|\xi'|^{2h}+d))^m+$$
$$+A_1(\xi')(D_t-2i(|\xi'|^{2h}+d))^{m-1}+\cdots+A_m(\xi').$$

Then the partial Fourier transform of $F_1(S)$ is

$$\widehat{F_1(S)}(\xi',t)=[(D_t-2i(|\xi'|^{2h}+d))^m+$$
$$+A_1(\xi')(D_t-2i(|\xi'|^{2h}+d))^{m-1}+\cdots+A_m(\xi')]\hat{U}(\xi',t)$$
$$=e^{-2(|\xi'|^{2h}+d)t}[D_t^m+A_1(\xi')D_t^{m-1}+\cdots+A_m(\xi')]\hat{T}(\xi',t)$$
$$=e^{-2(|\xi'|^{2h}+d)t}\widehat{F(T)}(\xi',t).$$

Consequently,

$$k_{F_1(S)}=\inf_{\xi'}k_{\widehat{F_1(S)}(\xi',t)}=\inf_{\xi'}k_{\widehat{F(T)}(\xi',t)}=k_{F(T)}.$$

Now the roots of $F_1(\xi',\tau)=0$ are

$$\tilde{\tau}_j(\xi')=2i(|\xi'|^{2h}+d)+\tau_j(\xi')\qquad(j=1,2,\cdots,m).$$

By considering separately the cases where $|\xi'|\geq c$ and where $|\xi'|\leq c$, we have

$$|\mathrm{Im}\,\tilde{\tau}_j(\xi')|\geq 2(|\xi'|^{2h}+d)-|\mathrm{Im}\,\tau_j(\xi')|>d.$$

Therefore we can apply Theorem 1 to infer that $k_S=k_{F_1(S)}$. Hence we conclude that $k_T=k_{F(T)}$.

Thus the proof of the theorem is complete.

Similarly, we can show that if we put

$$K_T = \sup\{t: x \in \operatorname{supp} T\}$$

then $K_T = K_{F(T)}$ for every $T \in \mathscr{S}'$ such that $K_T < +\infty$.

Theorem 2 does not hold for an element T of $\mathscr{S}'$ such that $k_T = -\infty$ in general. For example, take $F = D_t$ and let T be a non-zero constant. Then $k_T = -\infty$ and $k_{F(T)} = +\infty$.

As an illustration of our results, we consider the differential operator

$$P(D) = D_t^m + A_1(D_{x'})D_t^{m-1} + \cdots + A_m(D_{x'})$$

where $A_j(D_{x'})$ $(j = 1, 2, \cdots, m)$ are polynomials in $D_{x'}$ with constant coefficients. If the plane $t = 0$ is characteristic with respect to $P(D)$, then the differential equation $P(D)\,T = 0$ has a null solution with respect to the half space $[t \geq 0]$, that is, a C^∞-function which is 0 for $t < 0$ and whose support contains the origin ([1], p. 121). By making use of Theorem 2 we can assert that there exist no null solutions of $P(D)\,T = 0$ contained in $\mathscr{S}'$. In fact, let T be a null solution of $P(D)\,T = 0$. If $T \in \mathscr{S}'$, then by Theorem 2 we see that $k_T = k_{P(D)T}$, which contradicts the facts that $k_T = 0$ and $k_{P(D)T} = +\infty$. Therefore $T \notin \mathscr{S}'$.

For example, the equation

$$P(D)\,T = \frac{\partial T}{\partial t} - \sum_{j=1}^{n-1} \frac{\partial^2 T}{\partial x_j^2} = 0$$

has actually a null solution, since the plane $t = 0$ is characteristic with respect to $P(D)$. By the result stated above, there exist no null solutions contained in $\mathscr{S}'$.

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