J. Sci. Hiroshima Univ. Ser. A-I
30 (1966), 117-122

On the Integral Closure of a Domain

H. S. Butts and W. W. Smith

(Received March 10, 1966)

0. Introduction

Throughout this paper D will denote an integral domain with $1\neq 0$ and quotient field K. An element $x\in K$ is integral with respect to D provided there exist elements $a_0, a_1, \cdots, a_{n-1}$ in D such that $x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0=0$. The integral closure D^c of D in K consists of the elements of K which are integral with respect to D, and D is said to be integrally closed in case $D=D^c$. If $x\in K$, we say that x is "almost integral" over D provided there exists $d\in D$ such that $d\neq 0$ and $dx^n\in D$ for each positive integer n. If the set D^* of almost integral elements of K over D is equal to D, then D is called "completely integrally closed" in K.

In section 1 we determine conditions in order that D^c be a one-dimensional Prufer domain, an almost Dedekind domain, and a Dedekind domain.

In [12] Krull shows that if D is a valuation ring then D is completely integrally closed if and only if D is a (proper) maximal ring in K (i.e., a rank one valuation ring). It follows that an intersection of maximal rings in K is completely integrally closed, and Krull conjectured that the converse was also true. However, an example by Nakayama [14] shows the converse to be false. In section 2 we show that the converse is true in case D satisfies a certain finiteness condition.

In general our notation and terminology will be that of [1] and [2]. In particular, we use $\subset$ to mean "contained in or equal" and $<$ to denote proper containment. An ideal A of D is proper provided $(0)<A<D$. If J is a domain with quotient field K, then J^c will denote the integral closure of J in K and J^* will denote the "complete integral closure" of J in K (i.e., the set of "almost integral" elements of K over J).

1. The Domain D^c

A domain D is called a Prufer (almost Dedekind) domain provided the quotient ring D_P is a valuation ring (rank one discrete valuation ring) for each proper prime ideal P of D (see [3], [7] and [9]). The following theorem was first proved by Gilmer in [5], however we include it here for completeness and give a different proof.

Theorem 1: *Every domain D' such that $D\subset D'<K$ is one dimensional if*

and only if D^c is a one dimensional Prufer domain.

PROOF: Suppose that every domain D' such that $D \subset D' < K$ is one dimensional, and let P be a proper prime ideal of D^c. Let t be any element of K and suppose that neither t nor t^{-1} belong to D^c_P. Then, by the corollary on page 20 of [2], the ideal $PD^c[t]$ is a proper prime ideal of $D^c[t]$ and therefore maximal. If $t \in PD^c[t]$ then there exist $p_i \in P$ for $i=0, 1, \cdots, n$ such that $p_0 + p_1 t + \cdots + p_n t^n = t$ and $p_0 + (p_1 - 1)t + \cdots + p_n t^n = 0$. Since $p_1 - 1 \notin P$, it follows from the lemma on page 19 [2] that either t or t^{-1} belongs to D^c_P. If $t \notin PD^c[t]$, then $PD^c[t] + (t)D^c[t] = D^c[t]$ and hence there exist $x_i \in P$ for $i=0, \cdots, m$ and $y_i \in D^c$ for $i=0, 1, \cdots, n$ such that
$x_0 + x_1 t + \cdots + x_m t^m + (y_0 + y_1 t + \cdots + y_n t^n)t = 1$. Since $x_0 - 1 \notin P$ it follows that either t or t^{-1} belongs to D^c_P.

Conversely, suppose that D^c is a one dimensional Prufer domain and let D_v be any valuation ring such that $D \subset D_v < K$. Then $D^c \subset D_v < K$. Let P_v be the center of D_v in D^c. It is clear that $D^c_{P_v} \subset D_v$. Since D^c is a one dimensional Prufer domain, then $D^c_{P_v}$ is a rank one valuation ring (and therefore a maximal ring in K) and hence $D^c_{P_v} = D_v$. Since every valuation ring lying over D is rank one, then every domain D' such that $D \subset D' < K$ is one dimensional (see [11], Theorem 11.9, page 37).

COROLLARY 1: *Every valuation ring of K lying over D is rank one if and only if D^c is a one dimensional Prufer domain.*

THEOREM 2: *Every valuation ring of K lying over D is discrete and rank one if and only if D^c is an almost Dedekind domain.*

PROOF: Suppose that every valuation ring of K lying over D is discrete and rank one. It follows from Corollary 1 that D^c is a one dimensional Prufer domain, and therefore D^c_P is a discrete and rank one valuation ring for each proper prime P in D^c. Hence D^c is almost Dedekind.

Suppose D^c is almost Dedekind and let D_v be a valuation ring such that $D \subset D_v < K$. Then $D^c \subset D_v < K$. If P is the center of D_v in D^c, then $D^c_P \subset D_v$; since D^c_P is discrete and rank one, it follows that $D^c_P = D_v$ and every valuation ring of K lying over D is rank one and discrete.

We say that D has property π provided $\bigcap_{n=0}^{\infty} P^n = (0)$ for every proper prime ideal P in every domain J such that $D \subset J \subset K$.

COROLLARY 2: *The domain D has property π if and only if D^c is an almost Dedekind domain.*

LEMMA 1: *If D^c is a Dedekind domain then D is one dimensional and every proper ideal in D is contained in only a finite number of prime ideals of D.*

Proof: It follows from page 259 of [1] that D is one dimensional, and therefore two different proper prime ideals of D cannot be contained in the same proper prime ideal of D^c. If A is a proper ideal of D, then AD^c is a proper ideal of D^c which is contained in only a finite number of proper prime ideals in D^c. Hence A is contained in only a finite number of proper prime ideals in D.

Lemma 2: *If D^c is a Dedekind domain, then every proper ideal of D is an intersection of a finite number of pairwise co-maximal primary ideals in D (and hence every proper ideal of D is a product of a finite number of pairwise co-maximal primary ideals in D).*

Proof: Since proper prime ideals are maximal in D, then every ideal of D is equal to its kernel (see page 738 of [13]). Since there are only a finite number of prime ideal divisors of a proper ideal in D, then every proper ideal in D is an intersection of a finite number of pairwise comaximal primary ideals in D. (It is clear that this representation of a proper ideal as an intersection of pairwise comaximal primary ideals is unique).

A proper ideal N of D is called non-factorable provided $N=AB$, where A and B are ideals of D, implies that either $A=D$ or $B=D$.

Theorem 3: *If D^c is a Dedekind domain, then every proper ideal of D is a product of a finite number of non-factorable ideals in D.*

Proof: By lemma 2 it is sufficient to prove that every proper primary ideal in D is a product of non-factorable ideals in D. Let Q be a proper primary ideal and denote by M the radical of Q. Since D has property π, there exists a positive integer e such that $Q\subset M^e$ and $Q\not\subset M^{e+1}$. If $Q=A_1A_2\cdots A_n$, then $M=\sqrt{Q}\subset\sqrt{A_i}$ for $i=1, \cdots, n$. If A_1 is a proper ideal then $\sqrt{A_i}=M$ since M is maximal. It follows that $n\leq e$ and Q is a product of non-factorable ideals.

Remark: If D^c is a Dedekind domain, then it follows from lemma 2 that non-factorable ideals in D are primary. It follows from [8] that every proper ideal of D is a unique product of non-factorable ideals in D if and only if $D=D^c$.

Theorem 4: *D^c is a Dedekind domain if and only if*

1) *every valuation ring of K lying over D is discrete and rank one, and*

2) *if $0\neq x\in D$, then x is a unit in all except a finite number of valuation rings of K lying over D.*

Proof: If D^c is a Dedekind domain, then D^c is almost Dedekind and 1) follows from theorem 2; 2) follows from lemma 1.

If 1) and 2) hold, then D^c is almost Dedekind by theorem 2 (and hence one-dimensional) and every proper ideal of D^c is contained in only a finite number of prime ideals in D^c (see theorem 5, page 12 of [2]). It follows from Theorem 2.10 of [9] that every proper ideal of D^c is a product of prime powers and D^c is a Dedekind domain.

Mori and Nagata have proved that if D is Noetherian and one or two dimensional, then D^c is Noetherian ([10]). Hence if D is Noetherian and one dimensional then D^c is a Dedekind domain. If D^c is a Dedekind domain, then D is one-dimensional and has several "Noetherian properties" (e.g. $\cap A^n=(0)$ for every proper ideal A in D, every proper ideal is an intersection of a finite number of primary ideals, and every proper ideal is a product of a finite number of non-factorable ideals), however D need not be Noetherian as is shown by the following example (due to Robert W. Gilmer).

Let F be the prime field of characteristic 2 and $\bar{F}$ the algebraic closure of F. Let x be an indeterminate over $\bar{F}$ and set $J=\bar{F}[x]_{(x)}$. Denote by M the maximal ideal in the rank one discrete valuation ring J. Let D be the domain generated by F and M in J. It follows from theorem 6 and lemma 1 of [4] that D is not Noetherian, however D^c is a Dedekind domain.

2. Completely Integrally Closed Domains

We say that a domain J with quotient field K has property α provided there exists a collection F of valuations of K such that

1) J is the intersection of all of the valuation rings associated with the valuations of F, and
2) if $0\neq x\in K$, then $v(x)=0$ for all except a finite number of v in F.

We prove in corollary 1 that if D is a completely integrally closed domain with property α, then D is an intersection of (proper) maximal rings of K (we assume that $D\neq K$).

It is clear that $D^c\subset D^*$ and that $[D^*]^c=D^*$. However D^* may not be completely integrally closed, as is seen by an example due to Gilmer [6]. We show that if D^c has property α, then D^* is completely integrally closed if and only if $D^*=[D^c]^*$.

LEMMA 3: *If the domain $D(\neq K)$ has a proper minimal prime ideal, then D is contained in a (proper) maximal ring (i.e. a rank one valuation ring) of K.*

PROOF: Let P be a (proper) minimal prime ideal of D and let Γ be the collection of all valuation rings of K lying over D with center at P. Partially order Γ by inclusion and let Γ_0 be a maximal chain in Γ. Let $R=\cup D_v$, where the union is taken over all $D_v\in\Gamma_0$. It is clear that R is a (proper) maximal ring of K.

LEMMA 4: *Suppose that D^c has property α and that F is its associated* *family of valuations. If $0 \neq x \in [D^c]^*$ and x is not a unit in $[D^c]^*$, then there exists a valuation $v \in F$ such that x is contained in all of the proper prime ideals of D_v (the valuation ring of v) and hence D_v has a proper minimal prime ideal which contains x.*

PROOF: Let $v_1, \cdots, v_n$ be the valuations of F such that $v(x)=0$ for all $v \in F$ with $v \neq v_i$ $(i=1, \cdots, n)$, and let D_{v_i} be the valuation ring of v_i for $i=1, \cdots, n$. Suppose there exists a proper prime ideal P_i in D_{v_i} such that $x \notin P_i$ for $i=1, \cdots, n$. If $0 \neq d_i \in P_i \cap D^c$ for $i=1, \cdots, n$ then $d=d_1 d_2 \cdots d_n \neq 0$ and $d\left(\frac{1}{x}\right)^k \in D_{v_i}$ for every positive integer k (for $i=1, \cdots, n$). Since $\frac{1}{x} \in D_v$ for all $v \in F$ with $v \neq v_i$, then $d\left(\frac{1}{x}\right)^k \in \bigcap_{v \in F} D_v = D^c$ for every positive integer k. Thus $\frac{1}{x}$ is almost integral over D^c and hence $\frac{1}{x} \in [D^c]^*$, which is a contradiction. This proves the lemma.

As in the statement of lemma 4, suppose that D^c has property α and that F is its associated family of valuations. Now for each valuation ring D_v (with $v \in F$) which has a proper minimal prime ideal, let v' be the rank one valuation centered at that minimal prime. Let F' be the collection of all rank one valuations obtained in this manner. It is clear that $\bigcap_{v' \in F'} D_{v'}$ has property α (with F' as its associated family of valuations) and is completely integrally closed. As a result we have $D^c \subset [D^c]^* \subset \bigcap_{v' \in F'} D_{v'}$.

THEOREM 5: *If D^c has property α then $[D^c]^*$ is completely integrally closed and is the intersection of rank one valuation rings. In particular $[D^c]^* = \bigcap_{v' \in F'} D_{v'}$.*

PROOF: It is sufficient to prove that $\bigcap_{v' \in F'} D_{v'} \subset [D^c]^*$. Suppose $x \in \bigcap_{v' \in F'} D_{v'}$, let $v_1, \cdots, v_n$ be the valuations of F for which $v_i(x) \neq 0$. Index the v_i so that $v_i(x)>0$ for $i=1, \cdots, m$ and $v_i(x)<0$ for $m+1 \leq i \leq n$. Since the valuation rings D_{v_i} have no proper minimal prime (for $m+1 \leq i \leq n$) then there exist prime ideals $Q_i < P_i$ in D_{v_i} such that $\frac{1}{x} \in P_i \setminus Q_i$. Let d_i be a non-zero element of $Q_i \cap D^c$ for $m+1 \leq i \leq n$. Then $d_i x^k \in D_{v_i}$ for every positive integer k. Let $d=d_{m+1} \cdots d_n$. Then $d \neq 0$ and $dx^k \in D_{v_i}$ for $m+1 \leq i \leq n$ and for every positive integer k. It is clear that $dx^k \in Dv$ for all $v \in F$ with $v \neq v_i$ for $m+1 \leq i \leq n$, and hence $dx^k \in D_v$ for all $v \in F$ and for every positive integer k. Hence $dx^k \in D^c$ for every positive integer k, and therefore $x \in [D^c]^*$.

THEOREM 6: *If D^c has property α then D^* is completely integrally closed if and only if $D^* = [D^c]^*$.*

PROOF: This follows from theorem 5 since

$$D \subset D^c = \bigcap_{v \in F} D_v \subset D^* \subset [D^c]^* = \bigcap_{v' \in F'} D_{v'} \subset [D^*]^*.$$

COROLLARY 1: *If D has property α and is completely integrally closed, then D is an intersection of maximal rings.*

PROOF: Apply theorem 5 and the fact that a completely integrally closed domain is integrally closed.

COROLLARY 2: *If D^c has property α then $[D^*]^*$ is completely integrally closed.*

PROOF: This follows from theorem 5 and the proof of theorem 6.

COROLLARY 3: *If D^c has property α and the conductor of D in D^c is non-zero, then D^* is completely integrally closed.*

PROOF: Let $x \in [D^c]^*$ and let $f \neq 0$ be an element of the conductor. There exists $d \neq 0$ in D such that $dx^n \in D^c$ for each positive integer n. Therefore $fdx^n \in D$ for each n and hence $x \in D^*$. Hence $D^* = [D^c]^*$ and D^* is completely integrally closed by theorem 6.

References

[1] O. Zariski and P. Samuel, *Commutative Algebra*, vol. 1, D. van Nostrand Co., Inc., Princeton, 1958.

[2] ————, *Commutative Algebra*, vol. 2,D. van Nostrand Co., Inc., Princeton, 1960.

[3] R. W. Gilmer, *Integral domains which are almost Dedekind*, Proceedings Amer. Math. Soc., **15** (1964), pp. 813-818.

[4] ————, *Commutative rings containing at most two prime ideals*, Michigan Math. Jour., **10** (1963), pp. 263-268.

[5] ————, *Domains in which valuation ideals are prime powers*, submitted for publication.

[6] ————, *On the strong integral closure of an Integral Domain*, submitted for publication.

[7] R. Gilmer and J. Ohm, *Primary Ideals and Valuation Ideals*, Trans. Amer. Math. Soc. **117** (1965), pp. 235-250.

[8] H. S. Butts, *Unique factorization of ideals into nonfactorable ideals*, Proc. Amer. Math. Soc., **15** (1964), pp. 21.

[9] H. Butts and R. Phillips, *Almost Multiplication Rings*, Canadian Jour. of Math., **17** (1965), pp. 267-277.

[10] M. Nagata, *On the derived normal ring of Noetherian integral domains*, Mem. Coll. Sci Univ. Kyoto Ser. A, **29** (1955), pp. 293-303.

[11] ————, *Local rings*, Interscience, New York, 1962.

[12] W. Krull, *Allgemeine Bewertungstheorie*, Journal fur Mathematik, CLXVII (1932), pp. 160-196.

[13] ————, *Idealtheorie in Ringen Ohne Endlichkeit Bedingung*, Math. Annalen, **101** (1929), pp. 729-744.

[14] T. Nakayama, *On Krull's conjecture concerning completely integrally closed integrity domains* I, Proc. Imp. Acad. Tokyo **18** (1942), pp. 185-187; II, Proc. Imp. Acad. Tokyo **18** (1942), pp. 233-236; III, Proc. Japan Acad. (1946), pp. 249-250.

H. S. BUTTS
Louisiana State University
Baton Rouge, Louisiana, U.S.A.
and
W. W. SMITH
University of North Carolina
Chapel Hill, North Carolina, U.S.A.